

Derivative relationships between volume and surface area of compact regions in \mathbb{R}^p

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Abstract

We explore the idea that the derivative of the volume, V , of a region in \mathbb{R}^p with respect to r equals its surface area, A , where $r = pV/A$. We show that the families of regions for which this formula for r is valid, which we call homogeneous families, include all the families of similar regions. We determine equivalent conditions for a family to be homogeneous, provide examples of homogeneous families made up of non-similar regions, and offer a geometric interpretation of r in a few cases.

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1 Introduction

It is well known that there exists a remarkable derivative relationship between the area A and the perimeter P of a circle, namely

$$\frac{dA}{dr} = P,$$

where the variable r represents the radius of the circle. It is natural to wonder whether such a derivative relationship remains valid for other familiar shapes. At first glance, though, it does not even hold for the square when r represents the side length. However, it holds when r represents half of the side length, that is, the radius of the inscribed circle.

In a similar manner, the derivative of the volume function of a sphere is equal to the surface area, that is,

$$\frac{dV}{dr} = A$$

and this relationship still holds for cubes if r represents the radius of the inscribed sphere.

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We show that by choosing an appropriate variable to calculate volume and area, namely

$$r = \frac{pV}{A} \tag{1}$$

(as recently suggested by Tong [16]), we can generalize the derivative relationship to many compact regions in \mathbb{R}^p ($p \geq 2$).

Notice that, when we consider the derivative relationship of a given compact region, we actually consider a one-parameter family of similar compact regions. For example, the derivative relationship for a sphere involves considering a sphere that grows in radius, that is, a family of spheres.

Also, we can investigate families of non-similar regions. For example, consider a right circular cone whose base radius and height are functions of a certain parameter s . We can calculate the volume $V(s)$ and the surface area $A(s)$ as functions of s and then search for an appropriate change of variable $r(s)$ for which the derivative relationship holds.

In this general case of possibly non-similar regions, we show that the derivative relationship always holds for the change of variable

$$r(s) = \int \frac{V'(s)}{A(s)} ds. \tag{2}$$

In this paper we mainly investigate one-parameter families of regions for which the change of variable reduces to (1). We call these families *homogeneous* families and we show that a family is homogeneous if and only if its regions have the same isoperimetric ratio. In particular, any family of similar regions is homogeneous. We also show how to construct homogeneous families made up of non-similar regions.

The outline of this paper is as follows. In the next section we derive the change of variable formula (2). In Section 3 we provide characterizations of the class of homogeneous families. In Section 4 we show how to construct such families. Finally, in Section 5 we yield a geometric interpretation of the variable (1) for homogeneous families of star-like polyhedra.

Surprisingly, derivative relationships between volume and area of compact regions have not been widely investigated. To our knowledge, only a few researchers have worked on this interesting topic; see [6], [7], [11], [15], [16].

Throughout, we will use the notation \mathbb{R}_+ for the interval $(0, +\infty)$.

2 Derivative relationship: the general case

Let $p \geq 2$ be an integer. Consider a one-parameter family of compact regions in \mathbb{R}^p with boundaries of finite measures,

$$\mathcal{R} := \{R(s) \subset \mathbb{R}^p \mid s \in E\},$$

where E is an open interval of the real line. We assume that with this family is associated a differentiable function $V : E \rightarrow \mathbb{R}_+$ and a continuous function

$A : E \rightarrow \mathbb{R}_+$ such that, for any $s \in E$, the values $V(s)$ and $A(s)$ represent respectively the volume and the surface area of region $R(s)$.

Note that for plane figures in \mathbb{R}^2 , we replace the volume $V(s)$ and the area $A(s)$ with the area $A(s)$ and the perimeter $P(s)$, respectively.

The parameter s can represent either a linear dimension, or an angle, or may have no geometric meaning.

Example 2.1. Consider a family of cubes in \mathbb{R}^3 , with edge length $s \in \mathbb{R}_+$. In that case the volume and area functions are clearly given by $V(s) = s^3$ and $A(s) = 6s^2$, respectively. Of course, we could as well choose any positive function $\phi(s)$ of the parameter s to represent the edge length, thus leading to the new functions $V(s) = \phi(s)^3$ and $A(s) = 6\phi(s)^2$. In such an alternative representation the parameter s may have no geometric interpretation.

Under very general conditions, it is always possible to find an appropriate variable of differentiation leading to the derivative relationship between volume and surface area.

Proposition 2.1. *Suppose $V(s)$ is a strictly monotone and differentiable function in E and $A(s)$ is a continuous function in E . Then there is a differentiable change of variable $r(s) : E \rightarrow r(E)$, defined as*

$$r(s) := \int \frac{V'(s)}{A(s)} ds \quad (s \in E) \quad (3)$$

and unique within an additive constant $C \in \mathbb{R}$, such that

$$\frac{d}{dr} V[s(r)] = A[s(r)] \quad (r \in r(E)). \quad (4)$$

Proof. The sign of the derivative

$$r'(s) = \frac{V'(s)}{A(s)} \quad (s \in E) \quad (5)$$

is constant and $r(s)$ is a differentiable change of variable from E to $r(E)$.

By the chain rule, we then have

$$\frac{d}{dr} V[s(r)] = V'[s(r)] s'(r) = \frac{V'[s(r)]}{r'[s(r)]} = A[s(r)]$$

for all $r \in r(E)$. The uniqueness of $r(s)$ follows immediately from the latter equality. \square

From Eq. (3) we immediately see that the variable of differentiation r represents a linear dimension. Moreover, if $V(s)$ and $A(s)$ are replaced with

$$V_\phi(s) = V[\phi(s)] \quad \text{and} \quad A_\phi(s) = A[\phi(s)],$$

respectively, where ϕ is a differentiable function from E into itself, then $r(s)$ is simply replaced with

$$\begin{aligned} r_\phi(s) &= \int \frac{V'_\phi(s)}{A_\phi(s)} ds = \int \frac{V'[\phi(s)] \phi'(s)}{A[\phi(s)]} ds = \int \frac{V'(t)}{A(t)} dt \Big|_{t=\phi(s)} \\ &= r[\phi(s)], \end{aligned}$$

which clearly shows that the change of variable remains stable under any change of representation.

In Example 2.1, with the family of cubes of edge lengths s , we have

$$r(s) = \frac{s}{2} + C,$$

for a constant $C \in \mathbb{R}$. Letting $C = 0$, we observe that r represents the radius of the inscribed sphere. We then have

$$V[s(r)] = 8r^3 \quad \text{and} \quad A[s(r)] = 24r^2,$$

thus retrieving Eq. (4) with $E = r(E) = \mathbb{R}_+$.

Although the new variable r represents a length, a geometric interpretation of it is not always immediate, as the following example shows:

Example 2.2. Consider a family of rectangles with fixed length $a > 0$ and variable width $s > 0$. Then we have $A(s) = as$, $P(s) = 2s + 2a$, and

$$r(s) = \frac{a}{2} \ln(2s + 2a) + C.$$

In this case, no interpretation is known for the variable r .

Notice also that it is necessary that $V(s)$ be strictly monotone in E for $r(s)$ to be a change of variable. In situations where $V(s)$ is not strictly monotone in its domain, it is necessary to partition this domain into open subintervals E in which $V(s)$ is strictly monotone.

Example 2.3. Consider a family of rhombi in \mathbb{R}^2 with sides of fixed length $a > 0$ and a diagonal of variable length $s \in (0, 2a)$. The perimeter $P(s) = 4a$ is constant while the area

$$A(s) = s \sqrt{a^2 - \frac{s^2}{4}}$$

is strictly increasing in $(0, \sqrt{2}a)$ and strictly decreasing in $(\sqrt{2}a, 2a)$. In either of these subintervals, the change of variable is defined by

$$r(s) = \int \frac{A'(s)}{P(s)} ds = \frac{A(s)}{4a} + C,$$

for a constant $C \in \mathbb{R}$. Fixing $C = 0$, we merely have $A[s(r)] = 4ar$ and $P[s(r)] = 4a$. Moreover, we can easily see that r represents half of the radius of the inscribed circle (see final remark in Section 5).

Remark. The Minkowski's concept of surface area (see e.g. Bonnesen and Fenchel [3, §31]), which is based on the derivative relationship (4), is worth particular mention here. Let $R \in \mathbb{R}^p$ be a convex body of volume V_R and surface area A_R . For any $s > 0$, the Minkowski sum

$$R(s) := R + sB^p = \{x \in \mathbb{R}^p \mid d(x, R) \leq s\},$$

where B^p is the p -dimensional unit ball, is called the *outer parallel body* of R at distance s or, equivalently, the *s -neighborhood* of R . According to the *Steiner formula* (see e.g. Leichtweiß [10, p. 30] and Schneider [14, Chapter 4]), its volume can be expressed as a polynomial of degree p in s :

$$V_{R(s)} = \sum_{i=0}^p s^i \kappa_i V_{R,p-i}$$

where κ_i is the volume of the i -dimensional unit ball and $V_{R,p-i}$ is the *intrinsic* ($p-i$)-*volume* of R , with special cases $V_{R,p} = V_R$ and $\kappa_1 V_{R,p-1} = A_R$. It is then clear that

$$\lim_{s \rightarrow +0} \frac{V_{R(s)} - V_R}{s} = \left. \frac{dV_{R(s)}}{ds} \right|_{s=+0} = A_R$$

and that (see also Guggenheimer [9, Chapter 4])

$$\frac{dV_{R(s)}}{ds} = \left. \frac{dV_{R(s+t)}}{dt} \right|_{t=+0} = \left. \frac{dV_{[R(s)](t)}}{dt} \right|_{t=+0} = A_{R(s)}.$$

If we consider the family $\{R(s) \mid s \in \mathbb{R}_+\}$ with $V(s) := V_{R(s)}$ and $A(s) := A_{R(s)}$, we immediately retrieve Eq. (4) with $r = s$.

3 Homogeneous families

Consider a family \mathcal{R} of similar compact regions with boundaries of finite measures. Assume that $E = \mathbb{R}_+$ and that the parameter s represents a characteristic linear dimension of region $R(s)$, e.g., a diameter or an edge length. Then, under a dilation $s \mapsto ts$, the volume and area of that region are clearly magnified by the factors t^p and t^{p-1} , respectively. This means that the functions $V(s)$ and $A(s)$ fulfill the functional equations

$$V(ts) = t^p V(s) \quad \text{and} \quad A(ts) = t^{p-1} A(s) \quad (s, t \in \mathbb{R}_+),$$

and hence are homogeneous functions of degrees p and $p-1$, respectively, i.e., of the form

$$V(s) = k_1 s^p \quad \text{and} \quad A(s) = k_2 s^{p-1} \quad (s \in \mathbb{R}_+),$$

where k_1 and k_2 are positive constants.

Starting from this observation, Tong [16] noted that, for such homogeneous functions, the derivative relationship (4) holds for the change of variable

$$r(s) = p \frac{V(s)}{A(s)} \quad (6)$$

and the new variable r also represents a linear dimension.

Formula (6) can also be valid for families of non-similar regions (see Example 3.1).

A family of regions is said to be *homogeneous* if the change of variable in (6) ensures relation (4).

The following proposition yields equivalent conditions for a family to be homogeneous.

Proposition 3.1. *Suppose $V(s)$ is a strictly monotone and differentiable function in E and $A(s)$ is a continuous function in E . Let $r(s)$ be given by Eq. (3). Then the following assertions are equivalent:*

i) There exists a constant $C \in \mathbb{R}$ such that

$$r(s) = p \frac{V(s)}{A(s)} + C \quad (s \in E).$$

ii) There exists a constant $k > 0$ such that

$$A(s)^p = kV(s)^{p-1} \quad (s \in E).$$

iii) There exists a differentiable change of variable $\phi : E \rightarrow \phi(E)$ and constants $k_1, k_2 > 0$ such that

$$V(s) = k_1 \phi(s)^p \quad \text{and} \quad A(s) = k_2 \phi(s)^{p-1} \quad (s \in E).$$

Proof. $i) \Leftrightarrow ii)$ Since $V(s)$ is differentiable, so is $A(s)$. Then, from Eq. (5), we have

$$\begin{aligned} \exists C \in \mathbb{R} : r(s) = p \frac{V(s)}{A(s)} + C &\Leftrightarrow p \frac{d}{ds} \frac{V(s)}{A(s)} = \frac{V'(s)}{A(s)} \\ &\Leftrightarrow p \frac{d}{ds} \ln A(s) = (p-1) \frac{d}{ds} \ln V(s) \\ &\Leftrightarrow \exists k > 0 : A(s)^p = kV(s)^{p-1}. \end{aligned}$$

$ii) \Rightarrow iii)$ For any $s \in E$, define $\phi(s) = V(s)^{1/p}$. Then $V(s) = \phi(s)^p$ and

$$A(s) = k^{1/p} V(s)^{\frac{p-1}{p}} = k^{1/p} \phi(s)^{p-1}.$$

$iii) \Rightarrow ii)$ Clear. □

According to assertion (ii), Eq. (1) forces the functions $A(s)^p$ and $V(s)^{p-1}$ to be linearly dependent in E . Thus, it turns out that a family is homogeneous if and only if the *isoperimetric ratio* $A(s)^p/V(s)^{p-1}$ (introduced in Pólya [13]) is a constant function on E . On the other hand, assertion (iii) clearly means that $V(s)$ and $A(s)$ are homogeneous functions of degrees p and $p - 1$, respectively, up to the same change of variable $\phi(s)$. This justifies the terminology “homogeneous family”. Clearly, this function $\phi(s)$ represents a linear dimension and identifies with $V(s)^{1/p}$ up to a positive multiplicative constant.

We have seen in the beginning of this section that any family of similar regions is homogeneous whenever the parameter s represents a linear dimension. The following corollary shows that this property holds even if s does not represent a linear dimension.

Corollary 3.1. *Let \mathcal{R} be a family of compact regions in \mathbb{R}^p with boundaries of finite measures. Suppose that the associated function $V(s)$ is differentiable and strictly monotone. If the regions of \mathcal{R} are all similar then \mathcal{R} is a homogeneous family.*

Proof. Since the regions are all similar, the isoperimetric ratio $A(s)^p/V(s)^{p-1}$, which does not depend on the size (e.g., length of diameter) or $R(s)$, is a constant function on E .

Alternative proof. For any $s \in E$, let $\phi(s)$ be the diameter of region $R(s)$. Since the regions are all similar, the functions $V(s)$ and $A(s)$ are constant multiples of $\phi(s)^p$ and $\phi(s)^{p-1}$, respectively. \square

The following example shows that a homogeneous family need not be constructed from similar regions, even if the transformation carrying any region into any other one is angle-preserving.

Example 3.1. Consider the hexagons $R(s)$ whose inner angles all have a fixed amplitude $2\pi/3$ and the consecutive sides have lengths $a(s)$, $b(s)$, $c(s)$, $a(s)$, $b(s)$, and $c(s)$, respectively. Then it can be easily shown that

$$\begin{aligned} A(s) &= \frac{\sqrt{3}}{2}[a(s)b(s) + b(s)c(s) + c(s)a(s)], \\ P(s) &= 2[a(s) + b(s) + c(s)]. \end{aligned}$$

By choosing $a(s) = 1$, $b(s) = s$, and $c(s) = 1 + s + 2\sqrt{s}$, where $s \in \mathbb{R}_+$, we see that this particular family of hexagons is homogeneous. Moreover, even though the interior angles are fixed, the hexagons are not similar since the functions $a(s)$, $b(s)$, and $c(s)$ are not linearly dependent.

4 Finding homogeneous families

Let us investigate the following problem: Given a class \mathcal{C} of compact regions in \mathbb{R}^p with boundaries of finite measures, find homogeneous subfamilies, if any.

According to assertion (ii) of Proposition 3.1, any homogeneous family, with parameter $s \in E$, should satisfy the identity

$$\frac{A(s)^p}{V(s)^{p-1}} = k \quad (s \in E)$$

for some $k > 0$.

The admissible values of k are given by the well-known p -dimensional *isoperimetric inequality* (see e.g. [2], [4], [5]), which states that if R is a compact domain in \mathbb{R}^p with piecewise smooth boundary then

$$\frac{A_R^p}{V_R^{p-1}} \geq p^p \kappa_p \quad (7)$$

where

$$\kappa_p = \frac{\pi^{p/2}}{\Gamma(p/2 + 1)}$$

is the volume of the p -dimensional unit ball. Here the equality sign in (7) holds if and only if R is the p -dimensional unit ball.

Thus, the constant k is bounded below by $p^p \kappa_p$. For example, for $p = 2$ and $p = 3$, this lower bound is given by 4π and 36π , respectively.

Given a particular class \mathcal{C} of regions in \mathbb{R}^p , we can refine the lower bound of constant k by calculating

$$k_{\min}(\mathcal{C}) = \inf_{R \in \mathcal{C}} \frac{A_R^p}{V_R^{p-1}}$$

For example, if \mathcal{C} is the class of all n -gons in \mathbb{R}^2 , we have

$$k_{\min}(\mathcal{C}) = 4n \tan(\pi/n)$$

and this bound is achieved for the regular n -gons. In other words, the isoperimetric inequality for n -gons R in \mathbb{R}^2 is

$$\frac{P_R^2}{A_R} \geq 4n \tan(\pi/n)$$

with equality if and only if R is regular. In Table 1 we list results for some other examples. See also Florian [8] and Mitrinović *et al.* [12, Chapter 10] for recent surveys on isoperimetric inequalities for polytopes.

The following example demonstrates how to construct homogeneous families.

Consider the class \mathcal{C} of all parallelograms in \mathbb{R}^2 . Of course, any family of similar parallelograms is homogeneous, so let us try to find a homogeneous family that is not made up of similar parallelograms.

If we parameterize the parallelograms with side lengths $x_1 > 0$ and $x_2 > 0$, and angle $x_3 \in (0, \pi)$, then the corresponding area and perimeter functions are given by

$$A = x_1 x_2 \sin x_3 \quad \text{and} \quad P = 2x_1 + 2x_2 \quad (8)$$

p	Class \mathcal{C}	Best regions	$k_{\min}(\mathcal{C})$
2	triangles	equilateral triangles	$12\sqrt{3}$
2	right triangles	isosceles triangles	$2(2 + \sqrt{2})^2$
2	n -gons	regular n -gons	$4n \tan(\pi/n)$
3	rectangular parallelepipeds	cubes	216
3	right circular cylinders	height = diameter	54π
3	right circular cones	height = $\sqrt{2} \times$ diameter	72π
3	right square pyramids	height = $\sqrt{2} \times$ side	288
3	regular tori	apples with $r_1 = r_2$	$16\pi^2$

Table 1: Isoperimetric ratios for various regions

and hence

$$\frac{P^2}{A} = \frac{4(x_1 + x_2)^2}{x_1 x_2 \sin x_3}$$

Setting $z_2 = x_2/x_1 > 0$ and $z_3 = x_3 \in (0, \pi)$, we have

$$\frac{P^2}{A} = \frac{4(1 + z_2)^2}{z_2 \sin z_3}$$

Now, this latter expression has a global minimum in $\mathbb{R}_+ \times (0, \pi)$ at $(z_2, z_3) = (1, \pi/2)$, with value $k_{\min}(\mathcal{C}) = 16$ and has no maximum (z_2 can be made as close to zero as we wish). Of course any other parameterization of the parallelograms would provide the same value for $k_{\min}(\mathcal{C})$.

Therefore, for any fixed $k \geq 16$, the equation

$$\frac{4(x_1 + x_2)^2}{x_1 x_2 \sin x_3} = k$$

represents a level surface in the $x_1 x_2 x_3$ -space and each curve

$$\{(a(s), b(s), \theta(s)) \mid s \in E\}$$

along that surface represents a homogeneous family associated with the constant k . In particular, any curve in a plane $x_3 = c$ or any curve in a plane $x_2 = c x_1$ (where c is a positive constant) represents a family of similar parallelograms.

For example, if we fix $k = 32$, $a(s) = \sqrt{s}$, and $b(s) = s - \sqrt{s}$, then the third function $\theta(s)$ must be given by

$$\theta(s) = \arcsin \frac{s/8}{\sqrt{s} - 1}$$

throughout the open interval $(24 - 16\sqrt{2}, 24 + 16\sqrt{2})$. Interestingly, we observe that in this case

$$A(s) = s^2/8 \quad \text{and} \quad P(s) = 2s$$

are homogeneous functions.

As Eq. (8) shows, the functions A and P are homogeneous in the first two variables x_1 and x_2 . This is in accordance with the fact that these variables represent linear dimensions while x_3 represents an angle.

Let us generalize this approach. Consider a class \mathcal{C} of compact regions in \mathbb{R}^p with boundaries of finite measures. Consider also integers m and n such that $1 \leq m \leq n$ and suppose that the volume and area functions associated with \mathcal{C} are continuous functions from \mathbb{R}_+^n to \mathbb{R}_+ whose first m variables x_1, \dots, x_m represent linear dimensions and remaining variables x_{m+1}, \dots, x_n represent angles. In this case these functions are homogeneous of degrees p and $p-1$ in the first m variables, i.e., they fulfill the functional equations

$$\begin{aligned} V(tx_1, \dots, tx_m, x_{m+1}, \dots, x_n) &= t^p V(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \\ A(tx_1, \dots, tx_m, x_{m+1}, \dots, x_n) &= t^{p-1} A(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}_+^n$. The general solutions of these equations are given by (see e.g. Aczél and Dhombres [1, Chapter 20]):

$$\begin{aligned} V(x_1, \dots, x_m, x_{m+1}, \dots, x_n) &= x_1^p f\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}, x_{m+1}, \dots, x_n\right) \\ A(x_1, \dots, x_m, x_{m+1}, \dots, x_n) &= x_1^{p-1} g\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}, x_{m+1}, \dots, x_n\right) \end{aligned}$$

where $f : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}_+$ and $g : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}_+$ are arbitrary continuous functions (constants if $n = 1$).

Introducing the new variables

$$z_i = \begin{cases} x_i/x_1, & \text{if } i \leq m, \\ x_i, & \text{if } i > m, \end{cases}$$

the homogeneity condition is written

$$\frac{g(z_2, \dots, z_n)^p}{f(z_2, \dots, z_n)^{p-1}} = k, \quad \forall (z_2, \dots, z_n) \in \mathbb{R}_+^{n-1}.$$

Let us investigate the simple cases where $(m, n) = (1, 2)$ and $(m, n) = (2, 2)$. We have the following two immediate results:

Proposition 4.1. *Suppose the functions $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ are of the form*

$$V(x_1, x_2) = x_1^p f(x_2) \quad \text{and} \quad A(x_1, x_2) = x_1^{p-1} g(x_2)$$

where f^{p-1} and g^p are linearly dependent in no open interval of \mathbb{R}_+ . Then the homogeneity condition holds if and only if x_2 is constant.

Proposition 4.2. *Suppose the functions $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ are of the form*

$$V(x_1, x_2) = x_1^p f\left(\frac{x_2}{x_1}\right) \quad \text{and} \quad A(x_1, x_2) = x_1^{p-1} g\left(\frac{x_2}{x_1}\right)$$

where f^{p-1} and g^p are linearly dependent in no open interval of \mathbb{R}_+ . Then the homogeneity condition holds if and only if x_2/x_1 is constant.

Example 4.1. Consider a family of right circular cones in \mathbb{R}^3 , with slant height $l(s) \in \mathbb{R}_+$ and vertex angle $\theta(s) \in (0, \pi)$. This situation corresponds to the case $(m, n) = (1, 2)$ with

$$\begin{aligned} f(x) &= \frac{\pi}{3} \sin^2 \frac{x}{2} \cos \frac{x}{2} \\ g(x) &= \pi \sin \frac{x}{2} \left(1 + \sin \frac{x}{2}\right). \end{aligned}$$

As there is no open subinterval of $(0, \pi)$ in which f^2 and g^3 are linearly dependent, the family is homogeneous if and only if $\theta(s)$ is constant. The cones are then similar.

Example 4.2. Consider a family of right circular cones in \mathbb{R}^3 , with base radius $l_1(s) \in \mathbb{R}_+$ and height $l_2(s) \in \mathbb{R}_+$. This situation corresponds to the case $(m, n) = (2, 2)$ with

$$\begin{aligned} f(x) &= \frac{\pi}{3} x \\ g(x) &= \pi(1 + \sqrt{1 + x^2}). \end{aligned}$$

As there is no open subinterval of \mathbb{R}_+ in which f^2 and g^3 are linearly dependent, the family is homogeneous if and only if $l_2(s)/l_1(s)$ is constant. The cones are then similar.

5 Geometric interpretations of r

As we already observed in Example 2.2, a geometric meaning of the variable of differentiation r is not always apparent. However, for some homogeneous families, where $r(s)$ is given by Eq. (6), interpretations can be found.

For example, Emert and Nelson [7] proved that, for any family of similar circumscribing polytopes, the variable r represents the radius of the inscribed sphere. For an earlier work on regular polytopes, see Miller [11].

Other examples have been discussed recently by Dorff and Hall [6]. Among these, we have the following remarkable result, that was shown for families of similar regions in \mathbb{R}^2 and \mathbb{R}^3 . We state this result in \mathbb{R}^p and for homogeneous families. Also, Eq. (9) was previously unknown.

Proposition 5.1. *Let \mathcal{R} be a homogeneous family of n -faced polyhedra $R(s)$ that are star-like with respect to a point $T(s)$ in the interior of $R(s)$. Let $P_i(s)$ be the pyramid whose base is the i th facet of $R(s)$ and whose vertex is $T(s)$. Then*

$$r(s) = \sum_{i=1}^n \frac{A_i(s)}{A(s)} r_i(s) \tag{9}$$

and

$$\frac{1}{r(s)} = \sum_{i=1}^n \frac{V_i(s)}{V(s)} \frac{1}{r_i(s)} \tag{10}$$

where $V_i(s)$, $A_i(s)$, and $r_i(s)$ are respectively the volume of $P_i(s)$, the surface area of the base of $P_i(s)$, and the altitude from $T(s)$ of $P_i(s)$.

Proof. Since $P_i(s)$ is a p -dimensional pyramid, we have

$$V_i(s) = \frac{1}{p} A_i(s) r_i(s).$$

Then, as the family is homogeneous, we have, by (6),

$$r(s) = p \frac{V(s)}{A(s)} = p \sum_{i=1}^n \frac{V_i(s)}{A(s)} = \sum_{i=1}^n \frac{A_i(s)}{A(s)} r_i(s),$$

which proves (9) and

$$\frac{1}{r(s)} = \frac{1}{p} \frac{A(s)}{V(s)} = \frac{1}{p} \sum_{i=1}^n \frac{A_i(s)}{V(s)} = \sum_{i=1}^n \frac{V_i(s)}{V(s)} \frac{1}{r_i(s)},$$

which proves (10). \square

Eq. (9) simply means that the variable of differentiation $r(s)$ is the arithmetic mean of the altitudes from $T(s)$ of the pyramids $P_i(s)$, weighted by the relative areas of the corresponding facets. Similarly, Eq. (10) means that $r(s)$ is the harmonic mean of the altitudes from $T(s)$ of the pyramids $P_i(s)$, weighted by the relative volumes of these pyramids. Particularly, these both means do not depend upon the choice of $T(s)$.

For convex polytopes, Eq. (9) can be generalized as follows. Let $R \subseteq \mathbb{R}^p$ be an n -faced convex polytope and let $h_R : \mathbb{R}^p \rightarrow \mathbb{R}$ be its *support function*:

$$h_R(u) = \max_{x \in R} \{x \cdot u\},$$

where \cdot denotes the standard inner product on \mathbb{R}^p . Then, assuming that R has facet unit normals u_1, \dots, u_n and corresponding facet areas A_1, \dots, A_n , we have (see e.g. Leichtweiß [10, p. 22])

$$V_R = \frac{1}{p} \sum_{i=1}^n A_i h_R(u_i).$$

Considering a homogeneous family of polytopes, we have immediately

$$r(s) = \sum_{i=1}^n \frac{A_i(s)}{A(s)} h_{R(s)}[u_i(s)]$$

showing that $r(s)$ is the weighted arithmetic mean of numbers $h_{R(s)}[u_i(s)]$.

Remark. It can be easily proved that the function $A(s)$ of a family \mathcal{R} is constant if and only if

$$r(s) = \frac{V(s)}{A(s)} + C.$$

In this case, if the regions are star-like n -faced polyhedra as in Proposition 5.1, then Eqs. (9) and (10) become respectively

$$r(s) = \frac{1}{p} \sum_{i=1}^n \frac{A_i(s)}{A(s)} r_i(s)$$

and

$$\frac{1}{r(s)} = p \sum_{i=1}^n \frac{V_i(s)}{V(s)} \frac{1}{r_i(s)}$$

Example 2.3 illustrates these latter two formulas.

6 Conclusion

We have explored the idea of the derivative of the volume of a region in \mathbb{R}^p with respect to some variable r equaling its surface area for homogeneous families. This area of investigation is intriguing and appears not to have been previously studied. We have just skimmed the surface, and there are a lot of questions to be answered. For example, what other geometric interpretations are there for r ?

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