10 September 2014 **Limits**

(1) In this exercise we will develop a "bag of tricks" for evaluating limits of the form $\frac{0}{0}$ and $\frac{a}{0} - \frac{a}{0}$. When you encounter the forms $\frac{0}{0}$ or $\frac{a}{0} - \frac{a}{0}$, you must use some algebraic trick to evaluate the limit.

(a) Factor top and bottom, then cancel.

$$\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$$

(b) Rationalize the numerator or denominator

$$\lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h}$$

(c) Smash two terms together by finding a common denominator

$$\lim_{t\to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$$

(d) Using one of the three tricks above, evaluate the following limits:

(i) $\displaystyle \lim_{x\to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$

(ii) $\displaystyle \lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{x^2 - 3x + 2} \right)$

(iii) $\displaystyle \lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$

(2) Give examples of functions $f(x)$ and $g(x)$ for which neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exist, but $\lim_{x\to 0} f(x) + g(x)$ exists.

(3) (a) Sketch the graph of $y = |x|$.

(b) Fill in the blanks.

$$|x| = \begin{cases} & \text{if } x < 0 \\ & \text{if } x \geq 0 \end{cases}$$

(c) Evaluate the following limits, if they exist.

(i) $\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

(ii) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

(iii) $\lim_{x\to 2^-} \frac{x^2 + x - 6}{|x-2|}$

(4) Sketch the graph of $y = \tan x$ and use your graph to evaluate the following limits, if they exist:

(a) $\lim_{x\to \pi/2^-} \tan x$

(b) $\lim_{x\to \pi/2^+} \tan x$

(c) $\lim_{x\to \pi/2} \tan x$

(5) Sketch the graph of $y = \arctan x$ and use your graph to evaluate the following limits, if they exist:

(a) $\lim\limits_{x\to\infty} \arctan x$

(b) $\lim\limits_{x\to-\infty} \arctan x$

(c) $\lim\limits_{x\to-1} \arctan x$

(6) You will often encounter limits of the form $\frac{a}{0}$, where $a \neq 0$. These limits are always $\infty, -\infty$, or else the limit does not exist. I like to look at the denominator and determine whether it is approaching 0 from positive values or negative values, and write a small + or − next to the 0 to indicate that. Then, the sign of a will determine the answer. For instance, I might get something like

$$\lim_{x\to3^+} \frac{x^2-10}{3-x} \longrightarrow \frac{-1}{0^-} \longrightarrow +\infty$$

Evaluate the following limits:

(a) $\lim\limits_{x\to3^+} \dfrac{\log_3 x}{x-3}$

(b) $\lim\limits_{x\to\pi^+} \dfrac{\log_2 x}{\sin x}$

(7) Limits at ∞: when you encounter the limit of a rational function (polynomial over polynomial) at $\pm\infty$, it is best to multiply top and bottom by $\frac{1}{x^r}$, where r is the highest power of x appearing in the denominator. Then take the limit, remembering that $\frac{1}{x^a} \to 0$ as $x \to \pm\infty$ for any $a > 0$. Evaluate the following limits:

(a) $\lim\limits_{x\to\infty} \dfrac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^3-1}{\sqrt{9x^6-x}}$

(8) Use the squeeze theorem to show that
$$\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0.$$

(9) Compute the limits

(a) $\lim\limits_{x\to 0^+} \tan^{-1}(\log_5 x)$

(b) $\lim\limits_{x\to\infty} 3^{-2x}\cos x$ (hint: use the squeeze theorem)

(10) Evaluate
$$\lim_{x\to\infty} (\sqrt{x^2+ax} - \sqrt{x^2+bx})$$
Hint: rationalize the numerator.