4 October 2014 **Area**

(1) Estimate

$$\int_0^2 \frac{x}{1+x}\,dx$$

using a Riemann sum with 4 sub-intervals and left endpoints. Then repeat with right endpoints. Write your answers as fractions.

(2) Express

$$\int_0^2 \frac{x}{1+x}\,dx$$

as a limit of Riemann sums. Do not evaluate.

(3) Suppose that $0 < a < b$. Use geometry to compute

$$\int_a^b x\,dx.$$

(4) A nice, compact way to write $a_1 + a_2 + \ldots + a_n$ is

$$a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

The variable k is the index of summation. The sum above starts from $k = 1$ and ends at $k = n$. Sums satisfy the following properties:

$$(a) \sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k \qquad (c \text{ is anything that doesn't depend on } k)$$

$$(b) \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$(c) \sum_{k=1}^{n} 1 = (\text{think about it}) \qquad (\text{hint: write out } \textstyle\sum_{k=1}^{5} 1)$$

$$(d) \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$(e) \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Use the properties above to evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(3\frac{k^2}{n^3} - 4\frac{k}{n^2} + \frac{5}{n} \right)$$

(5) Consider the function $f(x) = x^2$ on the interval $[a, b] = [1, 3]$.

(a) Express $\int_a^b f(x)\,dx$ as a limit of Riemann sums.

(b) Use the properties of sums on the previous page to evaluate your expression in (a).

(6) With your group, discuss why each of the following properties of integrals is true. For some of them, you'll have to think about the definition as a Riemann sum; for others, you'll want to draw a picture.

(a) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(b) $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is a constant.

(c) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(d) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(e) $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(7) Use geometry and the properties in the previous problem to evaluate

$$\int_0^3 \left(\sqrt{9-x^2}+x-1\right)\,dx$$

(8) Let A_n be the area of a polygon with n equal sides inscribed in a circle of radius r (draw a picture). By diving the polygon into n triangles with central angle $2\pi/n$, show that

$$A_n = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$$

Show that $\lim_{n\to\infty} A_n = \pi r^2$ (hint: L'Hospital). You just found the area of a circle.