Doubly Close-to-Convex Functions

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Abstract

We introduce the class $L(\beta,\gamma)$ of holomorphic, locally univalent functions in the unit disk $\mathbb{D}=\{z:|z|<1\}$, which we call the class of doubly close-to-convex functions. This notion unifies the earlier known extensions [4], [1], [12]. The class $L(\beta,\gamma)$ appears to be linear invariant. First of all we determine the region of variability $\{w : w = \log f'(r), f \in L(\beta,\gamma)\}$ for fixed $z, |z| = r < 1$, which give us the exact rotation theorem. The rotation theorem and linear invariance allows us to find the sharp value for the radius of close-to-convexity and bound for the radius of univalence. Moreover, it was helpful as well in finding the sharp region for $\alpha \in \mathbb{R}$, for which the integral $\int_0^z (f'(t))^\alpha dt$, $f \in L(\beta,\gamma)$, is univalent. Because $L(\beta,\gamma)$ reduces to β-close-to-convex functions ($\gamma = 0$) and to convex functions ($\beta = 0$ and $\gamma = 0$), the obtained results generalize several corresponding ones for these classes. We improve as well the value of the radius of univalence for the class considered by Hengartner and Schober [4] from 0.345 to 0.577.

Key words:

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1 Introduction

We consider functions f that are holomorphic in $\mathbb{D} = \{z : |z| < 1\}$ with the normalization $f(0) = 0, f'(0) = 1$ and are locally univalent in $\mathbb{D}$ (i.e., $f'(z) \neq 0$ in $\mathbb{D}$). In particular, let S denote the class of all holomorphic and univalent functions with this normalization and let $S^c \subset S$ be the subclass consisting of convex functions. A function f is said to be close-to-convex of order $\beta \geq 0$ in $\mathbb{D}$ if there exists $g \in S^c$ and $\phi \in \mathbb{R}$ such that

$$\left|\arg e^{i\phi} \frac{f'(z)}{g'(z)}\right| \leq \beta \frac{\pi}{2}, \ z \in \mathbb{D}. \tag{1}$$

The class of close-to-convex functions of order β will be denoted by L_β [10], [5], [2]. We observe that $L_0 \equiv S^c$ and $L_1 \equiv L$, where L denotes the class of close-to-convex univalent functions in $\mathbb{D}$ [3]. If $\beta \in [0, 1]$, then L_β consists of functions that are univalent only in $\mathbb{D}$ and is a linear invariant family of order $(\beta + 1)$ (for all $\beta \geq 0$) in the sense of Pommerenke [10]. Hengartner and Schober [4] have studied the generalization of the class L by letting $\beta = 1$ and $g(z)$ to be a function which is convex in the direction of the imaginary axis in (1). Another generalization was considered in [1] and [12], where g was taken from the class of bounded boundary rotation. Here we extend these ideas by studying the more general class of doubly close-to-convex functions.

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2 Doubly close-to-convex functions

Definition 1 *Let* $\beta \geq 0$ *and* $\gamma \geq 0$ *be fixed. We say that a holomorphic, locally univalent function* f *in* $\mathbb{D}$ *with the normalization* $f(0) = 0, f'(0) = 1$ *belongs to the class* $L(\beta, \gamma)$ *if there exist* $g \in L_\gamma$ *and* $\phi \in \mathbb{R}$ *such that (1) holds. We call* $L(\beta, \gamma)$ *the class of doubly close-to-convex functions of order* (β, γ)*. Of course, we have that* $L(0,0) \equiv S^c, L(\beta, 0) \equiv L_\beta, L(0, \gamma) \equiv L_\gamma$*.*

The following lemmas follow almost directly from the definition.

Lemma 2 *A function* $f \in L(\beta, \gamma)$ *if and only if there exists a function* $h \in S^c$ *and two holomorphic functions* $p(z) = 1 + p_1 z + \cdots, q(z) = 1 + q_1 z + \cdots$ *in* $\mathbb{D}$ *such that* $\mathrm{Re}[\, e^{i\phi} p(z)] > 0$ *and* $\mathrm{Re}[\, e^{i\psi} q(z)] > 0$ *in* $\mathbb{D}$ *for some* $\phi, \psi \in \mathbb{R}$*, and*

$$f'(z) = h'(z) p^\gamma(z) q^\beta(z). \tag{2}$$

PROOF. By (1) we have $f'(z) = g'(z) q^\beta(z)$, where $g \in L_\gamma$ and $\mathrm{Re}[\, e^{i\psi} q(z)] > 0, z \in \mathbb{D}$ for some $\psi \in \mathbb{R}$ $(q(z) = 1 + q_1 z + q_2 z^2 + \cdots)$. On the other hand, $g \in L_\gamma$ if and only if $g'(z) = h'(z) p^\gamma(z)$, where $h \in S^c$ and $\mathrm{Re}[\, e^{i\phi} p(z)] > 0, z \in \mathbb{D}$ for some $\phi \in \mathbb{R}$ $(p(z) = 1 + p_1 z + \cdots)$. Therefore, we have (2).

Remark 3 *Formula (2) can be written in the form*

$$f'(z) = h'(z) \left(\frac{1 + e^{-i\phi} \omega_1(z)}{1 - \omega_1(z)} \right)^\gamma \left(\frac{1 + e^{-i\psi} \omega_2(z)}{1 - \omega_2(z)} \right)^\beta, \tag{2'}$$

where ω_1 *and* ω_2 *are holomorphic in* $\mathbb{D}$ *and satisfy the conditions of the Schwarz Lemma.*

Lemma 4 *The family* $L(\beta, \gamma)$ *is a linear invariant family of order* $(\beta + \gamma + 1)$*.*

PROOF. The proof of linear invariance is exactly the same as for L_β given in [5] or [2]. The order follows from the fact that from (1) $f(z) = z + a_2 z^2 + \cdots$, $g(z) = z + b_2 z^2 + \cdots$, and $q(z) + 1 + q_1 z + \cdots$, we have $a_2 = b_2 + \frac{1}{2}\beta q_1$. Since $|b_2| \le 1 + \gamma$ and $|q_1| \le 2$, we have $|a_2| \le \beta + \gamma + 1$.

The next theorem generalizes the classical result for close-to-convex functions from [6] and corresponding result from [12].

Theorem 5 *The region of variability* $G(z) = \{w : w = \log f'(z), f \in L(\beta, \gamma)\}$ *for fixed* $z = re^{i\phi} \in \mathbb{D}$, $0 < r < 1$, *is a closed and convex set whose boundary has the equation*

$$w = w(t) = \log \frac{(1 - re^{i\theta_2})^{\beta+\gamma}}{(1 - re^{i\theta_1})^{\beta+\gamma+2}}, t \in [0, 2\pi], \tag{3}$$

where

$$\theta_1 = \theta_1(t) = t - \arcsin(r \sin t);\ \ \theta_2 = \theta_2(t) = \pi + t + \arcsin(r \sin t). \tag{4}$$

PROOF. First observe that $G(z) = G(r), r = |z| < 1$, because the class $L(\beta, \gamma)$ is rotationally invariant. The set $G(r)$ is closed because the class $L(\beta, \gamma)$ is compact. The convexity of $G(r)$ is the consequence of the property that if $f_1, f_2 \in L(\beta, \gamma)$, then for all $\lambda \in [0, 1]$

$$f_\lambda(z) = \int_0^z [f_1(t)]^\lambda [f_2(t)]^{1-\lambda} dt \in L(\beta, \gamma).$$

Therefore, it will be enough to find the equation of the boundary of $G(r)$. By (2), it suffices to consider

$$f'(r) = h'(r) p^\gamma(r) q^\beta(r). \tag{5}$$

It is well known that the functions $h \in S^c$ corresponding to the boundary points of $\{w : w = h'(r), h \in S^c\}$ have the form

$$h(z) = \frac{z}{1 - ze^{i\tau}}\ , \theta \in [0, 2\pi],$$

and that the functions p corresponding to the boundary points of $\Big\{w : w = p(r), \operatorname{Re} p(z) > 0, z \in \mathbb{D}, p(0) = e^{i\delta}, |\delta| < \pi/2\Big\}$ have the form

$$p(z) = \frac{e^{i\delta} - ze^{i(s-\delta)}}{1 - ze^{is}},\ \ s \in [0, 2\pi].$$

The same is true for $q(z)$. These facts along with (2′) imply that the function f_0 corresponding to the boundary points of $G(r)$ has by the form

$$f_0'(r) = \frac{1}{(1 - \epsilon_5 r)^2} \cdot \left(\frac{1 - \epsilon_1 r}{1 - \epsilon_2 r}\right)^{\gamma} \left(\frac{1 - \epsilon_3 r}{1 - \epsilon_4 r}\right)^{\beta}, \tag{6}$$

where $\epsilon_j = e^{i\theta_j}, \theta_j \in [0, 2\pi], j = 1, 2, 3, 4, 5.$

The convexity of $G(r)$ implies that finding the boundary of $G(r)$ is equivalent to determining the maximum of the function

$$\begin{aligned} \operatorname{Re}[\, e^{-it} \log f'(r)] = \operatorname{Re}\{e^{-it}[-2 \log(1 - \epsilon_5 r) + \beta \log(1 - \epsilon_3 r) \\ + \gamma \log(1 - \epsilon_1 r) - \beta \log(1 - \epsilon_4 r) - \gamma \log(1 - \epsilon_2 r)]\} \end{aligned} \tag{7}$$

with respect to $\theta_j \in [0, 2\pi]$ for fixed $t \in [0, 2\pi]$, where t denotes the angle between the imaginary axis and supporting line to $G(r)$. Moreover, we observe from (7) that $G(r)$ is symmetric with respect to the real axis, because the image of the circle $\xi = 1 - re^{i\phi}, \phi \in [0, 2\pi]$, under the mapping $w = \log \xi$ is a convex curve symmetric about the real axis. Therefore one can restrict considerations to $t \in [0, \pi]$.

One can verify directly that the function

$$u(\theta) = \operatorname{Re}\{e^{-it} \log(1 - re^{i\theta})\}$$

attains its maximum for $\theta_2 = \theta_2(t)$ and minimum for $\theta = \theta_1(t)$ as given in (4).

Corollary 6 *If $f \in L(\beta,\gamma)$, then for $|z| = r < 1$ we have the following sharp bounds:*

$$|\arg f'(z)| \le 2(\beta+\gamma+1)\arcsin r; \tag{8}$$

$$\frac{(1-r)^{\beta+\gamma+2}}{(1+r)^{\beta+\gamma}} \le |f'(z)| \le \frac{(1+r)^{\beta+\gamma+2}}{(1-r)^{\beta+\gamma}}. \tag{9}$$

The extremal function has the form (6) with θ_1 and θ_2 given by (4) with an appropriate t.

PROOF. Using the symmetry of $G(r)$ we see that the $\max(\arg f'(r))$ is attained for $t = \pi/2$ while the bounds for $|f'(z)|$ is attained for $t = \pi$ and $t = 0$, which implies (8) and (9).

Theorem 7 *The radius of convexity of the class $L(\beta,\gamma)$ is equal to*

$$r_c(\beta,\gamma) = (\beta+\gamma+1) - \sqrt{(\beta+\gamma+1)^2 - 1}. \tag{10}$$

In particular, $r_c(1,1) = 3 - \sqrt{8}$, $r_c(1,0) = 2 - \sqrt{3}$ with these results being sharp.

The formula (10) follows from the Pommerenke result for linear invariant families [10, p. 133] and Lemma 2. The rotation theorem (8) and the linear invariance of the family $L(\beta,\gamma)$ determine the possibility of finding the radii of univalence and close-to-convexity for $L(\beta,\gamma)$.

Theorem 8 *The radius of univalence $r_u(\beta,\gamma)$ of the class $L(\beta,\gamma)$ satisfies the inequality $r_u(\beta,\gamma) \ge r_{\beta,\gamma}$, where*

$$r_{\beta,\gamma} = \tan\frac{\pi}{2(\beta+\gamma+1)} \ \text{if } \beta+\gamma > 1. \tag{11}$$

If $\beta+\gamma \le 1$, *then* $r_u(\beta,\gamma)=1$.

Corollary 9 *We have* $r_u(1,1) \ge \sqrt{3}/3 \cong 0.577$, *which improves the corresponding result for the class considered by Hengartner and Schober in [4], because their class of functions is a subclass of* $L(\beta,\gamma)$. *(The constant for* r_u *in [4] was approximately* 0.345.*)*

PROOF. If $\mathcal{M}$ is a linear invariant family, then Pommerenke [10] proved that $r_u(\mathcal{M}) \ge \hat{r} = r_0/(1+\sqrt{1-r_0^2})$, where $r_0 \in (0,1]$ is the radius of the disk $|z|<r_0$ in which $f(z)/z \ne 0, f \in \mathcal{M}$, and r_0 is determined from the equation

$$\max_{\substack{f\in\mathcal{M}\\ |z|=r<1}} |\arg f'(z)| = 2\pi.$$

From the bound in (8) we find that $r_0 = 1$ and $\hat{r} = 1$ if $\beta+\gamma \le 1$ and $r_0 = \sin\frac{\pi}{\beta+\gamma+1}$ if $\beta+\gamma>1$. By the above formula for $\hat{r} := r_{\beta+\gamma}$, equation (11) and Corollary 9 follow directly.

The result of Theorem 8 can be sharpened by the exact value of the radius of close-to-convexity which is the consequence of (8) and the following less known sharp result of Campbell and Ziegler [1, p. 19] (in our formulation):

Lemma 10 (A) *If* $\mathcal{M}$ *is a linear invariant family for which* $\max_{\substack{f\in\mathcal{M}\\ |z|=r<1}} |\arg f'(z)| = 2\tau \arcsin r$, *then the radius of close-to-convexity of* $\mathcal{M}$ *is 1 if* $1 \le \tau \le 2$ *and is the unique solution of the equation*

$$2\,\mathrm{arccot}\,\mathrm{w} - 2\tau\,\mathrm{arccot}(\tau\mathrm{w}) = -\pi \qquad (12)$$

where $w = \dfrac{1-r^2}{\sqrt{4\tau^2r^2-(1+r^2)^2}}$, *if* $\tau>2$.

Therefore we have the following sharp result.

Theorem 11 *Let $f \in L(\beta,\gamma)$. If $\beta+\gamma \le 1$, then f is close-to-convex univalent in $\mathbb{D}$. If $\beta+\gamma > 1$, then the radius of close-to-convexity $r_{cc}(\beta,\gamma)$ of $L(\beta,\gamma)$ is given by (12) with $\tau = (\beta+\gamma+1)$.*

Corollary 12 *We have*

$$r_{cc}(1,1) = \left\{12\sqrt{3} - 19 - 2\sqrt{198 - 114\sqrt{3}}\right\}^{1/2} \cong 0.553. \qquad (13)$$

PROOF. When $\beta = \gamma = 1$, then $\tau = 3$ and (12) can be reduced by the formula for $\cot 3\alpha$ and after some calculations to the equation

$$t^2 - 2(2\sqrt{13} - 19)t + 1 = 0, \quad t = r^2,$$

which yields (13). This value improves the result for r_u given in [4].

Formula (13) shows that $r_u(\beta,\gamma) > r_{cc}(\beta,\gamma)$ for the class $L(\beta,\gamma)$. However, they share the same region $\{(\beta,\gamma) : \beta+\gamma \le 1\}$ in which f is univalent.

3 Univalence of an integral operator of $L(\beta,\gamma)$

The univalence of some integral operators for univalent families like S, L, S^c, and in particular the univalence of

$$F_\alpha(z) = F_\alpha(f)(z) = \int_0^z (f'(t))^\alpha dt, \quad \alpha \in \mathbb{R}(\mathbb{C}) \qquad (14)$$

was studied in several papers. Here we solve the problem of univalence of (14) for $f \in L(\beta,\gamma)$ and $\alpha \in \mathbb{R}$ by applying the method from [11]. According to Pfaltzgraff's theorem [8] the integral in (14) is univalent for $f \in L(\beta,\gamma)$ if $|\alpha| \le \frac{1}{2(\beta+\gamma+1)}, \alpha \in \mathbb{C}$. However, for $\alpha \in \mathbb{R}$ the above region can be extended considerably and will be sharp.

We will use the following result.

Lemma 13 *The minimal invariant family containing the set* $\{F_\alpha(z) : f \in L(\beta,\gamma)\}$ *is the set of functions*

$$G_\alpha(z) = G_\alpha(f)(z) = \int_0^z \frac{(f'(t))^\alpha}{(1-\xi z)^{2-2\alpha}}\, dt, \quad \xi \in \mathbb{D} \setminus \{0\}, \alpha \in \mathbb{R}. \tag{15}$$

The order of the family $\{G_\alpha(f)\}$ *is equal to*

$$|\alpha|(\beta+\gamma+1) + |1-\alpha|.$$

PROOF. The first part of Lemma 13 holds for any invariant family and was proved in [11]. To calculate the order, notice that by Lemma 4

$$\begin{aligned}\sup_{f\in L(\beta,\gamma)} \frac{1}{2}|G''_\alpha(0)| &= \sup_{f\in L(\beta,\gamma)} |\alpha a_2 + (1-\alpha)\xi| \\ &= |\alpha|(\beta+\gamma+1) + |1-\alpha|.\end{aligned}$$

Theorem 14 *Let* $f \in L(\beta,\gamma)$ *and* $\alpha \in \mathbb{R}$. *The integral in (15) is univalent in the disk* $|z| \le r_u^\alpha(\beta,\gamma)$, *where*

$$r_u^\alpha(\beta,\gamma) \ge \min\left\{1; \tan \frac{\pi}{2[|\alpha|(\beta+\gamma+1)+|1-\alpha|]}\right\}. \tag{16}$$

The same conclusion holds for the integral (14).

PROOF. From (15) and (8) we obtain

$$\begin{aligned}|\arg G'_\alpha(z)| &\le |\alpha|\,|\arg f'(z)| + 2|1-\alpha|\;|\arg(1-\xi z)| \\ &\le 2\{|\alpha|(\beta+\gamma+1)+|1-\alpha|\}\cdot \arcsin r.\end{aligned} \tag{17}$$

The rest of the proof follows the same line of reasoning as in the proof of Theorem 8.

Using Lemma 13 and Lemma A with $\tau = |\alpha|(\beta+\gamma+1) + |1-\alpha|$ and the bound given in (17), we can find region for $\alpha \in \mathbb{R}$, when f is close-to-convex univalent in $\mathbb{D}$ which will strengthen and make sharp the conclusion given in (16). Namely, we have the following theorem by Lemma A.

Theorem 15 *If $f \in L(\beta,\gamma)$ and $\alpha \in \mathbb{R}$, then the integral in (15) is univalent and close-to-convex for all $\alpha \in \mathbb{R}$ such that $|\alpha|(\beta+\gamma+1) + |1-\alpha| \leq 2$. If $|\alpha|(\beta+\gamma+1) + |1-\alpha| > 2$ then the radius of close-to-convexity of (15) is the unique solution of equation (12) with $\tau = |\alpha|(\beta+\gamma+1) + |1-\alpha|$. The same conclusion holds for the integral in (14) with $f \in L(\beta,\gamma)$ and this is sharp.*

Corollary 16 *If $f \in L(\beta,\gamma)$, then the integral in (14) is univalent for*

$$\alpha \in \left[\frac{-1}{\beta+\gamma+2}\,,\,\frac{3}{\beta+\gamma+2}\right] \text{ if } \beta+\gamma \leq 1$$

and

$$\alpha \in \left[\frac{-1}{\beta+\gamma+2}\,,\,\frac{1}{\beta+\gamma}\right] \text{ if } \beta+\gamma \geq 1.$$

The result is sharp.

Putting $\beta = \gamma = 0$ and $\beta = 1, \gamma = 0$ we get the following results proved in [7] by different methods.

Corollary 17 *If $f \in S^c$, then the integral in (14) is univalent and close-to-convex for all $\alpha \in \left[-\frac{1}{2}, \frac{3}{2}\right]$ and this is sharp.*

Corollary 18 *If $f \in L$, then the integral in (14) is univalent and close-to-convex for all $\alpha \in \left[-\frac{1}{3}, 1\right]$ and this is sharp.*

Remark 19 *Does the class $L(\beta,\gamma)$ and in particular $L(1,1)$ or $L\left(\frac{1}{2},\frac{1}{2}\right)$ have any interesting geometric interpretation (like accessibility of $f(\mathbb{D})$ by angles* *from the complement of $f(\mathbb{D})$)?*

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