JENSEN POLYNOMIALS FOR THE RIEMANN XI FUNCTION

M. J. GRIFFIN, K. ONO, L. ROLEN, J. THORNER, Z. TRIPP, AND I. WAGNER

Abstract. We investigate Riemann's xi function $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ (here $\zeta(s)$ is the Riemann zeta function). The Riemann Hypothesis (RH) asserts that if $\xi(s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$. Pólya proved that RH is equivalent to the hyperbolicity of the Jensen polynomials $J^{d,n}(X)$ constructed from certain Taylor coefficients of $\xi(s)$. For each $d \geq 1$, recent work proves that $J^{d,n}(X)$ is hyperbolic for sufficiently large n. Here we make this result effective. Moreover, we show how the low-lying zeros of the derivatives $\xi^{(n)}(s)$ influence the hyperbolicity of $J^{d,n}(X)$.

1. Introduction and Statement of Results

We recall the Riemann xi function $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ (here $\zeta(s)$ is the Riemann zeta function). Define real numbers $\gamma(n)$ by the Taylor expansion

$$\psi(z) = \sum_{j=0}^{\infty} \frac{\gamma(j)}{j!} \cdot z^{2j} =: 8\xi\left(\frac{1}{2} + z\right). \tag{1.1}$$

It is known that $\gamma(n) \geq 0$ for all $n \geq 0$ (for example, see Section 4.4 of [2]). For integers $d, n \geq 0$, the degree d Jensen polynomial $J^{d,n}(X)$ associated to the n-th derivative $\xi^{(n)}(s)$ is

$$J^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) X^j. \tag{1.2}$$

A polynomial with real coefficients is *hyperbolic* if all of its zeros are real. Expanding on notes of Jensen, Pólya [10] proved that the Riemann Hypothesis (RH) is equivalent to the hyperbolicity of $J^{d,n}(X)$ for all $d, n \geq 0$. Since RH remains unproved, some research has focused on proving hyperbolicity for all $n \geq 0$ and small d. Csordas, Norfolk, and Varga [4] and Dimitrov and Lucas [6] proved hyperbolicity for $n \geq 0$ and $d \leq 3$. Chasse [3] proved hyperbolicity for $d \leq 2 \cdot 10^{17}$ and $n \geq 0$.

Recent work [8] provides a complementary treatment. For all $d \geq 1$, it is now known that there is an effectively computable threshold $N(d)$ such that $J^{d,n}(X)$ is hyperbolic for $n \geq N(d)$. Specifically, under a suitable transformation (see (2.2)), the polynomials $J^{d,n}(X)$ are closely modeled by the Hermite polynomials $H_d(X)$, where $\sum_{d=0}^{\infty} H_d(X)t^d/d! := e^{Xt-t^2}$. Thus for large n, the Jensen polynomials inherit hyperbolicity from the Hermite polynomials. Our main result is an effective upper bound for $N(d)$.

Theorem 1.1. *If $d \geq 1$ and $n \gg e^{8d/9}$, then $J^{d,n}(X)$ is hyperbolic.*

Key words and phrases. Riemann zeta function, Riemann hypothesis, Jensen polynomial.

The second author thanks the support of the Thomas Jefferson Fund and the NSF. The fourth author began this work while partially supported by a NSF Postdoctoral Fellowship.

2010 Mathematics Subject Classification: 11M26, 11M06.

Remark. *The proof of Theorem 1.1 actually shows that if* $d \geq 1$, $C > 1$, *and* $n \gg_C (32C^2+3)^{\frac{d}{4}}$, *then* $J^{d,n}(X)$ *is hyperbolic. Theorem 1.1 is obtained by choosing* $C = 1.00001$.

These results can be thought of as a proof of the *Gaussian Unitary Ensemble* (GUE) random matrix model prediction for the spacing of zeros of $\xi^{(n)}(s)$ as $n \to \infty$ (the "derivative aspect") (see [1]). Indeed, the zeros of the $H_d(X)$ satisfy Wigner's semicircular distribution, as do the eigenvalues of random Hermitian matrices. In light of this, it is natural to pursue other explicit relationships between the zeros of $\xi^{(n)}(s)$ and the Jensen polynomials.

For an integer $m \geq 0$, let RH_m to be the statement that if $\xi^{(m)}(s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$. It is well known that $\mathrm{RH} = \mathrm{RH}_0$ implies RH_m for all $m \geq 1$ (see [10]). The ideas of Pólya lead to the conclusion that $\xi^{(m)}(s)$ satisfies RH_m if and only if $J^{d,n}(X)$ is hyperbolic for $d \geq 1$ and $n \geq m$. For $T \geq 0$, we define $\mathrm{RH}_m(T)$ to be the statement that all zeros $\rho^{(m)}$ of $\xi^{(m)}(s)$ with $|\mathrm{Im}(\rho^{(m)})| \leq T$ satisfy $\mathrm{Re}(\rho^{(m)}) = \frac{1}{2}$. Our second result is a relationship between $\mathrm{RH}_m(T)$ and the hyperbolicity of $J^{d,n}(X)$ for $n \geq m$. This is a modest generalization of the work of Chasse; we include it for the sake of completeness. In what follows, $\lfloor x \rfloor$ denotes the usual floor function.

Theorem 1.2. *If* $\mathrm{RH}_m(T)$ *is true and* $d \leq \lfloor T \rfloor^2$, *then* $J^{d,n}(X)$ *is hyperbolic for all* $n \geq m$.

Platt [11] has verified $\mathrm{RH}_0(3.06 \times 10^{10})$. Therefore, Theorem 1.2 implies the following corollary.

Corollary 1.3. *If* $d \leq 9.36 \times 10^{20}$, *then* $J^{d,n}(X)$ *is hyperbolic for all* $n \geq 0$.

Remark. *One can generalize the notion of a Jensen polynomial by replacing the Taylor coefficients* $\gamma(n)$ *with other suitable arithmetic functions in* (1.2). *Questions of hyperbolicity for such polynomials can be of great arithmetic interest* [8]. *While some of the ideas presented here might apply in other settings, we restrict our consideration and only present the strongest conclusions for* $\xi(s)$ *that our methods appear to permit.*

In Section 2, we make use of results previously obtained in [8] to determine effective estimates for the coefficients of $J^{d,n}(X)$ under a certain normalization (see (2.3)). In Section 3, we prove Theorem 1.1 using a classical result of Turán. Finally, in Section 4, we prove Theorem 1.2.

2. Renormalized Jensen polynomials

In Section 5 of [8], it was shown that for each integer $d \geq 1$, there exists sequences of positive real numbers $\big(A(n)\big)$, $\big(\delta(n)\big)$, where $\delta(n) \sim \frac{1}{\sqrt{2n}}$, as well as sequences of real numbers $\big(g_3(n)\big)$, $\big(g_4(n)\big)$, $\ldots$ $\big(g_d(n)\big)$, such that

$$\log\Big(\frac{\gamma(n+j)}{\gamma(n)}\Big) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + \mathrm{o}\big(\delta(n)^d\big), \tag{2.1}$$

with $g_i(n) = \mathrm{O}\big(n^{1-i}\big) = \mathrm{o}\big(\delta(n)^i\big)$. These sequences were used to define renormalizations of the Jensen polynomials[1]

$$\widetilde{J}^{d,n}(X) := \frac{\delta(n)^{-d}}{\gamma(n)} J^{d,n}\Big(\frac{\delta(n)X\, -\, 1}{\exp(A(n))}\Big). \tag{2.2}$$

[1]In [8], these polynomials were denoted $\widehat{J}^{d,n}_{\gamma}(X)$.

We make use of a slight reformulation of these polynomials. Given a non-negative integer n and degree d, let $N := n + d$. We define

$$\widehat{J}^{d,n}(X) := \frac{\gamma(N)^{d-1}}{\gamma(N-1)^d \cdot \Delta(N)^d}\, J^{d,n}\Big(\frac{\gamma(N-1)}{\gamma(N)} \cdot \big(\Delta(N)\, X \,-\, 1\big)\Big), \tag{2.3}$$

where

$$\Delta(N) := \sqrt{\frac{1}{2}\left(1 - \frac{\gamma(N-2)\gamma(N)}{\gamma(N-1)^2}\right)}. \tag{2.4}$$

That $\Delta(N)$ is real is equivalent to the log-concavity of the $\gamma(n)$. This log-concavity follows from the hyperbolicity of degree 2 Jensen polynomials $J^{2,n}(X)$ proven in [4]. Furthermore, (15) and (18) of [8] imply that

$$\Delta(N) \sim \delta(N) \sim \frac{1}{\sqrt{2N}}. \tag{2.5}$$

Remark. *The polynomials $\widehat{J}^{d,n}(X)$ make use of the Taylor coefficients themselves, instead of the approximations used in* [8]*. Lemma 2.2 shows that the degree of $\widehat{J}^{d,n}(X) - H_d(X)$ is $\leq d-3$. We stress that these polynomials enjoy the same asymptotic properties as the renormalized Jensen polynomials in* [8]*, and so the results therein apply to them mutatis mutandis.*

The discussion above assumed $d \geq 1$ is a fixed integer. In order to discuss the asymptotic properties of all the $\widehat{J}^{d,n}(X)$, regardless of the degree, it is convenient to have an extension of (2.1), where the left hand side is expanded as an infinite convergent power series in j.

Theorem 2.1. *There is a sequence of functions $\{G_m(z)\}$, analytic for* $\mathrm{Re}(z) > 1$*, such that for all positive integers $j < N$ we have*

$$\log\left(\frac{\gamma(N-j)}{\gamma(N)}\right) = -\sum_{m=1}^{\infty} G_m(N)\Delta(N)^{2m-2} j^m. \tag{2.6}$$

Moreover, we have

$$G_2(N) = 1 + (1 - 3G_3(N))\Delta(N)^2 + O\big(\Delta(N)^4\big). \tag{2.7}$$

For $m \geq 2$, we have

$$\lim_{N\to\infty} G_m(N) = \frac{2^{m-1}}{m(m-1)}. \tag{2.8}$$

Proof. Following the notation in [8, Section 4], for $\mathrm{Re}(z) > 0$, we define

$$F(z) := \int_1^{\infty} (\log t)^z\, t^{-3/4}\, \theta_0(t)\, dt\,,$$

where $\theta_0(t) := \sum_{k=1}^{\infty} e^{-\pi k^2 t} = \frac{1}{2}(t^{-1/2} - 1) + t^{-1/2}\theta_0(1/t)$. Equation (13) of [8] gives

$$\gamma(N) = \frac{N!}{(2N)!} \cdot \frac{32\binom{2N}{2}F(2N-2) - F(2N)}{2^{2N-1}}.$$

The function $F(z)$ is holomorphic for $\mathrm{Re}(z) > 0$. Therefore, for fixed $N > 1$, we have that $\log\big(\frac{\gamma(N-j)}{\gamma(N)}\big)$ has a Taylor expansion in j. By varying N, we find the Taylor coefficents are themselves values of analytic functions in N. Therefore, the functions $G_m(z)$ can be chosen by dividing these analytic functions by the corresponding powers of $\Delta(N)$ to give (2.6).

Turning to the proof of (2.7), we may combine (2.4) and (2.6) to obtain a power series identity involving $\Delta(N)$ and the $G_m(N)$. Solving for $G_2(N)$ immediately gives (2.7).

To prove (2.8), we note that (2.6) clearly converges for j with $|j| < N-1$. Thanks to (2.1) and (2.5), where d can be chosen to be arbitrarily large, as $N \to \infty$ we have that $G_1(N) \sim A(N)$ and $G_2(N)\Delta(N)^2 \sim \delta(N)^2$. For the first claim, notice the sign change between (2.1) and (2.6). Moreover, for $m \geq 2$ we have

$$G_m(N)\Delta(N)^{2m-2} \sim (-1)^{m+1} g_m(N).$$

Using the asymptotics implied by [8, Equation 17] for $g_m(n)$, for $m \geq 2$ we find that

$$g_m(N) \sim \frac{(-1)^{m+1}}{m(m-1)} N^{1-m}.$$

Claim (2.8) now follows from (2.5). $\square$

We require precise asymptotic properties for the coefficients of the polynomials $\widehat{J}^{d,n}(X)$. For convenience, we define the coefficients $A_{d,k}(n)$ by

$$\widehat{J}^{d,n}(X) =: \sum_{k=0}^{d} A_{d,k}(n) X^{d-k}. \tag{2.9}$$

The following lemma shows that

$$A_{d,0}(n) = 1, \qquad A_{d,1}(n) = 0, \qquad A_{d,2}(n) = -d(d-1), \tag{2.10}$$

which are the first three coefficients of the Hermite polynomial $H_d(X)$.

Lemma 2.2. *Given a non-negative integer n and degree d, then we have that*

$$\deg\left(\widehat{J}^{d,n}(X) - H_d(X)\right) \leq d-3.$$

Proof. The conclusion follows easily from (2.3) and the original definition

$$J^{d,n}(X) = \gamma(N)X^d + d\gamma(N-1)X^{d-1} + \binom{d}{2}\gamma(N-2)X^{d-2} + O(X^{d-3}).$$

$\square$

The asymptotic properties of the remaining coefficients are critical in the sequel and are the content of the next theorem. In order to state the theorem and obtain explicit bounds that vary simply with k, we will let $C > 1$ and define N_C so that $\frac{1}{(2C(N-1))^{1/2}} \leq \Delta(N) \leq \frac{1}{N^{1/2}}$ for all $N \geq N_C$, which is possible since $\Delta(N) \sim \frac{1}{(2N)^{1/2}}$.

Theorem 2.3. *Assume that $d \geq 4$ is a positive integer, that $C > 1$ is fixed, and that $k > 2$. If n is a non-negative integer and $N = n + d$, then the following are true.*
(1) If $k = 2\ell$ is even and $N > \max(\ell^3, N_C, 64C^2\ell)$, then

$$A_{d,k}(n) = \frac{(-1)^\ell\, d!}{(d-k)!\,\ell!}\Big[1 + \ell(\ell-1)\left(-\frac{2}{3}(3\ell+2) + 2\ell G_3 - \frac{\ell-2}{2}G_3^2 - G_4\right)\cdot\Delta(N)^2 \,+\, \mathcal{E}_{1,k}(N)\Big],$$

where $|\mathcal{E}_{1,k}(N)| \ll k^6(4C)^k\Delta(N)^4$.

(2) If $k = 2\ell + 1$ is odd and $N > \max(\ell^3, N_C, 64C^2\ell)$, then

$$A_{d,k}(n) = \frac{(-1)^\ell\, d!}{(d-k)!\,\ell!}\Big[\ell\big(G_3(N) - 2\big)\cdot\Delta(N) \,+\, \mathcal{E}_{2,k}(N)\Big],$$

where $|\mathcal{E}_{2,k}(N)| \ll k^4(4C)^k\Delta(N)^3$.

2.1. **Proof of Theorem 2.3.** Using the functions $G_m(N)$ given by Theorem 2.1, we define

$$S(N,j) := \exp\left(\sum_{m=2}^{\infty} G_m(N)\Delta(N)^{2m-2}(j - j^m)\right) \tag{2.11}$$

$$=: \sum_{m=0}^{\infty} Q_m(N) j^m.$$

Then by (2.6), we have for integers j with $0 \le j \le N-1$ that

$$S(N,j) = \frac{\gamma(N-j)\gamma(N)^{j-1}}{\gamma(N-1)^j}.$$

Thanks to (2.3), we may rewrite the coefficients $A_{d,k}(n)$ as

$$A_{d,k}(n) = \binom{d}{k}\Delta(N)^{-k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} S(N,j). \tag{2.12}$$

Let $s_{m,k} := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^m$, so that the the expression above becomes

$$A_{d,k}(n) = \binom{d}{k}\Delta(N)^{-k}\sum_{m=0}^{\infty} s_{m,k}Q_m(N). \tag{2.13}$$

We have the following lemma about the size of the $s_{m,k}$.

Lemma 2.4. *Let $s_{m,k}$ be defined as above. Then $s_{m,k} = 0$ if $m < k$, and*

$$s_{k,k} = k!, \qquad s_{k+1,k} = k!\binom{k+1}{2}, \qquad s_{k+2,k} = k!\binom{k+2}{3}\frac{3k+1}{4}.$$

More generally, for $i \ge 1$ we have that

$$s_{k+i,i} = k!\binom{k+i}{1+i} P_i(k),$$

where $P_i(k)$ is some polynomial in k of degree $i-1$, satisfying $P_i(1) = 1$ and $P_i(k) \le k^{i-1}$ for all positive integers k.

Proof. If $f(x)$ is a rational function, the k-th difference $\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(j)$ is zero if and only if $f(x)$ is a polynomial of degree $\le k$. The expressions $s_{m,k}$ are precisely the k-th difference of the polynomials x^m. Thus $s_{m,k} = 0$ if $m < k$. We may also see this fact using the generating function

$$\sum_{m=0}^{\infty}\frac{s_{m,k}}{m!}\cdot X^m = (e^X - 1)^k.$$

This expression immediately gives $s_{m,k}$ for $m \leq k$. The generating function for the i-th difference of the $\frac{s_{k+i,k}}{(k+i)!}$ is given by

$$\sum_{j=0}^{i}(-1)^j \binom{i}{j} X^j \left(e^X - 1\right)^{k-j} = \left(e^X - 1\right)^{k-i}\left(e^X - X - 1\right)^i.$$

Since this expression has degree $k+i$, we see that $\frac{s_{k+i,k}}{(k+i)!}$ is a polynomial in k of degree i. For $i \geq 1$, we observe that $s_{i,0} = 0$, and $s_{i,1} = 1$. Thus, we can factor $s_{k+i,k}$ as

$$s_{k+i,k} \;=\; k! \prod_{j=0}^{i-1} \frac{(k+i-j)}{(1+i-j)} \cdot k \cdot P_i(k) \;=\; k! \binom{k+i}{i+1} \cdot P_i(k),$$

where $P_i(1) = 1$. A short calculation gives the claimed expressions for $s_{k+1,k}$ and $s_{k+2,k}$.

The assertion that $P_i(k) \leq k^{i-1}$ follows by comparing the Taylor coefficients of

$$\left(\frac{e^X - 1}{X}\right)^k = \sum_{i=0}^{\infty} \frac{k \cdot P_i(k)}{(i+1)!} X^i \quad \text{and} \quad \frac{e^{kX} - 1}{kX} = \sum_{i=0}^{\infty} \frac{k^i}{(i+1)!} X^i.$$

It suffices to show that the coefficients of $\left(\frac{e^{X/2} - e^{-X/2}}{X}\right)^k$ never exceed those of $\frac{e^{\frac{k}{2}X} - e^{-\frac{k}{2}X}}{kX}$. This follows from the stronger claim that for $k \geq 1$ the coefficients of

$$\left(\frac{e^{X/2} - e^{-X/2}}{X}\right)\left(\frac{e^{\frac{k}{2}X} - e^{-\frac{k}{2}X}}{kX}\right) = \frac{e^{\frac{k+1}{2}X} + e^{-\frac{k+1}{2}X} - e^{\frac{k-1}{2}X} - e^{-\frac{k-1}{2}X}}{kX^2}$$

never exceed those of $\frac{e^{\frac{k+1}{2}X} - e^{-\frac{k+1}{2}X}}{(k+1)X}$, or equivalently that for positive integers j,

$$\frac{2(k+1)}{(2j)!}\left(\left(\frac{k+1}{2}\right)^{2j} - \left(\frac{k-1}{2}\right)^{2j}\right) \leq \frac{2k}{(2j-1)!}\left(\frac{k+1}{2}\right)^{2j-1}.$$

This is immediate if $2j \geq \left(\frac{(k+1)^2}{2k}\right)$. Otherwise we have that $2j \leq \frac{k}{2} + 1$, and we expand $\left(\frac{k-1}{2}\right)^{2j} = \left(\left(\frac{k+1}{2}\right) - 1\right)^{2j}$ in the left hand side as a binomial. The first few terms are easily seen to be less than the right hand side, and the bound on j gives us that remaining alternating terms are strictly decreasing in absolute value and are easily bounded.

$\square$

Now that we have a formula for the $s_{m,k}$, we only need to find the asymptotics of Q_m to find the desired asymptotics of $A_{d,k}$. To do so, we first bound the functions G_m using the explicit formula for $\gamma(n)$ given in (16) of [8]. Namely, we have that

$$G_m(N)\Delta(N)^{2m-2} = -\frac{1}{m(m-1)N^{m-1}} - \frac{2(m-N)}{m(m-1)(N-1)^m} + \mathcal{E}_m(N), \tag{2.14}$$

where

$$
\begin{aligned}
|\mathcal{E}_m(N)| < {} & \frac{2N-2}{m!}\frac{\partial^m}{\partial j^m}\left[\frac{1}{L(2N-2j-2)} - \log(L(2N-2j-2))\right]_{j=0} \\
& + \frac{2}{(m-1)!}\frac{\partial^{m-1}}{\partial j^{m-1}}\left[\frac{1}{L(2N-2j-2)} - \log(L(2N-2j-2))\right]_{j=0} \\
& - \frac{1}{4\cdot m!}\frac{\partial^m}{\partial j^m}\left[L(2N-2j-2)\right]_{j=0} + \frac{1}{2mN^m} + \frac{3}{2m(N-1)^m} \\
& + \frac{1}{2\cdot m!}\frac{\partial^m}{\partial j^m}\left[\log(K(2N-2j-2))\right]_{j=0} - \frac{1}{m!}\frac{\partial^m}{\partial j^m}\left[\log(\beta(2N-2j-2))\right]_{j=0} \\
& - \frac{(12N+1)^m-(12N)^m}{mN^m(12N+1)^m} + \frac{(24N-23)^m-(24N-24)^m}{m(24N-23)^m(N-1)^m}.
\end{aligned}
$$

Recall that $L = L(n)$ is the solution to $n = L\left(\pi e^L + \frac{3}{4}\right)$, $K(n) = 2L^{-2}(1+L) + \frac{3}{4}$, $b_1(n) = \frac{24L^4+9L^3+16L^2+6L+2}{24(L+1)^3}$, and

$$\beta(2N-2j-2) = \frac{1 + \frac{b_1(2N-2j-2)}{2N-2j-2}}{1 + \frac{b_1(2N-2)}{2N-2}}.$$

This expression is computed in a similar way as in [8] and the inequality comes from cutting off the asymptotic expansion for γ and using Stirling's approximation. We can use Lambert's W-function to find an asymptotic expansion for L. Lambert's W-function is defined as the solution to $z = W(z)e^{W(z)}$ and has Taylor expansion $W(x) = \sum_{n\geq 1}\frac{(-n)^{n-1}}{n!}x^n$ (see [7], §14.3). For large x, we have the asymptotic expansion [5]

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + \frac{(\log\log x - 2)\log\log x}{2(\log x)^2} + O\Big(\frac{(\log\log x)^3}{(\log x)^3}\Big).$$

We then find that

$$
\begin{aligned}
\left|\frac{\partial^m}{\partial j^m}\left[L(2N-2j-2)\right]_{j=0}\right| &< \frac{(m-1)!}{(N-1)^m}, \\
\left|\frac{\partial^m}{\partial j^m}\left[\frac{1}{L(2N-2j-2)}\right]_{j=0}\right| &< \frac{2}{m(N-1)^{m-1}L(2N-2)^2}, \\
\left|\frac{\partial^m}{\partial j^m}\left[\log(L(2N-2j-2))\right]_{j=0}\right| &< \frac{2}{m(N-1)^{m-1}L(2N-2)}, \\
\left|\frac{\partial^m}{\partial j^m}\left[\log(K(2N-2j-2))\right]_{j=0}\right| &< \frac{m!}{(N-1)^m}, \text{ and} \\
\left|\frac{\partial^m}{\partial j^m}\left[\log(\beta(2N-2j-2))\right]_{j=0}\right| &< \frac{m!}{(N-1)^m}.
\end{aligned}
$$

We can bound the above terms in a similar way and reduce the expression to

$$\begin{aligned}|\mathcal{E}_m(N)| < \frac{2}{m(m-1)(N-1)^{m-1}L(2N-2)} + \frac{2}{m(m-1)(N-1)^{m-1}L(2N-2)^2}\\ + \frac{9+6m}{4m(N-1)^m} + \frac{2}{m(N-1)^{m+1}}.\end{aligned} \tag{2.15}$$

From our assumptions on N_C, it is easy to bound each of the individual summands of (2.14) and (2.15) to obtain $G_m(N) \ll (2C)^m$ for $N \geq N_C$, where the bound depends only on the choice of C and not on m or N. We will omit this dependence on C in the rest of the calculations.

Expanding (2.11) by direct symbolic calculation would yield the main terms for Q_m/Δ^m, but the error term would not be explicit in m. We obtain an explicit bound as follows.

Lemma 2.5. *Let m and ℓ be positive integers and $N \geq \max(\ell^3, N_C)$ for $C > 1$ fixed. Then*

$$\left|\frac{Q_m \ell!}{\Delta^m}\right| \ll (4C)^m \ell^{\ell - \frac{1}{2}m}.$$

Proof. From (2.11), we obtain

$$\begin{aligned}\frac{Q_m \ell!}{\Delta^m} &= \frac{\ell!}{\Delta^m}\sum_{\lambda \vdash m} \frac{(\widetilde{G}_1\Delta^2)^{\lambda_1}}{\lambda_1!}\frac{(-G_2\Delta^2)^{\lambda_2}}{\lambda_2!}\cdots\frac{(-G_m\Delta^{2m-2})^{\lambda_m}}{\lambda_m!}\\ &= \sum_{\lambda\vdash m}(-1)^{\ell(\lambda)-\lambda_1}\frac{\ell!}{\lambda_1!\lambda_2!\cdots\lambda_m!}\widetilde{G}_1^{\lambda_1}G_2^{\lambda_2}\cdots G_m^{\lambda_m}\Delta^{m-2\ell(\lambda)+2\lambda_1},\end{aligned} \tag{2.16}$$

where $\widetilde{G}_1 := \sum_{m=2}^{\infty} G_m \Delta^{2m-4} = 1 + O(\Delta^2)$, λ is a partition of m, λ_i is the number of parts of λ of size i, and $\ell(\lambda)$ is the length of the partition. In order to bound the absolute value, we will simply bound each summand by $O\left((2C)^m \ell^{\ell-\frac{1}{2}m}\right)$ and use that $p(m) \leq 2^m$. First, we note that $\widetilde{G}_1 \to 1$ and $G_i \ll (2C)^i$ tells us that $\widetilde{G}_1^{\lambda_1}G_2^{\lambda_2}\cdots G_m^{\lambda_m} \ll (2C)^{\lambda_1+2\lambda_2+\cdots+m\lambda_m} = (2C)^m$. Additionally, using that $\Delta \leq N^{-\frac{1}{2}} \leq \ell^{-\frac{3}{2}}$, we can obtain

$$\frac{\ell!}{\lambda_1!\lambda_2!\cdots\lambda_m!}\Delta^{m-2\ell(\lambda)+2\lambda_1} \leq \frac{\ell!}{\lambda_2!}\ell^{-\frac{3}{2}(m-2\ell(\lambda)+2\lambda_1)} \leq \ell^{\ell-\lambda_2-\frac{3}{2}(m-2\ell(\lambda)+2\lambda_1)}.$$

Using that $m = \lambda_1 + 2\lambda_2 + \cdots + m\lambda_m$ and $\ell(\lambda) = \lambda_1 + \cdots + \lambda_m$, one can see that $\lambda_2 + \frac{3}{2}(m - 2\ell(\lambda) + 2\lambda_1) \geq \frac{1}{2}m$, which completes the proof. $\square$

In order to address the main terms for Q_m/Δ^m, first consider the case when $m = 2\ell$. If we write out the terms with small powers of Δ in (2.16) and bound the remaining terms, we could instead obtain

$$\begin{aligned}\frac{(-1)^{\ell}Q_m\ell!}{\Delta^m} = G_2^{\ell} - \Bigg(&\ell G_2^{\ell-1}\frac{\widetilde{G}_1^2}{2!} + \ell(\ell-1)G_4G_2^{\ell-2} + \ell(\ell-1)G_3G_2^{\ell-2}\widetilde{G}_1\\ &+ \ell(\ell-1)(\ell-2)\frac{G_3^2}{2!}G_2^{\ell-3}\Bigg)\Delta^2 + O\left(m^6(4C)^m\Delta^4\right).\end{aligned} \tag{2.17}$$

The m^6 comes from taking out a Δ^4 in (2.16) when bounding the error. Similarly, when $m = 2\ell + 1$, we obtain

$$\frac{(-1)^\ell Q_m \ell!}{\Delta^m} = \left(G_2^\ell \tilde{G}_1 + \ell G_3 G_2^{\ell-1}\right)\Delta + O\left(m^4 (4C)^m \Delta^3\right). \tag{2.18}$$

Now, we can rewrite (2.13) using Lemma 2.4 as

$$A_{d,k}(n) = \frac{(-1)^\ell d!}{(d-k)!\ell!}\left(\frac{(-1)^\ell \ell! Q_k}{\Delta^k} + \sum_{i=1}^{\infty}\binom{k+i}{1+i}\frac{(-1)^\ell \ell! P_i(k) Q_{k+i}}{\Delta^k}\right), \tag{2.19}$$

where $\ell = \lfloor \frac{k}{2} \rfloor$. In the separate cases of $k = 2\ell$ and $k = 2\ell + 1$, one can write out the first few terms of (2.19) and obtain the main terms of the theorem by using the formulas (2.17) and (2.18) for Q_m, the formulas for $P_i(k)$ given in Lemma 2.4, and using the additional estimate $G_2^\ell = 1 + \ell(1 - 3G_3)\Delta^2 + O(\ell^2\Delta^4)$, which follows from (2.7).

To complete the proof, one can then bound the remaining terms in (2.19) using the general bounds in Lemmas 2.5 and 2.4. In the $k = 2\ell$ case for example, this yields

$$\sum_{i=4}^{\infty}\binom{k+i}{1+i} k^{i-1}(4C)^{k+i}\ell^{\ell-\frac{1}{2}(k+i)}\Delta^i = \frac{(4C)^k}{k}\sum_{i=4}^{\infty}\binom{k+i}{1+i}(4Ck\ell^{-\frac{1}{2}}\Delta)^i.$$

Note that the term in parentheses is $8C\ell^{1/2}\Delta \leq 8C\ell^{\frac{1}{2}}N^{-\frac{1}{2}} < 1$, so the sum does converge. We can interpret this as the remainder to the degree three approximation of

$$(4C)^{k+1}\ell^{-\frac{1}{2}}\Delta x^{-1}\left((1-x)^{-k} - 1\right)$$

at $x = 8C\ell^{1/2}\Delta$, which will give us the desired error bound. The same method works in the $k = 2\ell + 1$ case, which finishes the proof of Theorem 2.3.

3. Proof of Theorem 1.1

3.1. **Hyperbolicity of $J^{d,n}(X)$.** For each degree $d \geq 1$, the Jensen polynomials $J^{d,n}(X)$ are hyperbolic for sufficiently large n. This fact was established in [8] as a consequence of a newly established connection to Hermite polynomials.

The proof of Theorem 1 of [8] shows that these renormalized Jensen polynomials are modeled by *Hermite polynomials* $H_d(X)$, which we define (in a non-standard way) as the orthogonal polynomials for the measure $\mu(X) = e^{-X^2/4}$. Explicitly, the $H_d(X)$ are given by the generating function

$$\sum_{d=0}^{\infty} H_d(X)\cdot\frac{t^d}{d!} = e^{-t^2+Xt} = 1 + Xt + (X^2-2)\cdot\frac{t^2}{2!} + (X^3 - 6X)\cdot\frac{t^3}{3!} + \ldots, \tag{3.1}$$

or in closed form by $H_d(X) := \sum_{k=0}^{\lfloor d/2 \rfloor}\frac{(-1)^k d!}{k!(d-2k)!}\cdot X^{d-2k}$. Since the $H_d(X)$ are well-known to be hyperbolic, the following theorem is particularly satisfying.

Theorem 3.1. (Theorem 3 of [8]) *If $d \geq 1$, then*

$$\lim_{n\to+\infty}\widehat{J}^{d,n}(X) = H_d(X).$$

In particular, $J^{d,n}(X)$ is hyperbolic for all but possibly finitely many n.

3.2. **Proof of Theorem 1.1.** To prove Theorem 1.1, we need the following theorem of Turán.

Theorem 3.2. *Suppose that $G(z) \in \mathbb{R}[z]$ has degree n. If real numbers c_j are defined by*

$$G(z) = \sum_{j=0}^{n} c_j H_j(z),$$

and

$$\sum_{j=0}^{n-2} 2^j j! c_j^2 < 2^n (n-1)! c_n^2,$$

then the roots of $G(z)$ are real and simple.

Although the Hermite polynomials we have defined are different than those that typically appear in Turán's theorem, the result turns out to be the same. In order to apply this theorem, we first write $\widehat{J}^{d,n}(X)$ in the Hermite basis as

$$\widehat{J}^{d,n}(X) = \sum_{j=0}^{d} c_{d,n,j} H_{d-j}(X).$$

Using that

$$X^m = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{\ell!(m-2\ell)!} H_{m-2\ell}(X),$$

we can write the Hermite coefficients in terms of the $A_{d,k}(n)$ as

$$c_{d,n,j} = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(d-j+2i)!}{i!(d-j)!} A_{d,j-2i}(n). \tag{3.2}$$

In particular, $c_{d,n,0} = 1$ and $c_{d,n,1} = c_{d,n,2} = 0$, so we ultimately can rewrite the Turán condition in this case as

$$\sum_{j=3}^{d} 2^{-j} \frac{(d-j)!}{(d-1)!} c_{d,n,j}^2 < 1. \tag{3.3}$$

To prove the theorem, we only need to use (3.3) and find the asymptotics of $c_{d,n,j}$. We again have to consider cases. When $j = 2\ell$, we can use (3.2), the asymptotics for $A_{d,2i}$ given in Theorem 2.3, and the fact that $A_{d,0} = 1$ and $A_{d,2} = -d(d-1)$ to obtain

$$\begin{aligned} c_{d,n,j} &= \sum_{i=0}^{\ell} \frac{(d-2i)!}{(\ell-i)!(d-j)!} A_{d,2i}(n) \\ &= \frac{d!}{\ell!(d-j)!} \left[\sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^i + \sum_{i=2}^{\ell} \binom{\ell}{i} (-1)^i P(i) \Delta^2 + O\left(\sum_{i=2}^{\ell} \binom{\ell}{i} i^6 (4C)^{2i} \Delta^4 \right) \right], \end{aligned} \tag{3.4}$$

where

$$P(i) := i(i-1)\left(-\frac{2}{3}(3i+2) + 2iG_3 - \frac{i-2}{2} G_3^2 - G_4 \right).$$

It is clear that the first sum in brackets in (3.4) is 0 for $\ell \geq 1$. Similarly, since $P(i)$ is a polynomial of degree 3, the second sum is also 0 when $\ell \geq 4$. As a result, we can obtain

$$c_{d,n,j} \ll \frac{d!}{(d-j)!\ell!} \sum_{i=2}^{\ell} \binom{\ell}{i} i^6 (4C)^{2i} \Delta^4 \ll \frac{d!}{(d-j)!\ell!} \ell^6 (16C^2+1)^{\ell} \Delta^4.$$

We can find the same bound in the case that $\ell = 2$ and $\ell = 3$ by just bounding the main terms that we obtain. By substituting this in for the even j terms in (3.3) and using that $(2\ell)!/(\ell!)^2 \sim \sqrt{\pi} \ell^{-\frac{1}{2}} 2^{2\ell}$, one finds that the sum over the even terms can be bounded by $d^{\frac{25}{2}}(32C^2+3)^d \Delta^8$. A similar bound over the odd terms holds as well. Using that $\Delta^8 \sim N^{-4}$, the result follows.

4. Proof of Theorem 1.2

For convenience, we introduce some notation. For $0 < \delta < \pi/2$, we define

$$S(\theta, \delta) := \{z \in \mathbb{C}^{\times} \colon |\arg(z) - \theta| \leq \delta\}.$$

We define $C(\theta, \delta)$ to be the set of entire functions F with the property that there exist a sequence of complex numbers $(\beta_k)_{k \geq 1}$, an integer $q \geq 0$, and constants $c, \sigma \in \mathbb{C}$ such that

$$F(z) = c z^q e^{-\sigma z} \prod_{k=1}^{\infty} \Big(1 - \frac{z}{\beta_k}\Big),$$

where

$$\sum_{k=1}^{\infty} \frac{1}{|\beta_k|} < \infty, \qquad \beta_k, \sigma \in S(\theta, \delta).$$

Lemma 4.1. *Let $0 < \delta < \pi/2$. If $F \in C(\theta, \delta)$, then F is locally uniformly approximated by polynomials, each of whose zeros lie in $S(\theta, \delta)$, and conversely. Moreover, if $m \geq 1$ is an integer and the m-th derivative $F^{(m)}$ is not identically zero, then $F^{(m)} \in C(\theta, \delta)$.*

Proof. The first claim is proved in [9, Chapter VIII]. For the second claim, suppose that $F \in C(\theta, \delta)$ is non-constant. By the first claim, there exists a sequence of nonzero polynomials (g_n) which locally uniformly approximate F, and each zero of g_n lies in $S(\theta, \delta)$. By the Gauss-Lucas theorem, the zeros of g_n' belong to the convex hull of the set of zeros of g_n; thus each zero of g_n' lies in $S(\theta, \delta)$. Since the sequence (g_n') locally uniformly approximates F', it follows by the first claim that $F' \in C(\theta, \delta)$. For higher derivatives, we proceed by induction. □

Lemma 4.2. *If $\psi^{(m)} \in C(\pi, \delta)$, then $J^{d,m}(X)$ is hyperbolic for $d \leq |\sin(\delta)|^{-2}$.*

Proof. Since $\gamma(n)$ is positive for all $n \geq 0$ and the Taylor coefficients of $\psi^{(m)}$ are merely shifts of $8\gamma(n)$, the Taylor coefficients of $\psi^{(m)}$ are positive. Hence the lemma follows immediately from [3, Theorem 3.6] with $\varphi = \psi^{(m)}$. □

Proof of Theorem 1.2. We follow [3]. Let $m \geq 0$ be an integer. Suppose that $\mathrm{RH}_m(T)$ holds for some $T > \frac{1}{2}$. Then the zeros of $\psi^{(m)}$ in the rectangle $\{z \in \mathbb{C} \colon |\mathrm{Re}(z)| \leq T,\ |\mathrm{Im}(z)| < \frac{1}{2}\}$ are real. Consequently, the zeros of $\psi^{(m)}$ must lie in $S(0, \arctan(\frac{1}{2T})) \cup S(\pi, \arctan(\frac{1}{2T}))$. Hence the zeros of $\psi^{(m)}$ must lie in $S(\pi, 2\arctan(\frac{1}{2T}))$, and $\psi^{(m)} \in C(\pi, 2\arctan(\frac{1}{2T}))$. We see from Lemma 4.2 that $J^{d,m}(X)$ is hyperbolic for

$$d \leq \lfloor |\sin(2\arctan(\tfrac{1}{2T}))|^{-2} \rfloor = \left\lfloor T^2 + \frac{1}{2} + \frac{1}{16T^2} \right\rfloor.$$

Thus if $d \leq \lfloor T \rfloor^2$, then $J^{d,m}(X)$ is hyperbolic. Since $C(\theta, \delta)$ is closed under differentiation per Lemma 4.1, we also have that $\psi^{(m+1)} \in C(\pi, 2\arctan(\frac{1}{2T}))$. Thus by Lemma 4.2 again, $J^{d,m+1}(X)$ is hyperbolic for $d \leq \lfloor T \rfloor^2$. The theorem now follows by induction. $\square$

References

[1] E. Bombieri, *New progress on the zeta function: from old conjectures to a major breakthrough*, Proc. Natl. Acad. Sci. USA **116** (2019), no. 23, 11085?11086.

[2] K. Broughan, *Equivalents of the Riemann Hypothesis, Volume 2,* Cambridge Univ. Press, Cambridge, 2017.

[3] M. Chasse, *Laguerre multiplier sequences and sector properties of entire functions*, Complex Var. Elliptic Equ. **58** (2013), no. 7, 875-885.

[4] G. Csordas, T. S. Norfolk, and R. S. Varga, *The Riemann hypothesis and the Turán inequalities*, Trans. Amer. Math. Soc. **296** (1986), 521-541.

[5] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth. *On the Lambert W function.* Adv. Comput. Math. 5, 329-359, 1996.

[6] D. K. Dimitrov and F. R. Lucas, *Higher order Turán inequalities for the Riemann ξ-function* Proc. Amer. Math. Soc. **139** (2010), 1013-1022.

[7] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.23 of 2019-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds.

[8] M. Griffin, K. Ono, L. Rolen, and D. Zagier, *Jensen polynomials for Riemann's zeta function and suitable arithmetic sequences*, Proc. Natl. Acad. Sci., USA **116**, no. 23 (2019), 11103-11110.

[9] Levin, B. Ja. *Distribution of Zeros of Entire Functions*, *Trans. Math. Mono. Vol. 5*, Amer. Math. Soc., Providence, RI (1964); revised ed. 1980.

[10] G. Pólya, *Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen*, Kgl. Danske Vid. Sel. Math.-Fys. Medd. **7** (1927), 3-33.

[11] D. Platt, *Isolating some non-trivial zeros of zeta*, Math. Comp. **86** (2017), no. 307, 2449-2467.

Department of Mathematics, 275 TMCB, Brigham Young University, Provo, UT 84602
Email address: mjgriffin@math.byu.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22904
Email address: ko5wk@virginia.edu

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: larry.rolen@vanderbilt.edu

Department of Mathematics, University of Florida, Gainesville, FL 32601
Email address: jthorner@ufl.edu

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: zachary.d.tripp@vanderbilt.edu

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: ian.c.wagner@vanderbilt.edu