Divisibility Properties of Coefficients of Weight 0 Weakly Holomorphic Modular Forms

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In 1949, Lehner showed that certain coefficients of the modular invariant $j(\tau)$ are divisible by high powers of small primes. Kolberg refined Lehner's results and proved congruences for these coefficients modulo high powers of these primes. We extend Lehner's and Kolberg's work to the elements of a canonical basis for the space of weight 0 weakly holomorphic modular forms.

1. Introduction

A weakly holomorphic modular form of weight k for $SL_2(\mathbb{Z})$ is a holomorphic function f defined on the upper half-plane that satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

and is meromorphic at the cusp ∞. The function f will have a q-expansion of the form $f(\tau) = \sum_{n\geq n_0} a(n)q^n$, where $q = e^{2\pi i\tau}$ and $n_0 \in \mathbb{Z}$. If $n_0 \geq 0$, then f is a holomorphic modular form.

The modular function

$$j(\tau) = \frac{\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = q^{-1} + 744 + \sum_{n\geq 1} c(n)q^n$$

is a weakly holomorphic modular form of weight 0. The coefficients $c(n)$ of $j(\tau)$ are integers, and they play many important roles in mathematics. For instance, they appear as the degrees of a special graded representation of the Monster group.

In 1949 Lehner [7] showed that

$$c(2^a3^b5^c7^dn) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d}.$$

He also showed that similar results hold for the coefficients of other weight 0 weakly holomorphic modular forms, but only if the order of the pole at infinity is less than the prime under consideration.

Lehner's results for the function $j(\tau)$ were later refined by Kolberg [4,5] and Aas [1], who proved more specific congruences modulo large powers of the primes $p = 2$, 3, 5, and 7. These congruences give a lower bound for the power of the prime p dividing the coefficients of $j(\tau)$, and in many cases they give the exact power of p

dividing the coefficients. For the special case $c(p^a)$, their work shows that Lehner's congruences are the best possible.

In this paper we extend the results of Lehner, Kolberg, and Aas for the j-function to every function in a canonical basis for the space of weight 0 weakly holomorphic modular forms, removing Lehner's restriction on the order of the pole at ∞. In defining this basis we follow the notation of Duke and Jenkins [3]. For all integers $m \geq 0$, let $f_{0,m}(\tau)$ be the unique weight 0 weakly holomorphic modular form with Fourier expansion

$$f_{0,m}(\tau) = q^{-m} + \sum_{n=1}^{\infty} a_0(m,n)q^n.$$

Each of the $f_{0,m}(\tau)$ can be expressed as a monic polynomial in $j(\tau)$ with integer coefficients. For instance,

$$\begin{aligned} f_{0,0}(\tau) &= 1, \\ f_{0,1}(\tau) &= j(\tau) - 744, \\ f_{0,2}(\tau) &= j^2(\tau) - 1488\, j(\tau) + 159768. \end{aligned}$$

Note that all the $a_0(m,n)$ are integers. Additionally, for $m > 1$,

$$f_{0,m}(\tau) = f_{0,1}(\tau)|T_0(m),$$

where $T_0(m)$ is the normalized weight 0 Hecke operator of index m, which is m times the usual Hecke operator. These basis elements are the same as Zagier's J_m functions [9].

The main theorem of this paper gives congruences for the coefficients $a_0(m,n)$ of all the $f_{0,m}(\tau)$ that are similar to the congruences Kolberg and Aas showed for the coefficients $c(n)$ of the j-function.

A similar basis $\{f_{k,m}(\tau)\}$ can be defined for weakly holomorphic modular forms of any even weight k. Let

$$f_{k,m}(\tau) = q^{-m} + \sum_{n=\ell+1}^{\infty} a_k(m,n)q^n$$

for all integers $m \geq -\ell$, where $\ell = \left\lfloor \frac{k}{12} \right\rfloor - 1$ if $k \equiv 2 \pmod{12}$, and $\ell = \left\lfloor \frac{k}{12} \right\rfloor$ otherwise. Duke and Jenkins [3] showed that a Zagier-type duality exists between bases of this type, so that $a_k(m,n) = -a_{2-k}(n,m)$. In particular, this gives us $a_2(m,n) = -a_0(n,m)$. By use of this duality, our results for weight 0 modular forms are easily adapted to weight 2.

In section 2 of this paper we state the specific congruences given by Kolberg and Aas for the coefficients of $j(\tau)$, followed by the statement of our main theorem. The proof of the theorem will be given in section 3.

2. Statement of Results

We begin by stating Kolberg's and Aas's congruences for the coefficients $c(n)$ of $j(\tau)$. In all of the following, we assume that $(n,p)=1$.

For $p=2$:

$$c(2^a n) \equiv -2^{3a+8}3^{a-1}\sigma_7(n) \pmod{2^{3a+13}} \quad \text{if } a>0, \tag{2.1}$$
$$c(n) \equiv 20\sigma_7(n) \pmod{2^7} \quad \text{if } n\equiv 1 \pmod 8, \tag{2.2}$$
$$c(n) \equiv \frac{1}{2}\sigma(n) \pmod{2^3} \quad \text{if } n\equiv 3 \pmod 8, \tag{2.3}$$
$$c(n) \equiv -12\sigma_7(n) \pmod{2^8} \quad \text{if } n\equiv 5 \pmod 8. \tag{2.4}$$

For $p=3$:

$$c(3^a n) \equiv \mp 3^{2a+3}10^{a-1}\frac{\sigma(n)}{n} \pmod{3^{2a+6}} \quad \text{if } a>0,\ n\equiv \pm 1 \pmod 3, \tag{2.5}$$
$$c(n) \equiv 2\cdot 3^3\frac{\sigma(n)}{n} \pmod{3^7} \quad \text{if } n\equiv 1 \pmod 3. \tag{2.6}$$

For $p=5$:

$$c(5^a n) \equiv -5^{a+1}3^{a-1}n\sigma(n) \pmod{5^{a+2}} \quad \text{if } a>0, \tag{2.7}$$
$$c(n) \equiv 10n\sigma(n) \pmod{5^2} \quad \text{if } \left(\frac{n}{5}\right)=-1. \tag{2.8}$$

For $p=7$:

$$c(7^a n) \equiv 7^a5^{a-1}n\sigma_3(n) \pmod{7^{a+1}} \quad \text{if } a>0, \tag{2.9}$$
$$c(n) \equiv 2n\sigma_3(n) \pmod 7 \quad \text{if } \left(\frac{n}{7}\right)=1. \tag{2.10}$$

Equations (2.7) and (2.9) above are due to Aas [1]; the rest are due to Kolberg [4, 5,6]. In equations (2.8) and (2.10) above, $\left(\frac{n}{p}\right)$ denotes the Legendre symbol. No congruences have been given for $n\equiv -1 \pmod 8$, or for $n\equiv -1 \pmod p$ for the other primes. Kolberg suggests that no similar congruence exists in this case, and states that even determining the parity of $c(8n-1)$ may be comparable to determining the parity of the partition function [5].

The main theorem of this paper extends each of the congruences given above to the coefficients $a_0(m,n)$ of all basis elements $f_{0,m}$.

Theorem 2.1. *For each* $p\in\{2,3,5,7\}$, *let* $a_1,a_2\geq 0$, $a=|a_1-a_2|$, *and* $b_1,b_2\not\equiv 0 \pmod p$. *Then*

For $p = 2$:

$$
\begin{aligned}
a_0(2^{a_1}b_1, 2^{a_2}b_2) &\equiv -2^{3a+8}3^{a-1} \cdot b_1\sigma_7(b_1)\sigma_7(b_2) \pmod{2^{3a+13}} \\
&\qquad \textit{if } a_2 > a_1, && (2.11)\\
&\equiv -2^{4a+8}3^{a-1} \cdot b_1\sigma_7(b_1)\sigma_7(b_2) \pmod{2^{4a+13}} \\
&\qquad \textit{if } a_1 > a_2, && (2.12)\\
&\equiv 20b_1\sigma_7(b_1)\sigma_7(b_2) \pmod{2^7} \\
&\qquad \textit{if } a = 0, \quad b_1b_2 \equiv 1 \pmod{8}, && (2.13)\\
&\equiv \frac{1}{2}b_1\sigma(b_1)\sigma(b_2) \pmod{2^3} \\
&\qquad \textit{if } a = 0, \quad b_1b_2 \equiv 3 \pmod{8}, && (2.14)\\
&\equiv -12b_1\sigma_7(b_1)\sigma_7(b_2) \pmod{2^8} \\
&\qquad \textit{if } a = 0, \quad b_1b_2 \equiv 5 \pmod{8}. && (2.15)
\end{aligned}
$$

For $p = 3$:

$$
\begin{aligned}
a_0(3^{a_1}b_1, 3^{a_2}b_2) &\equiv \mp 3^{2a+3}10^{a-1}\frac{\sigma(b_1)\sigma(b_2)}{b_2} \pmod{3^{2a+6}} \\
&\qquad \textit{if } a_2 > a_1,\ b_1b_2 \equiv \pm 1 \pmod{3}, && (2.16)\\
&\equiv \mp 3^{3a+3}10^{a-1}\frac{\sigma(b_1)\sigma(b_2)}{b_2} \pmod{3^{3a+6}} \\
&\qquad \textit{if } a_1 > a_2,\ b_1b_2 \equiv \pm 1 \pmod{3}, && (2.17)\\
&\equiv 2 \cdot 3^3\frac{\sigma(b_1)\sigma(b_2)}{b_2} \pmod{3^7} \\
&\qquad \textit{if } a = 0,\ b_1b_2 \equiv 1 \pmod{3}. && (2.18)
\end{aligned}
$$

For $p = 5$:

$$
\begin{aligned}
a_0(5^{a_1}b_1, 5^{a_2}b_2) &\equiv -5^{a+1}3^{a-1}b_1^2b_2 \cdot \sigma(b_1)\sigma(b_2) \pmod{5^{a+2}} \\
&\qquad \textit{if } a_2 > a_1, && (2.19)\\
&\equiv -5^{2a+1}3^{a-1}b_1^2b_2 \cdot \sigma(b_1)\sigma(b_2) \pmod{5^{2a+2}} \\
&\qquad \textit{if } a_1 > a_2, && (2.20)\\
&\equiv 10b_1^2b_2 \cdot \sigma(b_1)\sigma(b_2) \pmod{5^2} \\
&\qquad \textit{if } a = 0,\ \left(\frac{b_1b_2}{5}\right) = -1. && (2.21)
\end{aligned}
$$

For $p = 7$*:*

$$\begin{aligned} a_0(7^{a_1}b_1, 7^{a_2}b_2) \equiv &\quad 7^a 5^{a-1} \cdot b_1^2 b_2 \cdot \sigma_3(b_1)\sigma_3(b_2) \pmod{7^{a+1}} \\ &\qquad \text{if } a_2 > a_1, \qquad (2.22) \\ \equiv &\quad 7^{2a}5^{a-1}b_1^2 b_2 \cdot \sigma_3(b_1)\sigma_3(b_2) \pmod{7^{2a+1}} \\ &\qquad \text{if } a_1 > a_2, \qquad (2.23) \\ \equiv &\quad 2b_1^2 b_2 \cdot \sigma_3(b_1)\sigma_3(b_2) \pmod{7} \\ &\qquad \text{if } a = 0, \left(\frac{b_1 b_2}{7}\right) = 1. \qquad (2.24) \end{aligned}$$

As with the coefficients $c(n)$ of the j-function, we have no congruences for $b_1 b_2 \equiv -1 \pmod{p}$. Note that if we take $a_1 = 0$ and $b_1 = 1$, these congruences reduce exactly to those of Kolberg and Aas.

3. Proof of the Theorem

3.1. *Preliminary Identities*

We begin by giving some propositions that will be useful in proving the theorem.

Proposition 1. *If* k *or* $(2-k) \in \{0, 4, 6, 8, 10, 14\}$*, then*

$$m^{k-1}a_k(m,n) = n^{k-1}a_k(n,m).$$

Proof. By the duality mentioned earlier, $a_k(m,n) = -a_{2-k}(n,m)$, so we need only prove the proposition for the case $k \in \{2, 4, 6, 8, 10, 14\}$. By Lewis and Zagier ([8], chapter IV, section 2) the $k-1$st derivative of $f_{2-k,m}(\tau)$ is a weakly holomorphic modular form on $SL_2(\mathbb{Z})$ of weight k, so it must be a linear combination of the $f_{k,m}(\tau)$. In fact, we find

$$\frac{1}{(2\pi i)^{k-1}}\frac{d^{k-1}}{d\tau^{k-1}}f_{2-k,m}(\tau) = -m^{k-1}q^{-m} + \sum_{n=1}^{\infty} n^{k-1}a_{2-k}(m,n)q^n = -m^{k-1}f_{k,m}(\tau).$$

This gives us $m^{k-1}a_k(m,n) = -n^{k-1}a_{2-k}(m,n) = n^{k-1}a_k(n,m)$. □

For $k = 0$, this gives us $na_0(m,n) = ma_0(n,m)$. A version of this weight 0 equation was used by Asai, Kaneko and Ninomiya [2].

Proposition 2. *For all positive integers* k*,*

$$\sum_{d|(m,n)} d^k \sigma_k\left(\frac{mn}{d^2}\right) = \sigma_k(m)\sigma_k(n)$$

Proof. For odd $k \geq 3$ the proposition follows from the application of the Hecke operator T_m to the appropriate Eisenstein series since these series are Hecke eigenforms. For the general proof we proceed algebraically.

We note first that $\sum_{d|(m,n)} d^k \sigma_k \left(\frac{mn}{d^2}\right)$ is multiplicative, so without loss of generality we reduce to the case $m = p^a$, $n = p^b$, with $a \leq b$. We rewrite the expression as

$$\begin{aligned}\sum_{j=0}^{a} p^{jk} \sigma_k(p^{a+b-2j}) &= \sum_{j=0}^{a} p^{jk} \left(\frac{p^{(a+b-2j+1)k} - 1}{p^k - 1}\right)\\ &= \frac{p^{(b+1)k} \sum_{j=0}^{a} p^{(a-j)k} - \sum_{j=0}^{a} p^{jk}}{p^k - 1}\\ &= \sigma_k(p^a)\sigma_k(p^b). \qquad \square\end{aligned}$$

Another important identity follows from standard formulas for the action of the Hecke operator. These give us

$$\begin{aligned}f_{0,1}(\tau)|T_0(m) &= \sum_n \left(\sum_{d|(m,n)} \frac{m}{d} a_0\left(1, \frac{mn}{d^2}\right)\right) q^n\\ &= q^{-m} + \sum_{n=1}^{\infty} \left(\sum_{d|(m,n)} \frac{m}{d} a_0\left(1, \frac{mn}{d^2}\right)\right) q^n\end{aligned}$$

which must be a weight 0 weakly homomorphic modular form. Therefore $f_{0,1}(\tau)|T_0(m) = f_{0,m}(\tau)$ (as noted earlier), and we have

$$a_0(m, n) = \sum_{d|(m,n)} \frac{m}{d} a_0\left(1, \frac{mn}{d^2}\right). \tag{3.1}$$

3.2. *Proof of the congruences*

The proofs of the congruences follow nearly identical steps for each prime. We give the full proof for $p = 2$, and include corresponding intermediate results for $p = 3$, 5, and 7. Throughout this section, let b_1 and b_2 be relatively prime to p.

Proposition 3. *Let $a = a_2 > 0$, $a_1 = 0$. Then*

$$a_0(b_1, 2^a b_2) \equiv -2^{3a+8} 3^{a-1} b_1 \sigma_7(b_1)\sigma_7(b_2) \pmod{2^{3a+13}}.$$

Proof.

We note that for odd d, $d^{-1} \equiv d^7 \pmod{2^5}$. We apply equation (3.1), Kolberg's congruence (2.1), and then reduce by Proposition 2 in that order:

$$\begin{aligned} 2^{-3a-8}a_0(b_1, 2^a b_2) &= 2^{-3a-8} \sum_{d|(b_1,b_2)} \frac{b_1}{d} a_0 \left(1, 2^a \frac{b_1\ b_2}{d^2}\right) \\ &\equiv -3^{a-1} b_1 \sum_{d|(b_1,b_2)} d^{-1} \sigma_7 \left(\frac{b_1 b_2}{d^2}\right) \\ &\equiv -3^{a-1} b_1 \sum_{d|(b_1,b_2)} d^{7} \sigma_7 \left(\frac{b_1 b_2}{d^2}\right) \\ &\equiv -3^{a-1} b_1 \sigma_7(b_1)\sigma_7(b_2) \pmod{2^5}. \end{aligned}$$ □

Following similar steps, we find that for $p = 3$,

$$a_0(b_1, 3^a b_2) \equiv \mp 3^{2a+3} 10^{a-1} \frac{1}{b_2} \sigma(b_1)\sigma(b_2) \pmod{3^{2a+6}} \text{ if } b_1 b_2 \equiv \pm 1 \pmod 3.$$

For $p = 5$, we use $d^{-3} \equiv d \pmod 5$ to get

$$a_0(b_1, 5^a b_2) \equiv -5^{a+1} 3^{a-1} b_1^2 b_2 \sigma(b_1)\sigma(b_2) \pmod{5^{a+2}}.$$

For $p = 7$, we use $d^{-3} \equiv d^3 \pmod 7$ to get

$$a_0(b_1, 7^a b_2) \equiv 7^a 5^{a-1} b_1^2 b_2 \sigma_3(b_1)\sigma_3(b_2) \pmod{7^{a+1}}.$$

Proposition 4. *If $a_2 \geq a_1$, then*

$$a_0(2^{a_1} b_1, 2^{a_2} b_2) \equiv a_0(b_1, 2^{a_2-a_1} b_2) \pmod{2^{3(a_2-a_1)+15}}$$

Proof. By applying $T_0(2^{a_1})$ to f_{0,b_1}, we find that

$$a_0(2^{a_1} b_1, 2^{a_2} b_2) = \sum_{d|2^{a_1}} \frac{2^{a_1}}{d} a_0(b_1, b_2 \frac{2^{a_2+a_1}}{d^2}).$$

When $d < 2^{a_1}$, Proposition 3 gives us

$$\frac{2^{a_1}}{d} a_0(b_1, b_2 \frac{2^{a_2+a_1}}{d^2}) \equiv 0 \pmod{2^{3(a_2-a_1+2)+8+1}},$$

so the previous expression reduces to $a_0(b_1, 2^{a_2-a_1} b_2) \pmod{2^{3(a_2-a_1)+15}}$. □

Similarly, for $p = 3$ we find that

$$a_0(3^{a_1} b_1, 3^{a_2} b_2) \equiv a_0(b_1, 3^{a_2-a_1} b_2) \pmod{3^{2(a_2-a_1)+11}}.$$

For $p = 5$,

$$a_0(5^{a_1} b_1, 5^{a_2} b_2) \equiv a_0(b_1, 5^{a_2-a_1} b_2) \pmod{5^{2(a_2-a_1)+6}}.$$

For $p = 7$,

$$a_0(b_1, 7^a b_2) \equiv 7^a 5^{a-1} b_1^2 b_2 \sigma_3(b_1)\sigma_3(b_2) \pmod{7^{a+1}}.$$

Propositions 3 and 4, and their counterparts for $p = 3$, 5, and 7, imply congruences (2.11), (2.16), (2.19), and (2.22). Along with Proposition 1, these imply congruences (2.12), (2.17), (2.20), and (2.23) respectively.

Following steps similar to those used in the proof of Proposition 3, we will now prove congruences (2.13)-(2.15). We note that for all odd b_1, b_2, and d, it is true that $\frac{b_1 b_2}{d^2} \equiv b_1 b_2 \pmod{2^3}$.

Proof of (2.13). Let $a_1 = a_2 = a$, and suppose $b_1 b_2 \equiv 1 \pmod{2^3}$. Again, $d^{-1} \equiv d^7 \pmod{2^5}$. Therefore

$$\begin{aligned} 2^{-2} a_0(2^a b_1, 2^a b_2) &\equiv 2^{-2} a_0(b_1, b_2) \\ &\equiv 5\, b_1 \sum_{d \mid (b_1, b_2)} d^{-1} \sigma_7 \left(\frac{b_1 b_2}{d^2} \right) \\ &\equiv 5\, b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^5}. \end{aligned}$$ □

Proof of (2.14). Let $a_1 = a_2 = a$, and suppose $b_1 b_2 \equiv 3 \pmod 8$. Since $b_1 b_2 \not\equiv 1 \pmod 8$, $b_1 b_2$ is not a square, which implies that $\frac{1}{2}\sigma \left(\frac{b_1 b_2}{d^2} \right)$ is an integer. For d odd, $d^{-1} \equiv d \pmod{2^3}$. Therefore,

$$\begin{aligned} a_0(2^a b_1, 2^a b_2) &\equiv a_0(b_1, b_2) \\ &\equiv \frac{1}{2} b_1 \sum_{d \mid (b_1, b_2)} d^{-1} \sigma \left(\frac{b_1 b_2}{d^2} \right) \\ &\equiv \frac{1}{2} b_1 \sigma(b_1) \sigma(b_2) \pmod{2^3}. \end{aligned}$$ □

Proof of (2.15). Let $a_1 = a_2 = a$, and suppose $b_1 b_2 \equiv 5 \pmod{2^3}$. Since $\frac{1}{2}\sigma \left(\frac{b_1 b_2}{d^2} \right)$ and $\frac{\sigma_7(b_1)\sigma_7(b_2)}{2}$ are both integers, we have

$$\begin{aligned} 2^{-3} a_0(2^a b_1, 2^a b_2) &\equiv 2^{-3} a_0(b_1, b_2) \\ &\equiv -\frac{3}{2} b_1 \sum_{d \mid (b_1, b_2)} d^{-1} \sigma_7 \left(\frac{b_1 b_2}{d^2} \right) \\ &\equiv -\frac{3}{2} b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^5}. \end{aligned}$$ □

The congruences (2.18), (2.21), and (2.24) are proven similarly, using the respective versions of Propositions 3 and 4. This concludes the proof of the theorem.

4. Concluding remarks

We conclude with a few remarks about certain interesting phenomena.

In his work for $p = 7$, Aas [1] makes special note of the fact that if $\left(\frac{n}{7}\right) = -1$, then $n\sigma_3(n) \equiv 0 \pmod 7$. With congruence (2.9), this shows that if $\left(\frac{n}{7}\right) = -1$, then the coefficient $c(7^a n)$ is divisible by 7^{a+1}. Similarly our congruences show that if either $\left(\frac{b_1}{7}\right)$ or $\left(\frac{b_2}{7}\right) = -1$, then $a_0(7^{a_1}b_1, 7^{a_2}b_2)$ is divisible by an extra power of 7 than is guaranteed by the Theorem 2.1 for the general case. Specifically, $a_0(7^{a_1}b_1, 7^{a_2}b_2)$ is divisible by 7^{a+1} if $a_2 > a_1$, and by 7^{2a+1} if $a_1 > a_2$. Additionally, if $a_1 = a_2$, and $\left(\frac{b_1}{7}\right) = \left(\frac{b_2}{7}\right) = -1$, congruence (2.23) shows that $a_0(7^{a_1}b_1, 7^{a_2}b_2) \equiv 0 \pmod 7$.

In many cases, Theorem 2.1 gives the exact power of p dividing the coefficient $a_0(m, n)$. In several cases where the p-divisibility is not best possible, we still find interesting results involving the expression inside the congruence. For example, if $v_2(x)$ is the 2-adic valuation of x, we have not found any counterexamples computationally to $v_2\left(a_0(b_1, 2^a b_2)\right) = 3a + 8 + v_2\left(\sigma_7(b_1)\sigma_7(b_2)\right)$, even when this valuation is much greater than $3a + 13$. This is the divisibility suggested in (2.11), with equality rather than congruence. If we let $m = 31$ and $n = 762$, we find

$$v_2\left(a_0(31, 762)\right) = 3 \cdot 1 + 8 + v_2\left(\sigma_7(31)\sigma_7(381)\right) = 25,$$

though the theorem only gives a congruence modulo 2^{18}. We find similar phenomena when $b_1 b_2 \equiv 1 \pmod 4$. For instance,

$$v_2\left(a_0(21, 889)\right) = 2 + v_2\left(\sigma_7(21)\sigma_7(889)\right) = 17,$$

although the theorem only guarantees divisibility by 2^8.

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