## JENSEN POLYNOMIALS FOR THE RIEMANN ZETA FUNCTION AND OTHER SEQUENCES

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ABSTRACT. In 1927 Pólya proved that the Riemann Hypothesis is equivalent to the hyperbolicity of Jensen polynomials for the Riemann zeta function  $\zeta(s)$  at its point of symmetry. This hyperbolicity has been proved for degrees  $d \leq 3$ . We obtain an asymptotic formula for the central derivatives  $\zeta^{(2n)}(1/2)$  that is accurate to all orders, which allows us to prove the hyperbolicity of a density 1 subset of the Jensen polynomials of each degree. Moreover, we establish hyperbolicity for all  $d \leq 8$ . These results follow from a general theorem which models such polynomials by Hermite polynomials. In the case of the Riemann zeta function, this proves the GUE random matrix model prediction in derivative aspect. The general theorem also allows us to prove a conjecture of Chen, Jia, and Wang on the partition function.

#### 1. INTRODUCTION AND STATEMENT OF RESULTS

Expanding on notes of Jensen, Pólya [1] proved that the Riemann Hypothesis (RH) is equivalent to the hyperbolicity of the Jensen polynomials for the Riemann zeta function  $\zeta(s)$  at its point of symmetry. More precisely, he showed that the RH is equivalent to the hyperbolicity of all Jensen polynomials associated to the sequence of Taylor coefficients  $\{\gamma(n)\}$  defined by

(1) 
$$\left(-1+4z^2\right)\Lambda\left(\frac{1}{2}+z\right) = \sum_{n=0}^{\infty}\frac{\gamma(n)}{n!}\cdot z^{2n},$$

where  $\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1-s)$ , and where we say that a polynomial with real coefficients is *hyperbolic* if all of its zeros are real, and where the *Jensen polynomial of degree d and* shift n of an arbitrary sequence  $\{\alpha(0), \alpha(1), \alpha(2), \ldots\}$  of real numbers is the polynomial

(2) 
$$J^{d,n}_{\alpha}(X) := \sum_{j=0}^{d} {d \choose j} \alpha(n+j) X^{j}.$$

Thus, the RH is equivalent to the hyperbolicity of the polynomials  $J^{d,n}_{\gamma}(X)$  for all non-negative integers d and n [2, 3, 1]. Since this condition is preserved under differentiation, to prove RH it would be enough to show hyperbolicity for the  $J^{d,0}_{\gamma}(X)^{-1}$ . Due to the difficulty of proving RH, research has focused on establishing hyperbolicity for all shifts n for small d. Previous to this paper, hyperbolicity was known for  $d \leq 3$  by work<sup>2</sup> of Csordas, Norfolk, and Varga, and Dimitrov and Lucas [5, 3].

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<sup>&</sup>lt;sup>1</sup>The hyperbolicity for  $J_{\gamma}^{\hat{d},\hat{0}}(X)$  has been confirmed for  $d \leq 2 \cdot 10^{17}$  by Chasse (cf. Theorem 1.8 of [4]).

<sup>&</sup>lt;sup>2</sup>These works use a slightly different normalization for the  $\gamma(n)$ .

Asymptotics for the  $\gamma(n)$  were obtained of Coffey and Pustyl'nikov [6, 7]. We improve on their results by obtaining an arbitrary precision asymptotic formula<sup>3</sup> (see Theorem 7), a result that is of independent interest. We will use this strengthened result to prove the following theorem for all degrees d.

**Theorem 1.** If  $d \ge 1$ , then  $J^{d,n}_{\gamma}(X)$  is hyperbolic for all sufficiently large n.

An effective proof of Theorem 1 for small d gives the following theorem.

**Theorem 2.** If  $1 \le d \le 8$ , then  $J^{d,n}_{\gamma}(X)$  is hyperbolic for every  $n \ge 0$ .

Theorem 1 follows from a general phenomenon that Jensen polynomials for a wide class of sequences  $\alpha$  can be modeled by the *Hermite polynomials*  $H_d(X)$ , which we define (in a somewhat non-standard normalization) as the orthogonal polynomials for the measure  $\mu(X) = e^{-X^2/4}$  or more explicitly by the generating function

(3) 
$$\sum_{d=0}^{\infty} H_d(X) \frac{t^d}{d!} = e^{-t^2 + Xt} = 1 + Xt + (X^2 - 2)\frac{t^2}{2!} + (X^3 - 6X)\frac{t^3}{3!} + \cdots$$

More precisely, we will prove the following general theorem describing the limiting behavior of Jensen polynomials of sequences with appropriate growth.

**Theorem 3.** Let  $\{\alpha(n)\}$ ,  $\{A(n)\}$  and  $\{\delta(n)\}$  be three sequences of positive real numbers with  $\delta(n)$  tending to zero and satisfying

(4) 
$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + o\left(\delta(n)^d\right) \quad \text{as } n \to \infty$$

for some integer  $d \ge 1$  and all  $0 \le j \le d$ . Then we have

(5) 
$$\lim_{n \to \infty} \left( \frac{\delta(n)^{-d}}{\alpha(n)} J^{d,n}_{\alpha} \left( \frac{\delta(n) X - 1}{\exp(A(n))} \right) \right) = H_d(X),$$

uniformly for X in any compact subset of  $\mathbb{R}$ .

Since the Hermite polynomials have distinct roots, and since this property of a polynomial with real coefficients is invariant under small deformation, we immediately deduce the following corollary.

**Corollary.** The Jensen polynomials  $J^{d,n}_{\alpha}(X)$  for a sequence  $\alpha \colon \mathbb{N} \to \mathbb{R}$  satisfying the conditions in Theorem 3 are hyperbolic for all but finitely many values n.

Theorem 1 is a special case of this corollary. Namely, we shall employ Theorem 7 to prove that the Taylor coefficients  $\{\gamma(n)\}$  satisfy the required growth conditions in Theorem 3 for every  $d \ge 2$ .

Theorem 3 in the case of the Riemann zeta function is the *derivative aspect Gaussian Unitary* Ensemble (GUE) random matrix model prediction for the zeros of Jensen polynomials. To make this precise, recall that Dyson, Montgomery, and Odlyzko [9, 10, 11] conjecture that the nontrivial zeros of the Riemann zeta function are distributed like the eigenvalues of random Hermitian matrices. These eigenvalues satisfy Wigner's Semicircular Law, as do the roots of the Hermite polynomials  $H_d(X)$ , when suitably normalized, as  $d \to +\infty$  (see Chapter 3 of [12]). The roots of  $J^{d,0}_{\gamma}(X)$ , as  $d \to +\infty$ , approximate the zeros of  $\Lambda\left(\frac{1}{2}+z\right)$  (see [1] or Lemma 2.2 of [13]), and so

 $<sup>^{3}</sup>$ Our results imply the results in [6, 7] after typographical errors are corrected.

GUE predicts that these roots also obey the Semicircular Law. Since the derivatives of  $\Lambda\left(\frac{1}{2}+z\right)$  are also predicted to satisfy GUE, it is natural to consider the limiting behavior of  $J^{d,n}_{\gamma}(X)$  as  $n \to +\infty$ . The work here proves that these derivative aspect limits are the Hermite polynomials  $H_d(X)$ , which, as mentioned above, satisfy GUE in degree aspect.

Returning to the general case of sequences with suitable growth conditions, Theorem 3 has applications in combinatorics where the hyperbolicity of polynomials determines the log-concavity of enumerative statistics. For example, see the classic theorem by Heilmann and Leib [14], along with works by Chudnovsky and Seymour, Haglund, Stanley, and Wagner [15, 16, 17, 18, 19], to name a few. Theorem 3 represents a new criterion for establishing the hyperbolicity of polynomials in enumerative combinatorics. The theorem reduces the problem to determining whether suitable asymptotics hold. Here we were motivated by a conjecture of Chen, Jia and Wang concerning the Jensen polynomials  $J_p^{d,n}(X)$ , where p(n) is the partition function. Nicolas [20] and Desalvo and Pak [21] proved that  $J_p^{2,n}(X)$  is hyperbolic for  $n \geq 25$ , and more recently, Chen, Jia, and Wang proved [22] that  $J_p^{3,n}(X)$  is hyperbolic for  $n \geq 94$ , inspiring them to state as a conjecture the following result.

**Theorem 4** (Chen-Jia-Wang Conjecture). For every integer  $d \ge 1$  there exists an integer N(d) such that  $J_n^{d,n}(X)$  is hyperbolic for  $n \ge N(d)$ .

The table below gives the conjectured minimal value for N(d) for  $d \leq 32$ . More precisely, for each  $d \leq 32$  it gives the smallest integer such that  $J_p^{d,n}(X)$  is hyperbolic for  $N(d) \leq n \leq 50000$ .

d	1	2	3	4	5	6	7	8
N(d)	1	25	94	206	381	610	908	1269
d	9	10	11	12	13	14	15	16
N(d)	1701	2210	2787	3455	4194	5018	5927	6917
d	17	18	19	20	21	22	23	24
N(d)	8004	9171	10435	11788	13232	14777	16407	18146
d	25	26	27	28	29	30	31	32

*Remark.* Larson and Wagner [23] have made the proof of Theorem 4 effective by a brute force implementation of Hermite's criterion (see Theorem C of [3]). They showed that the values in the table are correct for d = 4 and d = 5 and that  $N(d) \leq (3d)^{24d} (50d)^{3d^2}$  in general. The true values are presumably much smaller, and are probably of only polynomial growth, the numbers in the table being approximately of size  $N(d) \approx 10 d^2 \log d$ .

Theorem 4 suggests a natural generalization. As is well-known, the numbers p(n) are the Fourier coefficients of a modular form, namely

(6) 
$$\frac{1}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} \qquad (\Im(\tau) > 0, \ q = e^{2\pi i \tau}),$$

where  $\eta(\tau) = q^{1/24} \prod (1-q^n)$  is the Dedekind eta-function. Theorem 4 is then an example of a more general theorem about the Jensen polynomials of the Fourier coefficients of an arbitrary weakly holomorphic modular form, which for the purposes of this article will mean a modular form (possibly of fractional weight and with multiplier system) with real Fourier coefficients on the full modular group  $SL_2(\mathbb{Z})$  that is holomorphic apart from a pole of (possibly fractional) positive order at infinity. If f is such a form, we denote its Fourier expansion by<sup>4</sup>

(7) 
$$f(\tau) = \sum_{n \in -m + \mathbb{Z}_{>0}} a_f(n) q^n \qquad (m \in \mathbb{Q}_{>0}, \ a_f(-m) \neq 0)$$

Then we will prove the following theorem, which includes Theorem 4.

**Theorem 5.** If f is a weakly holomorphic modular form as above, then for any fixed  $d \ge 1$  the Jensen polynomials  $J_{a_f}^{d,n}(X)$  are hyperbolic for all sufficiently large n.

Our results are proved by showing that each of the sequences of interest to us (the partition function, the Fourier coefficients of weakly holomorphic modular forms, and the Taylor coefficients at  $s = \frac{1}{2}$  of  $4s(1-s)\Lambda(s)$ ) satisfies the hypotheses of Theorem 3, which we prove in Section 2. Actually, in Section 2 we prove a more general result (Theorem 6) that gives the limits of suitably normalized Jensen polynomials for an even bigger class of sequences having suitable asymptotic properties (but without necessarily the corollary about hyperbolicity). Theorem 5 giving the hyperbolicity for coefficients of modular forms (and hence also for the partition function) is proved in Section 3. In Section 4 we prove Theorem 7, which gives an asymptotic formula to all orders for the Taylor coefficients of  $\Lambda(s)$  at  $s = \frac{1}{2}$ , and in Section 5 we prove Theorems 1 and 2 for the Riemann zeta function by using these asymptotics to verify that the hypotheses of Theorem 3 are fulfilled by the numbers  $\gamma(n)$ . We conclude in Section 6 with some numerical examples.

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### 2. Proof of Theorem 3

We deduce Theorem 3 from the following more general result.

**Theorem 6.** Suppose that  $\{E(n)\}$  and  $\{\delta(n)\}$  are positive real sequences with  $\delta(n)$  tending to 0, and that  $F(t) = \sum_{i=0}^{\infty} c_i t^i$  is a formal power series with complex coefficients. For a fixed  $d \ge 1$ , suppose that there are real sequences  $\{C_0(n)\}, \ldots, \{C_d(n)\}, \text{ with } \lim_{n \to +\infty} C_i(n) = c_i \text{ for } 0 \le i \le d$ , such that for  $0 \le j \le d$  we have

(8) 
$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} = \sum_{i=0}^{d} C_i(n) \,\delta(n)^i j^i + o\left(\delta(n)^d\right) \quad \text{as } n \to +\infty$$

Then the conclusion of Theorem 3 holds with  $\exp(A(n))$  replaced by E(n) and  $H_d(X)$  replaced by  $H_{F,d}(X)$ , where the polynomials  $H_{F,m}(X) \in \mathbb{C}[x]$  are now defined either by the generating function  $F(-t) e^{Xt} = \sum H_{F,m}(X) t^m/m!$  or in closed form by  $H_{F,m}(X) := m! \sum_{k=0}^m (-1)^{m-k} c_{m-k} X^k/k!$ .

<sup>&</sup>lt;sup>4</sup>Note that with these notations we have  $p(n) = a_f(n - \frac{1}{24})$  for  $f = 1/\eta$ , but making this shift of argument is irrelevant for the applicability of Theorem 5 to Theorem 4, since the required asymptotic property is obviously invariant under translations of n.

Proof of Theorem 6 and Theorem 3. After replacing  $\exp(A(n))$  by E(n), the polynomial appearing on the left-hand side of (5) becomes

$$\frac{\delta(n)^{-d}}{\alpha(n)} J^{d,n}_{\alpha} \left( \frac{\delta(n) X - 1}{E(n)} \right) = \sum_{k=0}^{d} \binom{d}{k} \left[ \delta(n)^{k-d} \sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} \frac{\alpha(n+j)}{\alpha(n)E(n)^{j}} \right] X^{k}$$

Since  $0 \le j \le d$ , and since the error term in (8) is  $o(\delta(n)^d)$ , we may reorder summation and find that the limiting value as  $n \to +\infty$  of the quantity in square brackets satisfies

$$\lim_{n \to +\infty} \left[ \sum_{i=0}^{d} C_i(n) \,\delta(n)^{k-d+i} \, \sum_{j=k}^{d} (-1)^{j-k} \begin{pmatrix} d-k\\ j-k \end{pmatrix} j^i \right] = (-1)^{d-k} (d-k)! \, c_{d-k} \, d_{j-k} \,$$

because the inner sum, which is the (d - k)th difference of the polynomial  $j \mapsto j^i$  evaluated at j = 0, vanishes for i < d - k and equals (d - k)! for i = d - k. Theorem 6 follows, and Theorem 3 is just the special case  $E(n) = e^{A(n)}$  and  $F(t) = e^{-t^2}$ .

### 3. Proof of Theorem 5

Assume that f is a modular form of (possibly fractional) weight k on  $SL_2(\mathbb{Z})$  (possibly with multiplier system) and with a pole of (possibly fractional) order m > 0 at infinity, and write its Fourier expansion at infinity as in (7). It is standard, either by the circle method of Hardy– Ramanujan–Rademacher or by using Poincaré series (for example, see [24]), that the Fourier coefficients of f have the asymptotic form

(9) 
$$a_f(n) = A_f n^{\frac{k-1}{2}} I_{k-1}(4\pi\sqrt{mn}) + O(n^C e^{2\pi\sqrt{mn}})$$

as  $n \to \infty$  for some non-zero constants  $A_f$  (an explicit multiple of  $a_f(-m)$ ) and C, where  $I_{\kappa}(x)$  denotes the usual *I*-Bessel function. In view of the expansion of Bessel functions at infinity, this implies that  $a_f(n)$  has an asymptotic expansion to all orders in 1/n of the form

$$a_f(n) \sim e^{4\pi\sqrt{mn}} n^{\frac{2k-3}{4}} \exp\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right)$$

for some constants  $c_0, c_1, \ldots$  depending on f (and in fact only on m and k if we normalize the leading coefficient  $a_f(-m)$  of f to be equal to 1). This gives an asymptotic expansion

(10) 
$$\log\left(\frac{a_f(n+j)}{a_f(n)}\right) \sim 4\pi\sqrt{m} \sum_{i=1}^{\infty} {\binom{1/2}{i}} \frac{j^i}{n^{i-\frac{1}{2}}} + \frac{2k-3}{4} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}j^i}{i\,n^i} + \sum_{i,k\geq 1} c_k {\binom{-k}{i}} \frac{j^i}{n^{i+k}}$$

valid to all orders in n, and it follows that the sequence  $\{a_f(n)\}\$  satisfies the hypotheses of Theorem 3 with  $A(n) = 2\pi \sqrt{m/n} + O(1/n)$  and  $\delta(n) = (\pi/2)^{1/2} m^{1/4} n^{-3/4} + O(n^{-5/4})$ . Theorem 5 then follows from the corollary to Theorem 3.

# 4. Asymptotics for $\Lambda^{(n)}\left(\frac{1}{2}\right)$

Previous work of Coffey [6] and Pustyl'nikov [7] offer asymptotics<sup>5</sup> for the derivatives  $\Lambda^{(n)}(\frac{1}{2})$ . Here we follow a slightly different approach and obtain effective asymptotics, a result which is of independent interest. In order to describe our asymptotic expansion, we first give a formula

<sup>&</sup>lt;sup>5</sup>It is interesting to note that Hadamard previously obtained rough estimates for these derivatives in 1893. His formulas are correctly reprinted on p. 125 of [8].

for these derivatives in terms of an auxiliary function, whose asymptotic expansion we shall then determine.

Following Riemann, (cf. Chapter 8 of [25]) we have

$$\Lambda(s) = \int_0^\infty t^{\frac{s}{2}-1} \theta_0(t) \, dt = \frac{1}{s(s-1)} + \int_1^\infty \left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right) \theta_0(t) \, \frac{dt}{t},$$

where  $\theta_0(t) = \sum_{k=1}^{\infty} e^{-\pi k^2 t} = \frac{1}{2}(t^{-1/2} - 1) + t^{-1/2}\theta_0(1/t)$ . It follows that

(11) 
$$\Lambda^{(n)}\left(\frac{1}{2}\right) = -2^{n+2}n! + \frac{F(n)}{2^{n-1}}$$

for n > 0 (both are of course zero for n odd), where F(n) is defined for any real  $n \ge 0$  by

(12) 
$$F(n) = \int_{1}^{\infty} (\log t)^{n} t^{-3/4} \theta_{0}(t) dt$$

In particular if n is a positive integer, then the Taylor coefficients  $\gamma(n)$  defined in (1) satisfy

(13) 
$$\gamma(n) = \frac{n!}{(2n)!} \cdot \left(8\binom{2n}{2}\Lambda^{(2n-2)}\left(\frac{1}{2}\right) - \Lambda^{(2n)}\left(\frac{1}{2}\right)\right) = \frac{n!}{(2n)!} \cdot \frac{32\binom{2n}{2}F(2n-2) - F(2n)}{2^{2n-1}}$$

**Theorem 7.** If n > 0 then the function F(n) defined by (12) is given to all orders in n by the asymptotic expansion

$$F(n) \sim \sqrt{2\pi} \frac{L^{n+1}}{\sqrt{(1+L)n - \frac{3}{4}L^2}} e^{L/4 - n/L + 3/4} \left( 1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots \right) \qquad (n \to \infty),$$

where  $L = L(n) \approx \log(\frac{n}{\log n})$  is the unique positive solution of the equation  $n = L(\pi e^L + \frac{3}{4})$  and each coefficient  $b_k$  belongs to  $\mathbb{Q}(L)$ , the first value being  $b_1 = \frac{2L^4 + 9L^3 + 16L^2 + 6L + 2}{24(L+1)^3}$ .

**Example.** Here we illustrate Theorem 7. The two-term approximation

$$F(n) \approx \sqrt{2\pi} \, \frac{L^{n+1}}{\sqrt{(1+L)n - \frac{3}{4}L^2}} \, e^{L/4 - n/L + 3/4} \left(1 + \frac{b_1}{n}\right) =: \widehat{F}(n)$$

is sufficiently strong for the proof of Theorem 1. In particular, Theorem 7 and (13) imply

(14) 
$$\widehat{\gamma}(n) := \frac{n!}{(2n)!} 2^{6-2n} {2n \choose 2} \widehat{F}(2n-2) = \gamma(n) \left(1 + O\left(\frac{1}{n^{2-\varepsilon}}\right)\right).$$

Here are some approximations  $\widehat{\gamma}(n)$  obtained from this expression by numerically computing L using its defining equation above. This table illustrates the high precision of this formula.

n	$\widehat{\gamma}(n)$		$\gamma(n)$		$\gamma(n)/\widehat{\gamma}(n)$	
10	8	$1.6313374394 \times 10^{-17}$	8	$1.6323380490 \times 10^{-17}$	22	1.000613367
100	$\approx$	$6.5776471904 \times 10^{-205}$	$\approx$	$6.5777263785 \times 10^{-205}$	$\approx$	1.000012038
1000	$\approx$	$3.8760333086 \times 10^{-2567}$	$\approx$	$3.8760340890 \times 10^{-2567}$	$\approx$	1.00000201
10000		$3.5219798669 \times 10^{-32265}$		$3.5219798773 \times 10^{-32265}$	$\approx$	1.000000002
100000	$\approx$	$6.3953905598 \times 10^{-397097}$	$\approx$	$6.3953905601 \times 10^{-397097}$	$\approx$	1.000000000

Proof of Theorem 7. We approximate the integrand in (12) by  $f(t) = (\log t)^n t^{-3/4} e^{-\pi t}$  (from now on we consider *n* as fixed and omit it from the notations). We have  $t \frac{d}{dt} \log f(t) = \frac{n}{\log t} - \pi t - \frac{3}{4}$ , so f(t) assumes its unique maximum at t = a, where  $a = e^L$  is the solution in  $(1, \infty)$  of

$$n = \left(\pi a + \frac{3}{4}\right)\log a$$

We can then apply the usual saddle point method. The Taylor expansion of f(t) around t = a is given by

$$\frac{f((1+\lambda)a)}{f(a)} = \left(1 + \frac{\log(1+\lambda)}{\log a}\right)^n (1+\lambda)^{-3/4} e^{-\pi\lambda a} = e^{-C\lambda^2/2} \left(1 + A_3\lambda^3 + A_4\lambda^4 + \cdots\right),$$

where  $C = (\varepsilon + \varepsilon^2)n - \frac{3}{4}$  (here we have set  $\varepsilon = \frac{1}{\log a} = L^{-1}$ ) and the  $A_i$   $(i \ge 3)$  are polynomials of degree  $\lfloor i/3 \rfloor$  in n with coefficients in  $\mathbb{Q}[\varepsilon]$ . This expansion is found by expanding  $\log(f((1 + \lambda)a)) - \log(f(a))$  in  $\lambda$ . The linear term vanishes by the choice of a, the quadratic term is  $-C\lambda^2/2$ , and the coefficients of the higher powers of  $\lambda$  are all linear expressions in n with coefficients in  $\mathbb{Q}[\varepsilon]$ . Exponentiating this expansion gives the claimed expression for  $f((1 + \lambda)a)/f(a)$ , where the dominant term of each  $A_i$  is governed primarily by the exponential of the cubic term of the logarithmic expansion. The first few  $A_i$  are

$$A_{3} = \left(\frac{\varepsilon}{3} + \frac{\varepsilon^{2}}{2} + \frac{\varepsilon^{3}}{3}\right)n - \frac{1}{4}, \qquad A_{4} = -\left(\frac{\varepsilon}{4} + \frac{11\varepsilon^{2}}{24} + \frac{\varepsilon^{3}}{2} + \frac{\varepsilon^{4}}{4}\right)n + \frac{3}{16},$$
  

$$A_{5} = \left(\frac{\varepsilon}{5} + \frac{5\varepsilon^{2}}{12} + \frac{7\varepsilon^{3}}{12} + \frac{\varepsilon^{4}}{2} + \frac{\varepsilon^{5}}{5}\right)n - \frac{3}{20},$$
  

$$A_{6} = \left(\frac{\varepsilon^{2}}{18} + \frac{\varepsilon^{3}}{6} + \frac{17\varepsilon^{4}}{72} + \frac{\varepsilon^{5}}{6} + \frac{\varepsilon^{6}}{18}\right)n^{2} - \left(\frac{\varepsilon}{4} + \frac{91\varepsilon^{2}}{180} + \frac{17\varepsilon^{3}}{24} + \frac{17\varepsilon^{4}}{24} + \frac{\varepsilon^{5}}{2} + \frac{\varepsilon^{6}}{6}\right)n + \frac{5}{32}.$$

Plugging in  $t = (1 + \lambda)a$  immediately gives the asymptotic expansion

$$\int_{1}^{\infty} f(t) dt = a f(a) \int_{-1+1/a}^{\infty} e^{-C\lambda^{2}/2} \left( 1 + A_{3}\lambda^{3} + A_{4}\lambda^{4} + \cdots \right) d\lambda$$
$$= a f(a) \sqrt{\frac{2\pi}{C}} \left( 1 + \frac{3A_{4}}{C^{2}} + \frac{15A_{6}}{C^{3}} + \cdots + \frac{(2i-1)!!A_{2i}}{C^{i}} + \cdots \right)$$

(Here only the part of the integral with  $C\lambda^2 < B \log n$ , where B is any function of n going to infinity as n does, contributes.) This equality and the expression in Theorem 7 are interpreted as asymptotic expansions. Although these series themselves may not converge for a fixed n, we may truncate the resulting approximation at  $O(n^{-A})$  for some A > 0, and as  $n \to +\infty$ this approximation becomes true to the specified precision. Substituting into this expansion the formulas for C and  $A_i$  in terms of n we obtain the statement of the theorem with F(n) replaced by the integral over f(t), with only  $A_{2i}$  ( $i \leq 3k$ ) contributing to  $b_k$ . But then the same asymptotic formula holds also for F(n), since the ratio  $f(t)/\theta_0(t) = 1 + e^{-3\pi t} + \cdots$  is equal to  $1 + O(n^{-K})$ for any K > 0 for t near a.

### 5. Proof of Theorems 1 and 2

5.1. **Proof of Theorem 1.** For each  $d \ge 1$ , we employ Theorem 3 with sequences  $\{A(n)\}$  and  $\{\delta(n)\}$  for which

(15) 
$$\log\left(\frac{\gamma(n+j)}{\gamma(n)}\right) = A(n)j - j^2\delta(n)^2 + \sum_{i=3}^d g_i(n)j^i + o\left(\delta(n)^d\right)$$

for all  $0 \le j \le d$ , where  $g_i(n) = o(\delta(n)^i)$ . Stirling's formula, (13), and (14) gives

(16) 
$$\gamma(n) = \frac{e^{n-2}n^{n+\frac{1}{2}}(1+\frac{1}{12n})L^{\widehat{n}}}{2^{\widehat{n}-3}\widehat{n}^{\widehat{n}+\frac{1}{2}}(1+\frac{1}{12\widehat{n}})}\sqrt{\frac{2\pi}{K}} \cdot \exp\left(\frac{L}{4} - \frac{\widehat{n}}{L} + \frac{3}{4}\right)\left(1 + \frac{b_1(\widehat{n})}{\widehat{n}}\right)\left(1 + O\left(\frac{1}{n^{2-\varepsilon}}\right)\right),$$

where  $\hat{n} := 2n - 2$ ,  $L := L(\hat{n})$ , and  $K := K(\hat{n}) := (L(\hat{n})^{-1} + L(\hat{n})^{-2})\hat{n} - 3/4$ . The  $L(\hat{n})$  are values of a non-vanishing holomorphic function for  $\Re(n) > 1$ , and so for |j| < n - 1 we have the Taylor expansion

$$\mathcal{L}(j;n) := \frac{L(\widehat{n}+2j)}{L(\widehat{n})} = 1 + \sum_{m \ge 1} \ell_m(n) \frac{j^m}{m!}$$

If  $J = \lambda(n-1)$  with  $-1 < \lambda < 1$ , then the asymptotic  $L(n) \approx \log(\frac{n}{\log n})$  implies

$$\lim_{n \to +\infty} \mathcal{L}(J, n) = \lim_{n \to +\infty} \frac{L(\widehat{n}(\lambda + 1))}{L(\widehat{n})} = 1.$$

In particular, we have  $\ell_1(n) = \frac{2}{K \cdot L^2}$  and  $\ell_2(n) = \frac{-8(\hat{n}-3/4L)(1+L/2)}{K^3 \cdot L^5}$  and  $\ell_m(n) = o(\frac{1}{(n-1)^m})$ . By a similar argument applied to

$$\mathcal{K}(j;n) := \frac{K(\widehat{n}+2j)}{K(\widehat{n})} = 1 + \sum_{m \ge 1} k_m(n) \frac{j^m}{m!} \quad \text{and} \quad \mathcal{B}(j;n) := \frac{1 + \frac{b_1(\widehat{n}+2j)}{\widehat{n}+2j}}{1 + \frac{b_1(\widehat{n})}{\widehat{n}}} = 1 + \sum_{m \ge 1} \beta_m(n) \frac{j^m}{m!},$$

we find that  $\beta_m(n) = o\left(\frac{1}{(n-1)^{m+1}}\right)$ ,  $k_1(n) = \frac{2(L+1)}{K \cdot L^2} - \frac{2\hat{n}(L+2)}{K^2 L^4}$ , and  $k_m(n) = o\left(\frac{1}{(n-1)^m}\right)$  for  $m \ge 2$ . Let R(j;n) be the approximation for  $\gamma(n+j)/\gamma(n)$  obtained from (16). We then expand

 $\log R(j;n) =: \sum_{m \ge 1} g_m(n)j^m$ , with the idea that we will choose  $A(n) \sim g_1(n)$  and  $\delta(n) \sim \sqrt{-g_2(n)}$ . To this end, if  $J = \lambda(n-1)$  for  $-1 < \lambda < 1$ , then a calculation reveals that

(17) 
$$-(1+\lambda)\log(1+\lambda) = \lim_{n \to +\infty} \frac{\log R(J;n) - J \log\left(\frac{nL^2}{4\hat{n}^2}\right) - J}{n-1}.$$

Therefore,  $g_m(n) = O((n-1)^{1-m})$ , and algebraic manipulations give

$$g_1(n) = \log\left(\frac{nL^2}{4\hat{n}^2}\right) + \hat{n}\ell_1(n)\frac{L+1}{L} - \frac{2}{L} + \frac{\ell_1(n)\cdot L}{4} - \frac{k_1(n)}{2} + O\left(\frac{1}{n^{2-\varepsilon}}\right),$$
  
$$g_2(n) = -\frac{1}{\hat{n}} + \left(4\ell_1(n) + \hat{n}\ell_2(n)\right)\frac{L+1}{2L} - \hat{n}\ell_1(n)^2\frac{L+2}{2L} + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

Using the formulas for  $\ell_1(n)$ ,  $\ell_2(n)$ , and  $k_1(n)$  above, we define

(18) 
$$\delta(n) := \sqrt{\frac{1}{\hat{n}} - \frac{2}{L^2 \cdot K}} \quad \text{and} \quad A(n) := \log\left(\frac{nL^2}{4\hat{n}^2}\right) + \frac{L-1}{L^2 \cdot K} + \frac{\hat{n}(L+2)}{L^4 \cdot K^2}$$

The bounds for the  $g_m(n)$  and the asymptotics above imply the o(1) error term in (15), and also that for sufficiently large n we have  $0 < \delta(n) \to 0$ . Therefore, Theorem 3 applies, and its corollary gives Theorem 1.

### 5.2. Sketch of the Proof of Theorem 2. Let A(n) and $\delta(n)$ be as in (18). If we let

$$\widehat{J}_{\gamma}^{d,n}(X) := \frac{\delta(n)^{-d}}{\gamma(n)} \cdot J_{\gamma}^{d,n}\left(\frac{\delta(n) X - 1}{\exp(A(n))}\right) = \sum_{k=0}^{d} \beta_k^{d,n} X^k,$$

then Theorem 1 implies that  $\lim_{n\to+\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X) =: \sum_{k=0}^d h_k X^k$ . We have confirmed the hyperbolicity of the  $\widehat{J}_{\gamma}^{d,n}(X)$  for  $n \leq 10^6$  and  $4 \leq d \leq 8$  using Hermite's criterion (see Theorem C of [3]).

Using this criterion, we also chose vectors  $\varepsilon_d := (\varepsilon_d(d), \varepsilon_d(d-1), \dots, \varepsilon_d(0))$  of positive numbers and signs  $s_d, s_{d-1}, \dots, s_0 \in \{\pm 1\}$  for which  $\widehat{J}_{\gamma}^{d,n}(X)$  is hyperbolic if  $0 \leq s_k(\beta_k^{d,n} - h_k) < \varepsilon_d(k)$  for all k. To make use of these inequalities, for positive integers n and  $1 \leq j \leq 8$ , define real numbers C(n, j) by

(19) 
$$\frac{\gamma(n+j)}{\gamma(n)e^{A(n)j}} \cdot e^{\delta(n)^2 j^2} = 1 + \frac{C(n,j)}{n^{3/2}}.$$

Using an effective form of (16), it can be shown<sup>6</sup> that 0 < C(n, j) < 14.25 for all  $n \ge 7$  and  $1 \le j \le 8$ . Finally, we determined numbers  $M_{\varepsilon_d}$  for which the required inequalities hold for  $n \ge M_{\varepsilon_d}$ . The proof follows from the fact that we found suitable choices for which  $M_{\varepsilon_d} < 10^6$ .

**Example.** We illustrate the case of d = 4 using  $\varepsilon_4 := (0.041, 1.384, 0.813, 7.313, 0.804)$ . For  $n \ge 100$  the odd degree coefficients satisfy

$$0 < \beta_3^{4,n} < 28 \ \delta(n)$$
 and  $-145.70\delta(n) < \beta_1^{4,n} < 0$ ,

while the even degree coefficients satisfy

$$1 - 16.05 \ \delta(n)^2 < \beta_4^{4,n} < 1, \quad -12 < \beta_2^{4,n} < -12 + 16.20 \ \delta(n), \quad 12 - 16.01 \ \delta(n) < \beta_0^{4,n} < 12.00 \ \delta(n) < -10.01 \ \delta(n) < \beta_0^{4,n} < 12.00 \ \delta(n) < -10.01 \ \delta(n) < -10.00 \ \delta(n) < -10.$$

It turns out that  $M_{\varepsilon_4} := 104 < 10^6$ .

### 6. Examples

For convenience, we let the  $\widehat{J}_{\alpha}^{d,n}(X)$  denote the polynomials which converge to  $H_d(X)$  in (5). We now illustrate Theorem 5 with (6), where m = 1/24 and k = -1/2. Using (10), we may choose  $A(n) = \frac{2\pi}{\sqrt{24n-1}} - \frac{24}{24n-1}$  and  $\delta(n) = \sqrt{\frac{12\pi}{(24n-1)^{3/2}} - \frac{288}{(24n-1)^2}}$ . Although the one-term approximations of (10) given at the end of Section 3 also satisfy Theorem 3, the two-term approximations converge more quickly and better illustrate the result. With this data we observe indeed that the degree 2 and 3 partition Jensen polynomials are modeled by  $H_2(X) = X^2 - 2$  and  $H_3(X) = X^3 - 6X$ .

<sup>&</sup>lt;sup>6</sup>It turns out that  $\delta(6)$  is not real.

n	$\widehat{J}_{p}^{2,n}(X)$	$\widehat{J}_{p}^{3,n}(X)$
100	$\approx 0.9993X^2 + 0.0731X - 1.9568$	$\approx 0.9981X^3 + 0.2072X^2 - 5.9270X + 1.1420$
200	$\approx 0.9997X^2 + 0.0459X - 1.9902$	$\approx 0.9993X^3 + 0.1284X^2 - 5.9262X - 1.4818$
300	$\approx 0.9998X^2 + 0.0346X - 1.9935$	$\approx 0.9996X^3 + 0.0965X^2 - 5.9497X - 1.3790$
400	$\approx 0.9999X^2 + 0.0282X - 1.9951$	$\approx 0.9998X^3 + 0.0786X^2 - 5.9621X - 1.2747$
:	÷	:
$10^{8}$	$\approx 0.9999X^2 + 0.0000X - 1.9999$	$\approx 0.9999X^3 + 0.0000X^2 - 5.9999X - 0.0529$

The next table illustrates Theorem 1 for the Riemann zeta function using (18) in the case of degrees 2 and 3.

n	$\widehat{J}^{2,n}_{\gamma}(X)$	$\widehat{J}_{\gamma}^{3,n}(X)$
100	$\approx 0.9896X^2 + 0.3083X - 2.0199$	$\approx 0.9769X^3 + 0.7570X^2 - 5.8690X - 1.2661$
200	$\approx 0.9943X^2 + 0.2271X - 2.0061$	$\approx 0.9872X^3 + 0.5625X^2 - 5.9153X - 0.9159$
300	$\approx 0.9960X^2 + 0.1894X - 2.0029$	$\approx 0.9911X^3 + 0.4705X^2 - 5.9374X - 0.7580$
400	$\approx 0.9969X^2 + 0.1663X - 2.0016$	$\approx 0.9931X^3 + 0.4136X^2 - 5.9501X - 0.6623$
:	:	÷
$10^{8}$	$\approx 0.9999X^2 + 0.0003X - 2.0000$	$\approx 0.9999X^3 + 0.0009X^2 - 5.9999X - 0.0014$

Finally, we conclude with data for the degree 6 renormalized Jensen polynomials  $J_{\gamma}^{6,n}(X)$  which converge to  $H_6(X) = X^6 - 30X^4 + 180X^2 - 120$ .

n	$\widehat{J}_{\gamma}^{6,n}(X)$
100	$\approx 0.912X^6 + 3.086X^5 - 24.114X^4 - 55.652X^3 + 133.109X^2 + 151.696X - 85.419$
200	$\approx 0.950X^6 + 2.374X^5 - 26.625X^4 - 42.824X^3 + 153.246X^2 + 115.849X - 100.510$
300	$\approx 0.965X^6 + 2.011X^5 - 27.608X^4 - 36.282X^3 + 161.084X^2 + 97.843X - 106.295$
400	$\approx 0.973X^6 + 1.780X^5 - 28.139X^4 - 32.111X^3 + 165.303X^2 + 86.428X - 109.388$
:	· · · · · · · · · · · · · · · · · · ·
$10^{10}$	$\approx 0.999X^6 + 0.000X^5 - 29.999X^4 - 0.008X^3 + 179.999X^2 + 0.020X - 119.999X^2 + 0.020X - 119.99Y^2 + 0.020X - 119.99Y^2 + 0.020X - 119.99Y^2 + 0.020X - 100X - 100X$

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