A VARIATION ON THE STEINER PROBLEM:
EQUALLY SPACED POINTS ON A WIDE CONE

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Abstract. In this paper we provide a closed-form solution to a special case of modified Steiner problem. We prove that if \( n \) points lie on the unit circle on a wide cone of angle \((60^\circ n)\), then the minimal path necessary to connect the points is formed by connecting \( n - 1 \) sides for small \( n \) and connecting points to the center in pairs for large \( n \).

1. Introduction

The Steiner problem is a widely known problem that is stated very simply: given \( n \) points, find the least-length path required to connect the points. An algorithm to find this least-length path is known, but its complexity grows quickly as the number of points grow. It is interesting, then, to study special cases of the Steiner problem whose solutions have a closed form. Graham’s conjecture, which was proved in 1992 by Rubinstein and Thomas in [1], focuses on points on a circle. One consequence of the conjecture is that for \( n \) points equally spaced around a circle, \( n \geq 6 \), the shortest network connecting the points is formed by drawing \( n - 1 \) sides of the \( n \)-gon.

In 2005, Colleen Hughes and Christine Truesdell worked with Gary Lawlor on a slight variation of Graham’s problem. Given a wide cone of angle \( 480^\circ \) (see Definition 2.1), they tried to find the least-length path to connect 8 points equally spaced on a circle. Since this problem appears to be analogous to the Graham’s problem with 6 points, one might expect the result to be the same. However, Hughes and Truesdell showed that the least length path consists of four Y-shaped paths connecting to the origin, as in Figure 8. Unfortunately, their proof required the analysis of more than 90 different cases, and the complicated details prevented them from publishing their proof.

In 2006, the authors of this paper took an interest in the results of Hughes and Truesdell and looked for a way to generalize them. We tried to answer the question: given \( n \) points on a wide cone of angle \((60^\circ n)\), what is the minimal path necessary to connect the points? We found that for \( n \leq 7 \), the shortest network is connecting the points is formed by drawing \( n - 1 \) sides of the \( n \)-gon, as Graham showed for six points in the plane. For \( n \geq 8 \), the shortest network is connected by connecting Y-shaped paths to the origin, as Hughes and Truesdell showed for eight points. Furthermore, we discovered a simple proof that only considers two cases.
2. Background and Definitions

**Definition 2.1.** The *wide cone of angle* $\alpha$ is locally isometric to $\mathbb{R}^2$ except the origin, where it has angle $\alpha$.

**Definition 2.2.** A polygon (including its interior) $p_is_ip_is_ip_j$, where $p_i$ and $p_j$ are regular points and each $s_k$ is a Steiner point, is a *Steiner polygon* of a Steiner tree, also called the *Steiner polygon* of $p_i$ and $p_j$.

**Definition 2.3.** Two regular points are called *siblings* if they are connected to the same Steiner point, i.e. if their Steiner polygon has three vertices.

**Lemma 2.4.** Given a wide cone of angle $\alpha$ (in degrees) and $n$ points equally spaced on a circle centered at the origin, then if a Steiner polygon contains the origin, we have

$$\frac{\alpha(n - 1)}{60n} - 1 < k \leq \frac{\alpha(n - 2)}{60n} + 2,$$

where $k$ is the number of vertices in the Steiner polygon. In the special case where $\alpha/n = 60^\circ$, then $k = n$ or $k = n - 1$.

**Proof.** For the left-hand side of the equation, we note that the sum of the angles of a $k$-gon containing the origin of a wide cone must be $180k - \alpha$ degrees. Each of the $k - 2$ angles with vertices at Steiner points must measure $120^\circ$. Finally, since the origin is contained in the Steiner polygon, the sum of the two angles with vertices at regular points must be greater than $180 - \alpha/n$ degrees. This gives us the following inequality:

$$180k - \alpha > 120(k - 2) + 180 - \frac{\alpha}{n}$$

$$k > \frac{\alpha(n - 1)}{60n} - 1$$

(2)

Similarly, for the right-hand side of the equation we note that the sum of the two angles with vertices at regular points must be less than or equal to $360 - 2\alpha/n$ degrees, or the Steiner tree will not be contained in our $n$-gon. This gives us the following inequality:

$$180k - \alpha \leq 120(k - 2) + 360 - \frac{2\alpha}{n}$$

$$k \leq \frac{\alpha(n - 2)}{60n} + 2$$

(3)

Combining Equations (2) and (3), we get Equation (1), as desired.

Finally, if we let $\alpha = 60n$, Equation (1) reduces to $n - 2 < k \leq n$.

3. CASE 1: THE ORIGIN LIES IN THE INTERIOR OF A STEINER POLYGON

We will show that if the origin lies in the interior of some Steiner polygon, then there is only one possible topology for the Steiner tree. To do this, we will use Lemma 2.4 to show that there are actually only two ways that the origin could lie in the interior of a Steiner polygon. We will then work by contradiction to eliminate one of these. The following lemma will be of use:

**Lemma 3.1.** Let $p_1$, $p_2$, and $p_3$ be adjacent regular points such that $p_1$ and $p_2$ are siblings connected to a Steiner point $s_1$. If $\angle s_1p_2p_3 \leq 90^\circ$, then the Steiner polygon of $p_2$ and $p_3$ does not contain the origin in its interior.
The sum of the last two angles is \( 360^\circ - 2 \times 120^\circ = 120^\circ \). So certainly one of these angles is less than or equal to \( 90^\circ \), so by Lemma 3.1, either \( p_2 \) and \( p_1 \) are not siblings or \( p_n \) and \( p_{n-1} \) are not siblings.

Suppose, without loss of generality, that \( p_1 \) and \( p_2 \) are not siblings. Since the Steiner polygon of \( p_1 \) and \( p_n \) has \( n-1 \) sides, it has \( n-3 \) of the \( n-2 \) Steiner points as vertices. This forces \( p_2 \) and \( p_3 \) to be siblings, as shown in Figure 2. Suppose that \( s_0 \) is the Steiner point of \( p_2 \) and \( p_3 \) and that the Steiner polygon of \( p_1 \) and \( p_n \) is \( \Delta p_1 s_2 \ldots s_{n-3} p_n \). Since \( \angle p_2 s_0 p_3 = 120^\circ \) and \( s_1 s_0 \) bisects this angle, there is a point \( E_0 \) such that \( \Delta p_2 p_3 E_0 \) is equilateral and \( E_0 \) lies on \( s_1 s_0 \). Similarly, there is a point \( E_1 \) such that \( \Delta E_1 E_0 p_1 \) is a right triangle and \( E_1 \) lies on \( s_2 s_1 \). There is a point \( E_2 \) such that \( \Delta E_2 E_1 p_4 \) is a right triangle and \( E_2 \) lies on \( s_3 s_2 \). Finally, there is a point \( E_3 \) such that \( \Delta E_3 E_2 p_5 \) is a right triangle and \( E_3 \) lies on \( s_4 s_3 \). It is important that these \( E_i \) depend only on \( p_1, \ldots, p_5 \).

We know that \( s_1 s_2 \) must be above the origin, so \( \angle O E_1 p_1 > \tan^{-1} \left( \frac{\sqrt{3}}{3} \right) \). Furthermore, \( s_3 s_4 \) is rotated \( 60^\circ \) from \( s_1 s_2 \). Since \( \overrightarrow{E_3 s_3} \) must fall to the clockwise side of \( p_6 \), we have \( \angle E_3 p_6 p_1 \geq 180^\circ - \tan^{-1}(3\sqrt{3}) \). If \( p \) is the intersection of \( \overrightarrow{E_1 s_1} \) and \( \overrightarrow{E_3 s_3} \), then this forces \( \angle E_1 p p_6 < 60^\circ \), which is a contradiction.

Therefore, \( k = n \), so any Steiner polygon containing the origin in its interior has \( n \) vertices.

\[\Box\]
4. Case 2: The Origin Lies on the Boundary of a Steiner Polygon

Lemma 4.1. Let $p_1$ and $p_2$ be regular points such that the origin is not contained in their Steiner polygon $P$, and let $k$ be the number of sides of $P$. Then

1. $k \leq 5$, and
2. if $\angle p_1 \leq 60^\circ$ then $k = 4$ or $k = 3$

Proof. Since $P$ does not contain the origin, the sum of the measures of its angles is $180^\circ k - 360^\circ$ degrees. Since $k - 2$ of the angles measure $120^\circ$, the two remaining angles measure

$$\angle p_1 + \angle p_2 = 180^\circ k - 360^\circ - 120^\circ (k - 2) = 60^\circ (k - 2)$$

Since $\angle p_1$ and $\angle p_2$ are both smaller than $120^\circ$, $\angle p_1 + \angle p_2 < 240^\circ$, giving us $60^\circ (k - 2) < 240^\circ$. Therefore, $k < 6$, as desired. In the case that $\angle p_1 \leq 60^\circ$, then $\angle p_1 + \angle p_2 < 180^\circ$. This gives us $60^\circ (k - 2) < 180^\circ$, so $k < 5$, as desired. □

Lemma 4.2. If the center is part of a Steiner minimal tree then the center is a sibling with some regular point.

Proof. Suppose not. Then the center connects to a Steiner point $s_1$ which in turn connects to two Steiner points $s_2$ and $s_3$. By Lemma 4.1 no Steiner polygon can have more than five sides, so $s_2$ and $s_3$ must connect to regular points $p_2$ and $p_3$, as shown in Figure 3 (otherwise, the Steiner polygon that has $s_1, s_2, s_3, s_4$ as vertices would have more than five sides). Since the sum of the angles $\angle p_2$ and $\angle p_3$ in the Steiner polygon of $p_2$ and $p_3$ is $180^\circ$, we can assume without loss of generality that $\angle p_2 \geq 90^\circ$.

Now consider the regular point $p_1$, as shown in Figure 3. The ray $\overrightarrow{s_2s_4}$ is a reflection of the ray $\overrightarrow{s_2s_1}$ across $\overrightarrow{p_2s_2}$. Since $\angle p_2 \geq 90^\circ$, $\overrightarrow{s_2s_4}$ is either at or below $\overrightarrow{s_2s_1}$, so $\overrightarrow{s_2s_4}$ passes below $p_1$. Therefore there is a Steiner point somewhere on $\overrightarrow{s_2s_4}$ which means there must be a regular point below $\overrightarrow{s_2s_4}$. This cannot be, so the center must be a sibling with some regular point. □
Lemma 4.3. If $p_1s_1s_2p_2$ is the four-sided Steiner polygon of $p_1$ and $p_2$, then $\angle s_1p_1p_2 = \angle s_2p_2p_3$.

Proof. Since $\angle s_1p_1p_2$ and $\angle s_2p_2p_1$ are angles of the polygon $p_1s_1s_2p_2$, their measures sum to $360^\circ - 2 \cdot 120^\circ = 120^\circ$ (see Figure 4). Also, $\angle s_2p_2p_3 + \angle s_2p_2p_1 = \angle p_1p_2p_3 = 120^\circ$. Combining these equations gives us that $\angle s_1p_1p_2 = \angle s_2p_2p_3$. □

Figure 4. Steiner configuration for Lemma 4.3

Lemma 4.4. Any tree that connects the origin and a number of regular points has the snail topology.

Proof. By Lemma 4.2, the origin $O$ is a sibling with some regular point. Call this point $p_1$, and let $s_1$ be the Steiner point that connects to the origin and to $p_1$. Orient the graph so that $s_1$ lies counterclockwise of $Op_1$ as shown in Figure 5. Clearly, $\angle s_1p_1p_2 < 60^\circ$. By Lemma 4.1, the Steiner polygon of $p_1$ and $p_2$ has 3 or 4 sides. If it had 3 sides, the Steiner tree would end, since all Steiner points would be connected to three other points. Thus the Steiner polygon has four sides. Then by Lemma 4.3 $\angle s_1p_1p_2 = \angle s_2p_2p_3$. By induction, the Steiner polygon of each pair of consecutive regular points has four sides, and $\angle s_ip_ip_{i+1} = \angle s_{i+1}p_{i+1}p_{i+2}$. Therefore the points are connected with a snail topology. □

Figure 3. Topology for Lemma 4.2

5. Main Theorem

Theorem 5.1. Given a cone of angle $\alpha$ and $n$ points equally spaced on a circle centered at the origin such that $\alpha/n = 60^\circ$, then the least length network to connect the $n$ points is as follows:
Figure 5. Steiner configuration for Lemma 4.4

(1) If $n \leq 7$ then the least length network is formed by directly connecting the $n$ points with $n - 1$ edges and no added points.

(2) If $n \geq 8$ then the least length network is formed by connecting Y’s to the origin if $n$ is even, or a triple and some Y’s if $n$ is odd.

Proof. We have proven with Lemmas 3.2 and 4.4 that there are only two possibilities for our Steiner configuration: either we have an $n$-sided Steiner polygon or the union of several snail topologies. We will analyze the two options, assuming that the points are spaced one unit apart.

We can analyze the snail topology and see that a snail connecting $n$ points to the origin has total length $\sqrt{n^2 - n} + 1$. The function

$$\frac{\sqrt{n^2 - n} + 1}{n}$$

represents the cost per point to connect $n$ points to the origin (see Figure 6). We see that it is most efficient to connect points to the origin two at a time. Since the cost per point to connect 3 points with a 1-snail and a 2-snail is just over 0.9, it is best to connect 3 points using a 3-snail.

If we connect $n$ points using several 2-snails and a 3-snail if necessary, total length of our network will be

$$\frac{n}{2}\sqrt{3} \text{ or } \left(\frac{n - 3}{2}\right)\sqrt{3} + \sqrt{7}$$

if $n$ is even or odd respectively. Meanwhile, the length of the Steiner network in Case 1 is $n - 1$. It is easy to see that for $n \geq 8$, the former is the shortest, while for $n \leq 7$, the latter is more efficient. To further illustrate this, the cost per point of the two different methods is graphed in Figure 7.

References

Figure 6. The cost per point of connecting \( n \) points to the origin using a snail.

Figure 7. The cost per point of connecting \( n \) points to the origin using each of the possible methods.
Figure 8. The least length path to connect eight points on a 480° wide cone. Angles are not drawn to scale.