23 September 2014 **Exam 1 Review**

1. Suppose f is an **odd** function which is **one-to-one**. If $f(-4) = 2$ then which of the following points lie on the graph of $y = f^{-1}(x)$? (You must circle all correct answers)

   (f) $(-2, 4)$

   (h) $(2, -4)$

   Incorrect points were $(4, 2), (-4, 2), (-4, -2), (4, -2), (2, 4), (-2, -4)$.

2. Indicate whether each statement is true or false.

   T A function can have two different horizontal asymptotes.

   F If the line $x = 1$ is a vertical asymptote of $y = f(x)$ then f is not defined at 1.

   T If $\lim\limits_{x\to\infty} f(x) = 7$ then $\lim\limits_{x\to 4-} (f(-\ln(4-x)) - 5) = 2$.

   T The equation $\sin x - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.

   T If $\lim\limits_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$ exists then f is continuous at $x = 3$.

   F $\tan^{-1} x = \dfrac{\sin^{-1} x}{\cos^{-1} x}$.

3. Let $f(x) = x - x^2$.

   Use the definition of the derivative as a limit to find $f'(x)$.
   Show each step in your calculation and be sure to use proper terminology in each step.

   Solution.

   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h) - (x+h)^2] - [x - x^2]}{h} = \lim_{h\to 0} \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} \\ &= \lim_{h\to 0} \frac{h - 2xh - h^2}{h} = \lim_{h\to 0} 1 - 2x - h = 1 - 2x. \end{aligned}$$

   □

4. A population of bacteria starts out with 87 bacteria and grows by a factor of 17 every 3 hours. Find a formula for the number of bacteria after t hours.

   Solution.

   $$P(t) = P_0\, a^{rt} \implies P(t) = 87 \cdot 17^{t/3}$$

   □

5. What is the value of $\cos\big(2\tan^{-1}(7)\big)$?

   Solution. Let $\theta = \tan^{-1}(7)$ and draw a triangle to find that opp$= 7$, adj$= 1$, and hyp$= \sqrt{50}$. Then

   $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{50}}\right)^2 - \left(\frac{7}{\sqrt{50}}\right)^2 = -\frac{48}{50}$$

   □

6. Suppose f is an odd function. Is $g(x) = \sin f(x^3)$ odd, even, or neither? Convince me.

Solution. Replace x by $-x$ to find

$$g(-x) = \sin f((-x)^3) = \sin f(-x^3) = \sin -f(x^3) = -\sin f(x)^3 = -g(x),$$

so $g(x)$ is odd. $\square$

7. Solve for x in the equation below.

$$\log_3(x-4) + \log_3(x-1) = 2\log_3(5-x)$$

Solution.

$$\begin{aligned} \log_3(x-4) + \log_3(x-1) &= 2\log_3(5-x) \\ \log_3[(x-4)(x-1)] &= \log_3(5-x)^2 && \text{(log rules)} \\ (x-4)(x-1) &= (5-x)^2 && \text{(raise 3 to the power of each side)} \\ x^2 - 5x + 4 &= 25 - 10x + x^2 \\ 5x &= 21 \\ x &= 21/5 \qquad \square \end{aligned}$$

8. Find all horizontal asymptotes on the graph of $f(x) = 2^{-x^2}\cos x$

Solution. The horizontal asymptotes will be given by the limits of $f(x)$ as x approaches $\pm\infty$. Note that since $x^2 \geq 0$, both limits ($\pm\infty$) will be the same. We use the squeeze theorem:

$$\begin{gathered} -1 \leq \cos x \leq 1 \\ -e^{-x^2} \leq e^{-x^2}\cos x \leq e^{-x^2} \\ \lim_{x\to\pm\infty} -e^{-x^2} \leq \lim_{x\to\pm\infty} e^{-x^2}\cos x \leq \lim_{x\to\pm\infty} e^{-x^2} \\ 0 \leq \lim_{x\to\pm\infty} e^{-x^2}\cos x \leq 0. \end{gathered}$$

So the only horzontal asymptote is $y = 0$. $\square$

9. Evaluate the following limits without the use of derivatives. Show sufficient justification for each answer. For infinite limits you must state if it is ∞ or $-\infty$.

   (a) $\displaystyle\lim_{x\to 1} \frac{5^{x^3-x}}{\log_2(x+1)}$

   Solution. Plug in $x = 1$ and see if what you get makes any sense.

$$\lim_{x\to 1} \frac{5^{x^3-x}}{\log_2(x+1)} = \frac{5^0}{\log_2 1} = 1$$

$\square$

(b) $\lim_{x\to\infty} \dfrac{x^2-10}{\sqrt{x^6+4x^4+16}}$

Solution. Multiply top and bottom by the highest power of x in the denominator.

$$\lim_{x\to\infty} \frac{x^2-10}{\sqrt{x^6+4x^4+16}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{\sqrt{x^6}}} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{10}{x^3}}{\sqrt{1+\frac{4}{x^2}+\frac{16}{x^6}}} = \frac{0}{1} = 0 \qquad \square$$

(c) $\lim_{x\to 1^+} \left(\dfrac{1}{1-x}+\dfrac{1}{|1-x|}\right)$

Solution. When x is slightly larger than 1, the effect of the absolute value on $1-x$ is to multiply by a negative: $|1-x| = -(1-x)$. So we have

$$\lim_{x\to 1^+} \left(\frac{1}{1-x}+\frac{1}{|1-x|}\right) = \lim_{x\to 1^+} \left(\frac{1}{1-x}-\frac{1}{1-x}\right) = 0 \qquad \square$$

(d) $\lim_{x\to 0} \dfrac{\frac{1}{(3+h)}-\frac{1}{3}}{h}$

Solution. Get a common denominator on top.

$$\lim_{x\to 0} \frac{\frac{1}{(3+h)}-\frac{1}{3}}{h} = \lim_{x\to 0} \frac{3-(3+h)}{3(3+h)} \cdot \frac{1}{h} = \lim_{x\to 0} \frac{-h}{3h(3+h)} = \lim_{x\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9} \qquad \square$$