DIVISORS OF MODULAR PARAMETRIZATIONS OF ELLIPTIC CURVES

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ABSTRACT. The modularity theorem implies that for every elliptic curve E/\mathbb{Q} there exist rational maps from the modular curve $X_0(N)$ to E, where N is the conductor of E. These maps may be expressed in terms of pairs of modular functions X(z)and Y(z) where X(z) and Y(z) satisfy the Weierstrass equation for E as well as a certain differential equation. Using these two relations, a recursive algorithm can be used to calculate the q - expansions of these parametrizations at any cusp. Using these functions, we determine the divisor of the parametrization and the preimage of rational points on E. We give a sufficient condition for when these preimages correspond to CM points on $X_0(N)$. We also examine a connection between the algebras generated by these functions for related elliptic curves, and describe sufficient conditions to determine congruences in the q-expansions of these objects.

1. INTRODUCTION AND STATEMENT OF RESULTS

The modularity theorem [2, 12] guarantees that for every elliptic curve E of conductor N there exists a weight 2 newform f_E of level N with Fourier coefficients in \mathbb{Z} . The Eichler integral of f_E (see (3)) and the Weierstrass \wp -function together give a rational map from the modular curve $X_0(N)$ to the coordinates of some model of E. This parametrization has singularities wherever the value of the Eichler integral is in the period lattice. Kodgis [6] showed computationally that many of the zeros of the Eichler integral occur at CM points. Peluse [8] later proved several general cases confirming many of these conjectured zeros using the theory of Hecke operators and Atkin–Lehner involutions.

In [1], the authors use the modular parametrization of an elliptic curve to give a harmonic Maass form of weight 3/2 whose Fourier coefficients encode the vanishing of central *L*-values and *L*-derivatives of quadratic twists of the curve. The Birch and Swinerton-Dyer conjecture asserts that the order of vanishing of the central *L*-value of an elliptic curve is the rank of the curve. Kolyvagin [7] confirmed this conjecture if the order of vanishing is less than 2. Unfortunately, the result of [1] is only fully constructive if the modular parametrization is holomorphic on the upper half plane. Otherwise we must remove the singularities, a task which is difficult without knowledge of their locations.

For a modular function F for some subgroup Γ of $SL_2(\mathbb{Z})$, we consider the *modular* polynomial of F

(1)
$$\Phi_F(x) := \prod_{\gamma \in \Gamma \setminus SL_2(\mathbb{Z})} \left(x - F(\gamma z) \right) = \sum A_i(z) x^i.$$

One of our goals is to calculate the minimal divisor of (1) for F which are rational in terms of the coordinates functions (X(z), Y(z)) of a given modular parametrization of E, chosen so as to have poles at the divisor of the parametrization. We may calculate

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the divisor by calculating the divisor of the coefficient functions $A_i(z)$. In order to calculate the product in (1) we need the expansion of F at each of the cusps of Γ . Algorithms for calculating the coefficients of X(z) and Y(z) at the cusp infinity are described by Cremona [3], and we include a variation of that method that allows for the computation of coefficients at any cusp.

Example 1.1. For the elliptic curve

(11a1)
$$E: y^2 + y = x^3 - x^2 - 10x - 20$$

one can calculate that E has (5,5) and (5,-6) as points of order 5. If we set $F(z) = (X(z) - 5)^{-1}$, then F(z) has zeros only when z is an element of the complex lattice associated to E, and poles only when z is mapped to one of these 5-torsion points. Computing the divisor of $\Phi_F(X)$, we find that

$$X(z) = 5 \implies (j(z) + 24729001)(j(z) + 32768) = 0.$$

If $z = \frac{1+\sqrt{-11}}{2}$, then j(z) = -32768. Since j(z) is invariant under the action of $SL_2(\mathbb{Z})$ while F is only $\Gamma_0(11)$ invariant, we look at the $\Gamma_0(11) \setminus SL_2(\mathbb{Z})$ orbit of z to find

$$z_0 = \frac{-11 + \sqrt{-11}}{55} \implies (X(z_0), Y(z_0)) = (5, 5).$$

Thus the point z_0 is a preimage of the rational point (5,5), and is a CM point on $X_0(11)$.

The points of $X_0(N)$ are in correspondence with pairs (e, c) where e is an elliptic curve and $c \subset e$ is a cyclic subgroup of order N (See Appendix C.13 of [10]). Using this description, we give a sufficient condition for when a point \mathcal{P} lying above a rational point P on E is a CM point. The proof is given in section 3.

Theorem 1.2. Fix an elliptic curve E/\mathbb{Q} of conductor N and P a point on E. Let \mathcal{P} a point on $X_0(N)$ that maps to P under some modular parametrization, and which is in correspondence to the pair (e, c) where e is an elliptic curve over a number field K. For each $m \parallel N$, either e admits an m-isogeny defined over K or e has CM by an order of discriminant D where $0 \leq -D \leq 4m$ and D is a square (mod 4m).

In section 4 we consider the question, given an elliptic curve E, when are the coefficients of these parametrizations contained in some prime ideal \mathfrak{p} of a number ring \mathcal{O} ? One sufficient condition we give is that the elliptic curves are isogenous, and have congruent coefficients mod p for some prime p lying below \mathfrak{p} . Another sufficient condition we provide is a bound similar to Sturm's bound that implies that every coefficient of the parametrizations are in \mathfrak{p} after a certain finite number of coefficients are.

2. Elliptic Curves

Given an elliptic curve E, we denote the periods of E by ω_1, ω_2 , and the period lattice they generate by Λ_E . The Weierstrass \wp function is defined in terms of Λ_E and a complex variable z as follows:

$$\wp(z, \Lambda_E) := \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda_E \\ \lambda \neq 0}} \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}.$$

The \wp -function $\wp(z, \Lambda_E)$ is even as a function of z, and its defining series is absolutely convergent and doubly periodic with periods ω_1, ω_2 . The functions $\wp(z, \Lambda_E)$ and $\wp'(z, \Lambda_E)$ satisfy the relation

(2)
$$\wp'(z,\Lambda_E)^2 = 4\wp(z,\Lambda_E)^3 - g_2\wp(z,\Lambda_E) - g_3$$

where

$$g_2 = g_2(\Lambda_E) = 60 \sum_{\substack{\lambda \in \Lambda_E \\ \lambda \neq 0}} (\lambda)^{-4}$$

and

$$g_3 = g_3(\Lambda_3) = 140 \sum_{\substack{\lambda \in \Lambda_E \\ \lambda \neq 0}} (\lambda)^{-6}.$$

Also associated E is the canonical differential

$$\omega = m f_E(z) dz,$$

where m is the Manin constant and f_E is the weight two cusp form associated to E. The Eichler integral is then defined as

(3)
$$\varepsilon(z) = \int_{z}^{i\infty} \omega = \int_{z}^{i\infty} m f_E(\tau) d\tau.$$

The function $\varepsilon(z)$ is not modular, but if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ acts as usual on the upper-half plane, then

$$\frac{d}{dz}(\varepsilon(\gamma z) - \varepsilon(z)) = \frac{d}{dz}2\pi i \int_{\gamma z}^{z} mf_{E}(\tau)d\tau$$
$$= 2\pi i m \left(f_{E}(z) - (cz+d)^{2}f_{E}(z)(cz+d)^{-2}\right) = 0$$

where the second to last equality follows from the fundamental theorem of calculus and the modularity of f_E . So $\varepsilon(z)$ is *almost* modular, in that the difference $\varepsilon(\gamma z) - \varepsilon(z)$ depends only on γ , and not on z. Denote this difference by

$$C(\gamma) := \varepsilon(\gamma z) - \varepsilon(z).$$

One readily verifies that $C : \Gamma_0(N) \to m\Lambda_E$ is a group homomorphism. Eichler and Shimura [4,9] showed that when the Manin constant is 1, that C is actually an isomorphism.

For any $\lambda \in \mathbb{C}$ such that $\lambda \in \text{End}(E)$, we have that $\lambda \Lambda_E \subseteq \Lambda_E$. So it is possible to define

$$\wp_{\lambda}(z, \Lambda_E) := \lambda^2 \wp(\lambda z, \Lambda_E) = \wp(z, \frac{1}{\lambda} \Lambda_E),$$

where the extra factor λ^2 normalizes \wp_{λ} to have a leading coefficient of q^{-2} in its Fourier expansion. Similarly,

$$\wp_{\lambda}'(z,\Lambda_E) := \lambda^3 \wp'(\lambda z,\Lambda_E) = \wp'(z,\frac{1}{\lambda}\Lambda_E).$$

With this notation we define

$$X_{\lambda}(z) = m^2 \wp_{\lambda}(\varepsilon(z), \Lambda_E) - \frac{a_1^2 + 4a_2}{12},$$
$$Y_{\lambda}(z) = \frac{m^3}{2} \wp_{\lambda}'(\varepsilon(z), \Lambda_E) - \frac{a_1 m^2}{2} \wp_{\lambda}(\varepsilon(z), \Lambda_E) + \frac{a_1^3 + 4a_1 a_2 - 12a_3}{24}$$

for E given in general Weierstrass form with the convention that if the subscript λ is omitted we take $\lambda = 1$. Note that if E is given in Wierstrass short form then

$$X_{\lambda}(z) := m^2 \wp_{\lambda}(\varepsilon(z), \Lambda_E) \quad Y_{\lambda}(z) := \frac{m^3}{2} \wp_{\lambda}'(\varepsilon(z), \Lambda_E).$$

By construction $X_{\lambda}(z), Y_{\lambda}(z)$ satisfy the Wierstrass equation for the elliptic curve. Importantly, $X_{\lambda}(z)$ and $Y_{\lambda}(z)$ are modular over $\Gamma_0(N)$ since

$$\wp_{\lambda}(\varepsilon(\gamma z), \Lambda_{E}) = \wp_{\lambda}(\varepsilon(z) + C(\gamma), \Lambda_{E}) = \wp_{\lambda}(\varepsilon(z), \Lambda_{E})$$

where the final equality holds because $\lambda C(\gamma) \in \Lambda_E$. A similar calculation holds for $Y_{\lambda}(z)$ as well as the parametrizations for the general form.

3. EXPANSIONS AT OTHER CUSPS

The first step in computing the coefficient functions A_i in (1) is to compute the q-expansions of each of the factors $(x - F(\gamma z))$ for x a formal variable and $\gamma \in SL_2(\mathbb{Z})$. Since we are interested specifically in F that are rational functions of $X_{\lambda}(z)$ and $Y_{\lambda}(z)$ it suffices to calculate the q-expansions for $X(\gamma z)$ and $Y(\gamma z)$. These coefficients are determined by two relations,

(4)
$$qX' = (2Y + a_1X + a_3)f_E$$

known as the invariant differential of E (see section III of [10]), and the elliptic curve relation

(5)
$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6.$$

A recursive algorithm was given by Cremona [3] using these two relations to calculate the expansions of X(z) and Y(z). Acting on (3) and (4) by a matrix $\gamma \in SL_2(\mathbb{Z})$ gives relations that allow us to recursively calculate the coefficients of modular parametrizations around cusps other than infinity. There are, however, a few complications we examine below.

If we let $q_N(z) = e^{\frac{2\pi i}{N}z}$, we can write the expansions of the modular parametrizations at a cusp ρ with width w as $X_{\lambda}(\gamma z) = \sum_{n=-2}^{\infty} b_n q_w^n$ and $Y_{\lambda}(\gamma z) = \sum_{n=-3}^{\infty} d_n q_w^n$. Note that b_i, d_i might be zero for i = -3, -2, -1 if neither X nor Y have poles at ρ . By examining the first few terms if the Laurent series of \wp_{λ} and \wp'_{λ} and evaluating them at $\varepsilon(\gamma z)$ we can calculate b_{-2} and d_{-3} . So our inductive set up will be to assume that we know the b_i coefficients for $-2 \leq i \leq n-1$ and the d_j coefficients for $-3 \leq j \leq n-2$ and use this information to calculate b_n and d_{n-1} . Letting c_n denote the coefficient of q_w^n of $f_E(\gamma z)$, relation (3) gives us that

$$\frac{1}{w}\sum_{n=-2}^{\infty}nb_nq_w^n = \left(2\sum_{n=-3}^{\infty}d_nq_w^n + a_1\sum_{n=-2}^{\infty}b_nq_w^n + a_3\right)\sum_{n=1}^{\infty}c_nq_w^n.$$

Comparing the coefficients of q_w^n gives us one linear relation between b_n and d_{n-1}

$$nb_n = 2w\sum_{k=-3}^{n-1} c_{n-k}d_k + a_1w\sum_{k=-2}^{n-1} c_{n-k}b_k + a_3wc_n.$$

Comparing the q_w^{n-4} term in (4) gives us

$$\sum_{k=-3}^{n-1} d_{n-4-k}d_k + a_1 \sum_{k=-3}^{n-4} b_{n-4-k}d_k + a_3d_{n-4} = \sum_{k=-2}^n \sum_{j=-2}^{n-2-k} b_{n-4-k-j}b_jb_k + a_2 \sum_{k=-2}^{n-2} b_{n-4-j}b_j + a_4b_{n-4} + a_6^*$$

where a_6^* indicates that this term is present only if n - 4 = 0. This gives a second linear relation between d_{n-1} and b_n , which allows us to solve for d_{n-1} and b_n uniquely whenever the determinant of the system is not 0, i.e. when $-2nd_{-3}^2 + 6wc_1b_{-2}^2 \neq 0$. Supposing that $X_{\lambda}(z)$ has a pole at ρ , (so that neither d_{-3} nor b_{-2} are 0), then

$$-2n(d_{-3})^2 + 6wc_1(b_{-2})^2 = 0 \implies n = \frac{3wc_1(b_{-2})^2}{(d_{-3})^2}.$$

So this recursive process will not fail if we can find the first $\frac{3wc_1(b_{-2})^2}{(d_{-3}^2)}$ nontrivial terms of X(z) and Y(z) via the Laurent series expansions of \wp_{λ} and \wp'_{λ} . Note that when $\rho = \infty$, we have that $w = c_1 = b_{-2} = d_{-3} = 1$ so that Cremona's algorithm doesn't fail with simply 3 known terms of the Laurent expansion of $\wp_{\lambda}(\varepsilon(z))$.

However, if there are no poles at ρ , then $d_{-3} = b_{-2} = 0$, and the determinant will be 0 for all n. So when calculating the q_w -expansions around cusps without poles, we need to compare other powers of q_w to get information about such systems. Fortunately, we can simply compare powers of q_w^n in (3) and (4) to get that a system with determinant $n(2d_0 + a_1b_0 + a_3)$.

Interestingly, this determinant is zero when $2d_0 + a_1b_0 + a_3 = 0$, i.e when the constant terms of the expansions give a point of order 2 on E. This is seen most easily by looking at (3), and observing that $2d_0 + a_1b_0 + a_3 = 0$ corresponds to a vertical tangent line on E. However, this is easily rectified. We first take $2d_0 + a_1b_0 + a_3 = 0$ as a hypothesis and compare powers of q_w^n in (3) and powers of q_w^n in (4) exactly like the previous case. The main difference is that since $2d_0 + a_1b_0 + a_3 = 0$, this gives us a system in the unknowns b_n and d_{n-1} instead of in terms of b_n and d_n . So by examining 3 cases we can effectively calculate the q_w -expansions of the modular parametrizations X(z) and Y(z) around any cusp.

Now that we can efficiently calculate these q-expansions for $X(\gamma z), Y(\gamma z)$ it is possible to construct

$$\Phi_F(x) := \prod_{\gamma \in \Gamma_0(N) \setminus SL_2(\mathbb{Z})} \left(x - F(\gamma z) \right) = \sum A_i(z) x^i$$

where x is a formal variable and F is any rational function in $X_{\lambda}(z)$ and $Y_{\lambda}(z)$. Note that by construction, the coefficients of $\Phi_F(x)$ are modular functions which are invariant under the action of $SL_2(\mathbb{Z})$, and so are rational functions in Klein's *j*-function.

In practice, in order to compute the minimal divisor of $\Phi_F(x)$ it is computationally advantageous to compute each of the functions $F(\gamma z)$ and then use symmetric polynomials to calculate the necessary coefficient functions until we locate all the poles of F.

Example 3.1. Consider the elliptic curve

(26b1)
$$E: y^2 + xy + y = x^3 - x^2 - 3x + 3.$$

The point (1,0) lies on E and has (1,-2) as its inverse. Then looking at the function $F(z) = \frac{Y(z)+2}{X(z)-1}$, we see that F has a simple pole at the values $z \in \mathcal{H}$ that map (X(z), Y(z)) to (1,0). Note that the conductor of E is 26, and $[SL_2(\mathbb{Z}) : \Gamma_0(26)] = 42$. Calculating the trace of Φ_F (or the coefficient $A_{41}(z)$) we get

$$\sum_{\gamma \in \Gamma_0(26) \setminus \operatorname{SL}_2(\mathbb{Z})} F(\gamma z) = \frac{-j(z)^2 + 54688j(z) - 37627200}{j(z) - 54000}$$

Testing the 42 cosets of $\Gamma_0(26)$ in $\operatorname{SL}_2(\mathbb{Z})$ gives us that for $z_0 = \frac{-7+\sqrt{-3}}{52}$, $(X(z_0), Y(z_0)) = (1,0)$. Thus the preimage of the rational point (1,0) is a CM point on $X_0(26)$.

Using this theory we are able to give a condition for when a point P on an elliptic curve E is the image of a CM point \mathcal{P} on the modular curve and prove Theorem 1.2.

Proof. Suppose that m exactly divides N and let $\mathcal{P}_2 = (e_2, c_2)$ be the image of $\mathcal{P}_1 = (e_1, c_1)$ under the Atkin-Lehner involution $W_m = \begin{pmatrix} am & b \\ cN & dm \end{pmatrix}$ for integers a, b, c, d. The matrix W_m imposes a rational map from $X_0(N)$ to itself, so if e_1 is not isomorphic to e_2 , then W_m is a rational isogeny of the curves e_1 and e_2 . If e_1 is isomorphic to e_2 and we write the periods for e_1, e_2 as ω_{11}, ω_{12} and ω_{21}, ω_{22} respectively, then W_m takes $\tau_1 = \frac{\omega_{12}}{\omega_{11}}$ to $\tau_2 = \frac{\omega_{22}}{\omega_{21}}$. However, since $e_1 \cong e_2$, there must be a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $SL_2(\mathbb{Z})$ such that $W_m \tau_1 = \tau_2 = A\tau_1$. This gives a quadratic relation that τ_1 satisfies, namely

$$(am\tau_1 + b)(\gamma\tau_1 + \delta) = (\alpha\tau_1 + \beta)(cN\tau_1 + dm).$$

Expanding and collecting like terms gives

$$(am\gamma - c\alpha N)\tau_1^2 + (b\gamma + am\delta - cN\beta - dm\alpha)\tau_1 + b\delta - dm\beta = 0.$$

The discriminant of this quadratic is

$$D = (b\gamma + am\delta - cN\beta - dm\alpha)^2 - 4(am\gamma - c\alpha N)(b\delta - dm\beta)$$

= $b^2\gamma^2 + a^2m^2\delta^2 + c^2N^2\beta^2 + d^2m^2\alpha^2$
+ $2b\gamma am\delta - 2b\gamma cN\beta - 2b\gamma dm\alpha - 2am\delta cN\beta - 2adm^2\alpha\delta + 2cN\beta dm\alpha$
- $4(am\gamma b\delta - am^2d\beta\gamma - cNb\alpha\delta + c\alpha Ndm\beta).$

We collect like terms and use the fact that $det(W_m) = adm^2 - cNb = m$ to get

$$D = b^{2}\gamma^{2} + a^{2}m^{2}\delta^{2} + c^{2}N^{2}\beta^{2} + d^{2}m^{2}\alpha^{2}$$

- $2b\gamma am\delta + 2b\gamma cN\beta - 2b\gamma dm\alpha - 2am\delta cN\beta + 2adm^{2}\alpha\delta - 2cN\beta dm\alpha$
- $4(m\alpha\delta - m\beta\gamma).$

Factoring and using that $det(A) = \alpha \delta - \beta \gamma = 1$ gives that

$$D = (b\gamma - am\delta + cN\beta - dm\alpha)^2 - 4m.$$

Thus D is a square mod 4m. Since τ_1 is in the upper half plane, we must have that D < 0. However, since $(b\gamma - am\delta + cN\beta - dm\alpha)^2$ is non-negative, it follows that $-4m \leq D < 0$.

Example 3.2. We return to the curve

(26b1)
$$E: y^2 + xy + y = x^3 - x^2 - 3x + 3$$

of conductor 26 and index 42. Consider the points (1, -2) and (3, 2) with inverses (1, 0) and (3, -6) on E. Then the functions F and G given by

$$F(z) = \frac{Y(z) - 0}{X(z) - 1}, \quad G(z) = \frac{Y(z) + 6}{X(z) - 3}$$

have simple poles for z such that (X(z), Y(z)) = (1, -2) or (3, 2) respectively. We calculate specific coefficient functions of $\Phi_F = \sum A_i(z)x^i$ and $\Phi_G = B_i(z)x^i$ to determine the location of these poles in the upper half plane:

$$A_{41}(z) = \frac{-j(z)^2 + 288156 \cdot j(z) - 199626768}{j(z) - 287496},$$

$$B_{40}(z) = \frac{j(z)^3 - 3214 \cdot j(z)^2 + 2726620 \cdot j - 274323456}{j(z) - 1728}$$

Thus $\Phi_F(z)$ has poles only when j(z) = 287496, i.e when z is in the $SL_2(\mathbb{Z})$ orbit of $\sqrt{-4}$, and G(z) has poles only when j(z) = 1728 i.e when z is in the $SL_2(\mathbb{Z})$ orbit of $\sqrt{-1}$. Comparing the actions of the coset representatives of $\Gamma_0(26)$, we find that $z_0 := \frac{-5+\sqrt{-1}}{52}$ satisfies (X(z), Y(z)) = (1, -2), and $z_1 = \frac{5+\sqrt{-1}}{13}$ satisfies (X(z), Y(z)) = (3, 2).

Examining the action of the Atkin-Lehner involutions W_2 and W_{13} , we find that $F_2 = F(W_2 z)$, and $G_2 = G(W_2 z)$ have coefficient functions

$$A_{40}(z) = \frac{-j(z)^2 + 3235 \cdot j(z) - 2655936}{j(z) - 1728}, \qquad B_{41}(z) = \frac{-42 \cdot j(z) + 21954240}{j(z) - 287496}$$

while $F_{13} := F(W_{13}z)$ and $G_{13} := G(W_{13}z)$ have coefficient functions

$$A_{41}(z) = \frac{-j(z)^2 + 288156 \cdot j(z) - 199626768}{j(z) - 287496},$$

$$B_{40}(z) = \frac{j(z)^3 - 3214 \cdot j(z)^2 + 2726620 \cdot j - 274323456}{j(z) - 1728}.$$

Thus since W_2 exchanges the poles of F and G, Theorem 1.2 gives that the points z_0 , z_1 correspond to isogenous elliptic curves on $X_0(26)$. Additionally, since W_{13} fixes z_0 and z_1 , Theorem 1.2 also tells us they are both CM points on $X_0(26)$ whose orders have discriminants that must be squares mod 52. In fact, the minimal polynomial of z_0 is $104z^2 - 20z + 1$ which has discriminant $-16 \equiv 6^2 \mod 52$, and the minimal polynomial for z_1 is $13z^2 - 10z + 2$ which has discriminant $-4 \equiv 10^2 \mod 52$.

Example 3.3. Theorem 1.2 can also be visualized in the following way. Consider again the elliptic curve $E: y^2 + y = x^3 - x^2 - 10x - 20$ of conductor 11, and the fundamental domain F_{11} in figure 1 for the congruence subgroup $\Gamma_0(11)$.

This fundamental domain has been constructed by taking $\operatorname{SL}_2(\mathbb{Z})$ coset representatives of the form $\binom{0}{1} \binom{0}{j}$ for $-5 \leq j \leq 5$, with each j labeled in the corresponding hypertriangle. The associated newform of E is $f_E = q - 2q^2 - q^3 + 2q^4 \dots$ Taking complex values z on the boundary of F_{11} and calculating $\varepsilon(z) = \int_z^{i\infty} m f_E(\tau) d\tau$ gives the image in Figure 2. The resulting image tiles the plane in a parallelogram-type pattern, with the same periods as E. The points A, B and C have been labeled at 2/5, 3/5 and 4/5 times the real period of E respectively. They correspond to the points (5, -6), (5, 5) and (16, 60) on E respectively. The action of W_{11} interchanges the two cusps in Figure 2 (∞ located at the origin, and 0 located at the value .2538... on



FIGURE 1. fundamental domain F_{11} for $\Gamma_0(11)$

FIGURE 2. Eichler integral over the boundary of F_{11}

the real line which is 1/5 the real period of E). Up to translation by the real period, we see that W_{11} interchanges the points A and C but fixes point B. By Theorem 1.2 we conclude that the preimages of the points (5, -6) and (16, 60) on $X_0(11)$ give isogenous elliptic curves, while the preimage of (5, 5) on $X_0(11)$ must be a CM point as we saw in Example 1.1.

4. Congruences Between Generated Algebras

Consider the elliptic curves E_1 , E_2 given by

(14a1)
$$E_1: y^2 + xy + y = x^3 + 4x - 6,$$

(14a2)
$$E_2: y^2 + xy + y = x^3 - 36x - 70.$$

These curves have coefficients that are congruent mod 8 and interestingly, if we look at the q-expansions of the row reduced basis elements of $\mathbb{Q}[X(z), Y(z)]$, we notice a similar phenomenon.

Basis over $E_1, X = X_{E_1}(z), Y = Y_{E_1}(z)$	q-expansion							
1				1				
X(z) - 2	q^{-2}	$+q^{-1}$	+2q	$+2q^{2}$	$+3q^{3}$	$+\cdots$		
-Y(z) - 2X(z) - 2	q^{-3}	$+2q^{-1}$	+5q	$+4q^{2}$	$+2q^{3}$	$+\cdots$		
$X(z)^{2} + 2Y(z) - X(z) + 2$	q^{-4}	$-q^{-1}$	-2q	$+8q^{2}$	$+5q^{3}$	$+\cdots$		
$-Y(z)X(z) - 3X(z)^{2} + 2Y(z) + 3X(z) - 2$	q^{-5}		-2q	$-4q^{2}$	$+18q^{3}$	$+\cdots$		
$X(z)^{3} + 3X(z)Y(z) - 5Y(z) + 2X(z) - 6$	q^{-6}	$-2q^{-1}$	+4q	$-7q^{2}$	$-6q^{3}$	$+\cdots$		

Basis over $E_2, X = X_{E_2}(z), Y = Y_{E_2}(z)$			q-ex	pansion		
1				1		
X(z) - 2	q^{-2}	$+q^{-1}$	+2q	$10q^2$	$-5q^{3}$	$+\cdots$
-Y(z) - 2X(z) - 2	q^{-3}	$+2q^{-1}$	-3q	$-4q^{2}$	$+2q^{3}$	$+\cdots$
$X(z)^2 + 2Y(z) - X(z) - 14$	q^{-4}	$-q^{-1}$	+14q		$+29q^{3}$	$+\cdots$
$-Y(z)X(z) - 3X(z)^{2} + 2Y(z) + 3X(z) + 38$	q^{-5}		+6q	$-28q^{2}$	$-14q^{3}$	$+\cdots$
$ X(z)^{3} + 3X(z)Y(z) - 5Y(z) - 22X(z) - 6 $	q^{-6}	$-2q^{-1}$	-12q	$+25q^{2}$	$+138q^{3}$	$+\cdots$

The coefficients of the q-expansions are also congruent mod 8. This is not simply a consequence of the congruence of the equations of E_1 and E_2 . For example, the curves

(15a3)
$$E_3: y^2 + xy + y = x^3 + x^2 - 5x + 2,$$

(15a4)
$$E_4: y^2 + xy + y = x^3 + x^2 + 35x - 28$$

are congruent mod 10, but the q expansions of the X term of their optimal modular parametrizations are

$$X_{E_3}(z) = q^{-2} + q^{-1} + 1 + 2q + 3q^2 + q^3 + \dots - 6q^{11} + \dots ,$$

$$X_{E_4}(z) = q^{-2} + q^{-1} + 1 + 2q - 5q^2 + 9q^3 + \dots + 7q^{11} + \dots .$$

Comparing the q^2 terms shows that any congruence between these two parametrizations must divide 8, and comparing the q^{11} terms shows that any such congruence must divide 13. Thus we conclude that there are *no* nontrivial congruences between the parametrizations. So when do congruences in the elliptic curve equation give rise to congruences in the generated algebras?

If we assume that the two elliptic curves E_1 and E_2 given by

$$E_1 : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

$$E_2 : y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2 + \alpha_4 x + \alpha_6,$$

are isogenous, then their period lattices will intersect nontrivially in a lattice Λ_3 , corresponding to an elliptic curve E_3 with integral model

$$y^{2} + \beta_{1}xy + \beta_{3}y = x^{3} + \beta_{2}x^{2} + \beta_{4}x + \beta_{6}.$$

Thus the difference

$$g(z) := \wp(z, \Lambda_1) - \wp(z, \Lambda_2)$$

is an even, elliptic function with period lattice Λ_3 . If we let $\{r_i\}$ represent the complex numbers such that $\wp(r_i, \Lambda_3)$ is a zero of g(z) in a fundamental parallelogram of Λ_3 and let $\{t_j\}$ be the values in Λ_3 such that $\wp(t_j, \Lambda_3)$ is a pole of g(z) (repeated according to multiplicities) except possibly at the origin (even if the origin is a zero or pole of g), then the function

$$\frac{\prod_{i} \left(\wp(z, \Lambda_3) - \wp(r_i, \Lambda_3) \right)}{\prod_{i} \left(\wp(z, \Lambda_3) - \wp(t_j, \Lambda_3) \right)}$$

is monic, and has the same zeros and poles as g(z) except possibly at 0. However, a classical argument shows that the product must have the same zero or pole as g(z) at 0 as well (see [5] for example). Thus

(6)
$$g(z) = \wp(z, \Lambda_1) - \wp(z, \Lambda_2) = C \frac{\prod_i (\wp(z, \Lambda_3) - \wp(r_i, \Lambda_3))}{\prod_j (\wp(z, \Lambda_3) - \wp(t_j, \Lambda_3))}$$

for some constant C. Since

$$\wp(z,\Lambda_1) - \wp(z,\Lambda_2) = \frac{g_2(\Lambda_1) - g_2(\Lambda_2)}{20} z^2 + \frac{g_3(\Lambda_1) - g_3(\Lambda_2)}{28} z^4 + \cdots$$

we see that

$$C = C(\Lambda_1, \Lambda_2) = \begin{cases} \frac{g_2(\Lambda_1) - g_2(\Lambda_2)}{20} & \text{if } g_2(\Lambda_1) \neq g_2(\Lambda_2) \\ \frac{g_3(\Lambda_1) - g_3(\Lambda_2)}{28} & \text{if } g_2(\Lambda_1) = g_2(\Lambda_2). \end{cases}$$

With this notation we have the following.

Theorem 4.1. Suppose that E_1, E_2 are two isogenous elliptic curves over \mathbb{Q} . Also assume that the coordinates of the torsion points of order dividing N in $\overline{\mathbb{Q}}$ are algebraic integers. Then there is an explicit natural number $D(\Lambda_1, \Lambda_2)$ so that the q-expansion of $X_{E_1} - X_{E_2}$ is congruent to a constant mod $C(\Lambda_1, \Lambda_2)/D(\Lambda_1, \Lambda_2)$.

Proof. Evaluating equation (6) at $\varepsilon(z)$, and adding the appropriate constant to both sides of the equality gives

$$X_{E_1}(z) - X_{E_2}(z) = \wp(\varepsilon(z), \Lambda_1) + \frac{a_1^2 - 4a_2}{12} - \wp(\varepsilon(z), \Lambda_2) - \frac{\alpha_1^2 - 4\alpha_2}{12}$$
$$= C \frac{\prod_i (\wp(\varepsilon(z), \Lambda_3) - \wp(r_i, \Lambda_3))}{\prod_j (\wp(\varepsilon(z), \Lambda_3) - \wp(t_j, \Lambda_3))} + \frac{a_1^2 - \alpha_1^2 + 4\alpha_2 - 4a_2}{12}$$
$$= C \frac{\prod_i X_{E_3} - R_i}{\prod_j X_{E_3} - T_j} + \frac{a_1^2 - \alpha_1^2 + 4\alpha_2 - 4a_2}{12}$$

where $R_i = \wp(r_i, \Lambda_3) - \frac{\beta_1^2 - 4\beta_2}{12}$ and $T_j = \wp(t_j, \Lambda_3) - \frac{\beta_1^2 - 4\beta_2}{12}$. The final equality follows from In fact that $X_{E_3} = \wp(z, \Lambda_3) + \frac{\beta_1^2 - 4\beta_4}{12}$ so that the fraction cancels out of the X_{E_3} term and the R_i or T_j term.

The T_j 's are x-coordinates of torsion points of order dividing N because the poles of g(z) occur at lattice points of either Λ_1 or Λ_2 . By hypothesis, these coordinates are algebraic integers. Since the q-expansions of both X_{E_1} and X_{E_2} are both integers, we also have that each of $\wp(r_i, \Lambda_3)$ must be algebraic. So we define $D = D(\Lambda_1, \Lambda_2) = \prod_i D_i$ where D_i is the minimal natural number so that $D_i R_i$ is an algebraic integer. Thus

$$X_{E_1}(z) - X_{E_2}(z) = \frac{C}{D} \frac{\prod_i D_i X_{E_3} - D_i R_i}{\prod_j X_{E_3} - T_j}.$$

Since the formal product $(\prod_j X_{E_3} - T_j)^{-1}$ has algebraic integer coefficients, and since $D_i R_i$ is an algebraic integer for all *i*, the above shows that all but the constant term of the *q*-expansion of $X_{E_1}(z) - X_{E_2}(z)$ are congruent to zero mod C/D.

Example 4.2. Let's return to the curves E_1 , E_2 (Cremona labels 14a1 and 14a2) where we found a congruence mod 8 between the q-expansions for their modular parametrizations. The period lattices for E_1 , E_2 are given by the generators

 $(z_{11}, z_{12}) \approx (1.981341, .990670 + 1.325491i), \quad (z_{21}, z_{22}) \approx (.990670, 1.325491i),$

and so we see that $\Lambda_{E_1} \subseteq \Lambda_{E_2}$. So we can write $\wp(z, \Lambda_2)$ as a rational function in $\wp(z, \Lambda_1)$. A quick calculation shows that in fact,

$$\wp(z,\Lambda_1) - \wp(z,\Lambda_2) = \frac{8}{13/12 - \wp(z,\Lambda_1)}$$

Since $X_{E_1}(z) = \wp(\varepsilon(z), \Lambda_1) - 1/12$, we conclude that

$$X_{E_1}(z) - X_{E_2}(z) = \frac{8}{1 - X_{E_1}}$$

Since X_{E_1} has integer coefficients, this makes the congruence mod 8 between X_{E_1} and X_{E_2} now apparent.

Example 4.3. Using Theorem 4.1 we can now see why the curves

(15a3)
$$E_3: y^2 + xy + y = x^3 + x^2 - 5x + 2,$$

(15a4)
$$E_4: y^2 + xy + y = x^3 + x^2 + 35x - 28.$$

had only the trivial congruence mod 1 even though their expressions share a congruence mod 10. These curves are isogenous and $\Lambda_3 \subseteq \Lambda_4$, so we can write the difference $X_{E_4} - X_{E_3}$ as a rational function in terms of X_{E_3} . Since $g_2(\Lambda_{E_3})/20 = 241/240$ and $g_2(\Lambda_{E_4})/20 = -1679/240$, we see that C = (241 + 1679)/240 = 8. Also, we compute that

$$X_{E_4} - X_{E_3} = C \frac{-(X_{E_3} - \frac{3}{4})(X_{E_3} - \frac{3}{2})}{(X_{E_3} - 1)(X_{E_3})^2}.$$

So we see that D = 8 as well. Thus C/D = 1.

While Theorem 4.1 describes many congruent algebras, it does not describe all congruences that we noticed computationally on curves of conductor less than 100. For example, the curves

(96a3)
$$E_1: y^2 = x^3 + x^2 - 32x + 60$$

(48a5)
$$E_2: y^2 = x^3 + x^2 - 384x + 2772.$$

are not isogenous over \mathbb{Q} , so Theorem 4.1 doesn't tell us of any congruences between the two algebras. However, looking at the difference of the *q*-expansions of the modular parametrizations of the *x* coordinates of these two curves gives

$$-68q + 780q^3 - 5020q^5 + 24140q^7 - 96712q^9 + 340500q^{11} - 1086568q^{13} + O(q^{15}).$$

So we see that this form appears to be 0 mod 4. In fact, computationally we can confirm that a large number of coefficients are divisible by 4. We would like to be able to tell that all of the coefficients are congruent to 0 by looking at some finite number of terms in the q-expansion. To this end, we give a generalization of Sturm's bound that applies to meromorphic modular forms. The argument is essentially the same, but

we give a proof for completeness. For a modular form with q-expansion $f = \sum a_n q^n$ we denote

$$\operatorname{ord}_{\mathfrak{p}} f := \operatorname{ord}_{\infty}(f \mod \mathfrak{p}) = \min\{n : a_n \notin \mathfrak{p}\}$$

and observe that since \mathfrak{p} is a prime ideal, $\operatorname{ord}_{\mathfrak{p}}(fg) = \operatorname{ord}_{\mathfrak{p}}(f) + \operatorname{ord}_{\mathfrak{p}}(g)$. We also denote by $M_k^{!!}(\Gamma, \mathcal{O})$ the collection of meromorphic modular forms of weight k over Γ with coefficients in \mathcal{O} . Finally, let $f^{[\gamma]_k}$ denote $(cz+d)^{-k}f(\gamma z)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. With this notation we prove the following.

Lemma 4.4. Let \mathfrak{p} be a prime ideal in the ring of integers \mathcal{O} of a number field K. Further suppose that $f \in M_k^{ll}(\Gamma, \mathcal{O})$ and $|\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})| = m$. Finally, let Ω be the set of points on $X_0(N)$ where f has poles. Then

$$\operatorname{ord}_{\mathfrak{p}}(f) + \sum_{\tau \in \Omega} \operatorname{ord}_{\tau}(f) > \frac{km}{12}$$

implies that $f \equiv 0 \pmod{\mathfrak{p}}$.

Proof. We start with the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We first note that since f is meromorphic, ord_{τ} $f < \infty$ for all $\tau \in \Omega$. Also, since the coefficients of f are elements of \mathcal{O} , for each of the finite complex numbers $\tau_i \in \Omega \cap \Gamma \setminus \mathcal{H}$, we can pick relatively prime algebraic integers α_i , β_i so that $\beta_i j(z) - \alpha_i$ has a zero of order at least 1 at τ_i . So

$$g(z) := f(z) \prod_{i} (\beta_i j(z) - \alpha_i)^{-\operatorname{ord}_{\tau_i} f}$$

has poles only at infinity, and is modular over $SL_2(\mathbb{Z})$. Thus Sturm's theorem applies giving $g(z) \equiv 0 \mod \mathfrak{p}$ since

$$\operatorname{ord}_{\mathfrak{p}}(g) = \operatorname{ord}_{\mathfrak{p}}(f) - \sum_{\tau_i \in \Omega} \operatorname{ord}_{\tau_i}(f_i) \operatorname{ord}_{\mathfrak{p}}(\beta_i j + \alpha_i)$$
$$\geq \operatorname{ord}_{\mathfrak{p}}(f) + \sum_{\tau_i \in \Omega} \operatorname{ord}_{\tau_i}(f) > \frac{k}{12}.$$

The first inequality holds since α_i and β_i are relatively prime algebraic integers in \mathcal{O} , implies that each of the terms $(\beta_i j + \alpha_i)$ has order $0, -1 \mod \mathfrak{p}$ corresponding to $\beta_i \in \mathfrak{p}$ or not. Thus $g \equiv 0 \pmod{\mathfrak{p}}$ which implies that $f \equiv 0 \pmod{\mathfrak{p}}$. This concludes the proof in the case that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

If Γ is an arbitrary congruence subgroup, we first pick N so that $\Gamma(N) \subseteq \Gamma$ with m coset representatives γ_{ℓ} for $\Gamma(N)$ and we set $L = K(\zeta_N)$. Since $f \in M_k^{\parallel}(\Gamma(N), L)$ and $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the functions $f^{[\gamma_{\ell}]_k}$ are elements of $M_k^{\parallel}(\Gamma(N), L)$. Furthermore, the denominators of the fourier coefficients of $f^{[\gamma_{\ell}]_k}$ are bounded because each is a finite L-linear combination of some integral basis of a finite dimensional subspace of $M_k^{\parallel}(\Gamma(N), L)$. Note that in general $M_k^{\parallel}(\Gamma(N), L)$ is not finite dimensional; however, if we restrict ourselves to the subspace that has poles of the same order and at the same locations as those of f and $f^{[\gamma_{\ell}]_k}$, then this subspace is finite dimensional. Thus we can pick constants $A_{\ell} \in L^{\times}$ so that each of the functions $\mathrm{ord}_{\mathfrak{P}}(A_{\ell}f^{[\gamma_{\ell}]_k}) = 0$ for some prime ideal \mathfrak{P} lying over \mathfrak{p} . Letting γ_1 be the identity matrix, the function

$$G(z) := f(z) \prod_{\ell=2}^{m} A_{\ell} f^{[\gamma_{\ell}]_{k}}$$

is a meromorphic modular form of weight km over $SL_2(\mathbb{Z})$ with coefficients in \mathcal{O}_L . Then

$$\operatorname{ord}_{\mathfrak{p}}(G) \ge \operatorname{ord}_{\mathfrak{p}}(G) \ge \operatorname{ord}_{\mathfrak{p}}(f) + \sum_{\tau \in \Omega} \operatorname{ord}_{\tau}(f) > \frac{km}{12},$$

where the first equality follows because $\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p}$. We conclude that $G \equiv 0 \pmod{\mathfrak{P}}$ from the $\mathrm{SL}_2(\mathbb{Z})$ case. Since each of the functions $A_{\gamma_\ell} f^{[\gamma_\ell]_k}$ were chosen such that $\mathrm{ord}_{\mathfrak{P}}(A_\ell f^{[\gamma_\ell]_k}) = 0$, this gives $G \equiv 0 \pmod{\mathfrak{p}}$ and so $f \equiv 0 \pmod{\mathfrak{p}}$. See theorem 9.18 in [11] to compare the above to the proof of Sturm's theorem for elements of $M_k(\Gamma, \mathcal{O})$.

Corollary 4.5. If X_{E_1} and X_{E_2} are modular parametrizations for the x coordiantes of elliptic curves E_1 and E_2 of conductor N_1 and N_2 with modular degrees d_1 and d_2 respectively, then if $\operatorname{ord}_p(X_{E_1} - X_{E_2}) > 2(d_1 + d_2)$, then $X_{E_1} \equiv X_{E_2} \mod p$.

Proof. The number of poles of X_{E_i} is at most $2d_i$ counting multiplicities. Thus the corollary follows immediately from Theorem 4.4 applied to the difference $X_{E_1} - X_{E_2}$ which is modular over $\Gamma_0(\operatorname{lcm}(N_1, N_2))$ since

$$\operatorname{ord}_p(X_{E_1} - X_{E_2}) + \sum_{\tau \in \omega} \operatorname{ord}_\tau(X_{E_1} - X_{E_2}) > 2(d_1 + d_2) - 2(d_1 + d_2) = 0 = \frac{km}{12}.$$

Note that this bound is independent of both N_1 and N_2 since the weight k of the modular parametrizations is zero. We obtain a better estimate if we know a priori the locations of the poles of X_{E_i} and if they cancel in the difference $X_{E_1} - X_{E_2}$.

Corollary 4.4 gives us an easy way for determining if two related parametrizations are congruent mod \mathfrak{p} . Returning to our earlier example with the curves

(96a3)
$$E_1: y^2 = x^3 + x^2 - 32x + 60,$$

(48a5) $E_2: y^2 = x^3 + x^2 - 384x + 2772,$

since the modular degree of both E_1 and E_2 is 8, computing 2(8+8) = 32 coefficients of the difference function and observing that they are congruent to 0 mod 4 is sufficient to prove that all of the coefficients are congruent mod 4.

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