# Idempotent chains and bounded generation of $\mathrm{SL}_2$

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Abstract. We investigate connections between association chains of idempotents and bounded generation of $\mathrm{SL}_2$ by elementary matrices. This enables us to identify many classes of matrix rings that have universal bounds on the length of association chains. In particular, many matrix rings have the property that every regular element is special clean. We also improve the usual criterion for checking perspectivity via association chains.

## 1. Introduction

Relations between idempotents in the endomorphism ring of a module are known to provide useful information on the direct summands of the module. For instance, as pointed out by T. Y. Lam in [16, Section 8], a module $M$ has internal cancellation if and only if isomorphic idempotents in $R = \mathrm{End}(M)$ have isomorphic complements; equivalently, by (21.16) in [15], isomorphic idempotents in $R$ are conjugate in $R$. For more on modules with internal cancellation, see [9].

Besides isomorphy and conjugacy, a third important relation between idempotents has emerged in the context of rings and modules, that of association. Idempotents $e, f$ in a ring $R$ are *left associate* if they generate the same left ideal $Re = Rf$. One dually defines right association, and, as the two relations do not commute in general, the relation they jointly generate is described in terms of chains of alternating left and right associate idempotents.

These association chains in rings were first studied by Diesl, Dittmer, and the third author in connection with perspectivity of summands of a module; see [4]. (In regular semigroups these association chains were studied much earlier, under the name of $E$-chains, in [21].) It was shown that isomorphic summands of a module are always perspective (i.e., they share a common direct sum complement) if and only if isomorphic idempotents in the endomorphism ring are connected by two association chains of length 3, one starting with a left association, and the other with a right association (see Proposition 2.1(3)). In this paper, we show that one of the two chains suffices—the other comes for free; see Theorem 3.8 and Corollary 3.9.

It is now known that association chains are connected to many more diverse concepts such as Bass's stable range one condition, quasi-continuous modules, special clean elements, strongly IC rings, and numerous generalizations of perspectivity; see, for instance, [11, 12, 18]. In this paper, we find some rather interesting connections between association chains and the bounded generation of $\mathrm{SL}_2$ by elementary matrices. For instance, if $S$ is Dedekind-finite (i.e., all one-sided units are two-sided) and the nontrivial idempotents of $R = \mathbb{M}_2(S)$ are isomorphic to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then all isomorphic idempotents in $R$ are connected by an association chain of bounded length if and only if $\mathrm{SL}_2(S)$ has bounded generation by elementary matrices;

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see Theorem 3.2. Quite surprisingly, we prove that for large classes of rings, association chains have a bounded length of 4, which is just past where perspectivity holds.

In regards to bounded generation of $\mathrm{SL}_2$, we should mention that it has been known for quite some time that, under the generalized Riemann hypothesis (GRH), if $S$ is a (certain type of) localization of a ring of integers of a number field, and the unit group $\mathrm{U}(S)$ is infinite, then $\mathrm{SL}_2(S)$ has bounded generation [2, Corollary 2.3]. Recently the GRH assumption was removed, and a uniform bound (independent of the number field) on the number of elementary matrices was given [20, Theorem 1.1]; that bound is 9.

An outline of the paper follows. In Section 2, we introduce basic properties about idempotents, we explain notations related to association chains, and we mention some standard results about these chains that have appeared in the literature. In Section 3, we begin our investigation into the connections between association chains, elementary matrices, and division chains. Given any idempotent $e \in \mathrm{idem}(R)$, the Peirce decomposition with respect to $e$ provides a way to view $R$ as a matrix ring. In turn, this allows us to characterize association chains in terms of products of elementary transition matrices in a corresponding Peirce decomposition; see Theorem 3.1. Consequently, using well known equivalences between products of elementary matrices and division chains, we show in Theorem 3.2 that a bound on the termination lengths of division chains is equivalent to a (slightly longer) bound on the lengths of association chains. Numerous consequences follow.

An important open problem is to minimize the lengths of these chains. We end Section 3 with a discussion of how further improvements can be made by converting left association chains to right association chains, and vice versa. It was recently shown that such conversions always occur for association chains of length 2, and in Theorem 3.8 we show that such conversions also occur for chains of length 3. This simplifies a standard criterion for perspectivity; see Corollary 3.9.

We end the paper in Section 4, by translating the previous work from Peirce decompositions to full matrix rings. Our main result there, Theorem 4.3, shows that if a ring $S$ has bounded stable range, and if $S$ is a projective-free ring (e.g., $S$ is a commutative PID), then association chains for idempotents in $\mathbb{M}_m(S)$ have length bounded by 4, as long as $m$ is sufficiently large. An explicit bound in terms of the stable range is obtained.

## 2. Basic properties of association chains

We assume that the reader is familiar with those basic facts about idempotents in rings that can be found in Section 21 of [14], and so we will quickly review the information we will find most useful.

Given a ring $R$, we let $\mathrm{idem}(R)$ denote the set of idempotents in $R$. Given $e, f \in \mathrm{idem}(R)$, we write $e \cong f$ when $e$ and $f$ are *isomorphic* in $R$, which means that $eR \cong fR$ as right $R$-modules (or, equivalently, that $Re \cong Rf$ as left $R$-modules). It is well known that an isomorphism $e \cong f$ is equivalent to the existence of elements $a, b \in R$ satisfying the four equalities $ab = e$, $ba = f$, $aba = a$, and $bab = b$. The latter two equalities express the fact that $a$ and $b$ are a pair of *reflexive inverses*.

We say that $e, f \in \mathrm{idem}(R)$ are *conjugate* in $R$ when there exists a unit $u \in \mathrm{U}(R)$ such that $f = u^{-1}eu$. (The ring in which an isomorphy or conjugacy is taking place will always be clear from context.) As mentioned previously, it is well known that $e$ and $f$ are conjugate exactly when they are isomorphic and their complement idempotents $1 - e$ and $1 - f$ are isomorphic; see [15, (21.16)].

As mentioned in the introduction, two idempotents $e, f \in \mathrm{idem}(R)$ are *left associate* if $Re = Rf$, and we write $e \sim_\ell f$ in this case; *right association* is denoted by $e \sim_r f$. Left association of idempotents is equivalent to the pair of equalities $ef = e$ and $fe = f$, which is purely semigroup-theoretic. In rings, there are a number of other equivalent conditions, as given by (the left-right dual of) [15, (21.4)]; also see [22, Lemma 4.2].

Of particular importance is the fact that $e \sim_\ell f$ is equivalent to the existence of some (unique) unit $u \in 1 + (1-e)Re \subseteq \mathrm{U}(R)$ such that $f = ue$. Note that $eu^{-1} = e$ for such a unit, and hence $f = ueu^{-1}$ is a conjugate of $e$. Thinking of this another way, the set $(1-e)Re$ parameterizes the left associates of $e$.

Given some $n \in \mathbb{N}$, a *left $n$-chain* from $e$ to $f$ consists of a sequence of idempotents $e_0, e_1, \ldots, e_n \in \mathrm{idem}(R)$ that are related in the alternating fashion

$$e = e_0 \sim_\ell e_1 \sim_r e_2 \sim_\ell \cdots e_n = f.$$

(The word "left" comes from the fact that the first pair of idempotents $e_0$ and $e_1$ are *left* associate.) When $n$ is small, such as $n = 2$ or $n = 3$, we will write $e \sim_{\ell r} f$, respectively $e \sim_{\ell r \ell} f$, thus suppressing the intermediate idempotents. We define *right $n$-chains* dually, and we write $e \approx f$ to denote that $e$ and $f$ are connected by some (left or right) association chain. It is easy to see that $\approx$ is an equivalence relation, being the transitive closure of the union of $\sim_\ell$ and $\sim_r$.

In the ring $\mathbb{M}_2(\mathbb{Z})$, any two nontrivial idempotents are connected by an association chain, but the minimal length of such chains becomes unbounded [4, Proposition 6.9]. When pairs of isomorphic idempotents are connected by association chains of small *bounded* length, the ring exhibits very strong conditions. The following proposition summarizes information found in the literature.

**Proposition 2.1.** *Let $M$ be a right module, and let $R = \mathrm{End}(M)$ be its endomorphism ring. Let $\mathscr{P}(n)$ denote the statement that any two isomorphic idempotents of $R$ are connected by both a left and a right association chain of length $n \in \mathbb{N}$. The following are true:*

(1) *$\mathscr{P}(1)$ holds if and only if isomorphic idempotents are equal, if and only if isomorphic summands of $M$ are equal.*
(2) *$\mathscr{P}(2)$ holds if and only if idempotents of $R$ are central modulo the Jacobson radical, if and only if isomorphic summands of $M$ share all their complements.*
(3) *$\mathscr{P}(3)$ holds if and only if $R$ is a perspective ring, if and only if isomorphic summands of $M$ are perspective.*
(4) *$\mathscr{P}(4)$ holds if and only if the (von Neumann) regular elements of $R$ are special clean, if and only if isomorphic summands of $M$ have the property that any complement of one summand will be perspective to some complement of the other summand.*

*Proof.* (1) Idempotents are both left and right associate if and only if they are equal, as proved in [4, Lemma 6.2(4)], which yields the first equivalence. The second equivalence follows from the fact that isomorphic summands arise as the images of isomorphic idempotents.

(2) The first equivalence comes from [11, Theorem 3.13]. The second equivalence comes from [11, Theorem 3.13 and Theorem 3.17].

(3) The first equivalence follows from [4, Lemma 6.3] together with the left-right symmetry result [11, Lemma 3.6]. The second equivalence is [5, Corollary 5.2].

(4) The first equivalence comes from [11, Proposition 3.18(1), Theorem 4.1] (again using Lemma 3.6 of that paper for left-right symmetry). The second equivalence follows by modifying [11, Theorem 3.17] from 2-chains to 4-chains. ☐

As it will be useful later, we describe a simple consequence of the $\mathscr{P}(n)$ properties.

**Lemma 2.2.** *If $R$ is a ring satisfying $\mathscr{P}(n)$ for some $n \in \mathbb{N}$, then $R$ satisfies internal cancellation, and hence it is Dedekind-finite (i.e., every one-sided unit is two-sided).*

*Proof.* Suppose $e, f \in \mathrm{idem}(R)$ with $e \cong f$. By hypothesis, $e \approx f$, and in particular $e$ and $f$ are conjugate. Internal cancellation follows from [9, (1.4)], and as mentioned on page 204 of that paper, Dedekind-finiteness is a well known and obvious consequence of internal cancellation. □

Let $\mathscr{Q}(n)$ denote the statement that any two conjugate idempotents of $R$ are connected by both a left and a right association chain of length $n \in \mathbb{N}$; this is a slight weakening of the $\mathscr{P}(n)$ condition. Surprisingly, the class of rings satisfying $\mathscr{Q}(4)$ is quite extensive, including the endomorphism rings of (arbitrary dimensional) vector spaces, and more generally quasi-continuous modules [11, Proposition 3.18(2) and Theorem 4.11]; note that such rings do not generally satisfy $\mathscr{P}(4)$ since they are generally not Dedekind-finite. In this paper we will describe large classes of rings satisfying the stronger property $\mathscr{P}(4)$. First, we need to connect idempotent chains with Morita contexts.

## 3. Idempotents and Morita contexts

Recall that a *Morita context* consists of two rings $S$ and $T$, together with two bimodules ${}_SP_T$ and ${}_TQ_S$, as well as two bilinear maps $P \otimes_T Q \to S$ and $Q \otimes_S P \to T$ that satisfy the appropriate associativity conditions to make

$$R = \begin{pmatrix} S & P \\ Q & T \end{pmatrix}$$

into a ring under the usual matrix addition and multiplication operations. The reader is directed to [13, Section 18C] for additional information on this construction. This matrix ring data can be described globally as follows.

Let $R$ be a ring, and let $e \in \mathrm{idem}(R)$ be any idempotent. This gives rise to the matrix representation, sometimes called the *Peirce decomposition* with respect to $e$, given by

$$R' = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Taking $S = eRe$, $T = (1-e)R(1-e)$, $P = eR(1-e)$, and $Q = (1-e)Re$, we may view $R'$ as a Morita context. Conversely, every Morita context arises this way. There is an isomorphism $R' \to R$ given by the rule

$$\begin{pmatrix} s & p \\ q & t \end{pmatrix} \mapsto s + p + q + t.$$

(See Exercises 18–20 of Section 18 from [13] for further details. These exercises are worked out in [17].) For simplicity, it is standard to identify $R'$ with $R$, via this isomorphism. In the remainder of the paper we will also follow this convention.

Given $e \in \mathrm{idem}(R)$, the left associates of $e$ are exactly those idempotents whose Peirce representation (with respect to $e$) is of the form $\left(\begin{smallmatrix} 1 & 0 \\ q & 0 \end{smallmatrix}\right)$ for some $q \in (1-e)Re$. (The 1 in the upper left corner is the identity of $eRe$, which is $e$.) Thus, it is quite easy to recognize left and right associates when working in such contexts.

Isomorphic idempotents also behave well in such contexts. Indeed, take $e, f \in \mathrm{idem}(R)$ with $e \cong f$. We may fix a pair of reflexive inverses $a, b \in R$ with $ab = e$ and $ba = f$. From the fact that $aba = a$, we see that $ea = a$. In terms of the Peirce representation (with respect to $e$), the matrix $a$ has a zero second row. Similarly, $bab = b$ implies that $b$ has a zero second column. So we can write

$$a = \begin{pmatrix} x & p \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} y & 0 \\ q & 0 \end{pmatrix}$$

with $x, y \in eRe$, $p \in eR(1-e)$, and $q \in (1-e)Re$. Moreover, $xy + pq = ab = e = 1_{eRe}$.

The following criterion, for recognizing when isomorphic idempotents are association chained, has a nice interpretation in the corresponding Peirce decomposition.

**Theorem 3.1** (cf. [19, Theorem 2.5])**.** *Let $R$ be a ring, and let $n \in \mathbb{N}$. If $a, b \in R$ are reflexive inverses, then there exists a left $(n+2)$-chain from $f = ba$ to $e = ab$ if and only if there exist $z_1, z_2, \ldots, z_n$ with*

$$z_i \in \begin{cases} (1-e)Re & \textit{if } i \textit{ is odd} \\ eR(1-e) & \textit{if } i \textit{ is even,} \end{cases}$$

*and $ea(1+z_n)(1+z_{n-1})\cdots(1+z_2)(1+z_1)e \in \mathrm{U}(eRe)$.*

*Proof.* We work by induction on $n$. When $n = 0$, this is exactly [19, Proposition 2.2], which handles the base case.

Note that the existence of a left $(n+3)$-chain from $f$ to $e$ is equivalent to the existence of some idempotent $g \in \mathrm{idem}(R)$ such that $f$ is connected by a left $(n+2)$-chain to $g$, and $g \sim_\ell e$ or $g \sim_r e$ (according to whether $n$ is even or odd). Equivalently, there exists some

$$z_{n+1} \in \begin{cases} (1-e)Re & \text{if } n+1 \text{ is odd} \\ eR(1-e) & \text{if } n+1 \text{ is even,} \end{cases}$$

such that $f$ is connected by a left $(n+2)$-chain to $g = (1+z_{n+1})e(1-z_{n+1})$. In other words, after conjugating there is a left $(n+2)$-chain from

$$(1-z_{n+1})f(1+z_{n+1}) = [(1-z_{n+1})b][a(1+z_{n+1})]$$

to $e = [a(1+z_{n+1})][(1-z_{n+1})b]$. This is equivalent, by applying our inductive hypothesis to the reflexive pair $a' = a(1+z_{n+1})$ and $b' = (1-z_{n+1})b$, to the existence of $z_1, z_2, \ldots, z_n$ belonging to $(1-e)Re$ or $eR(1-e)$, according to the parity of the subscripts, with

$$ea'(1+z_n)\cdots(1+z_2)(1+z_1)e = ea(1+z_{n+1})(1+z_n)\cdots(1+z_2)(1+z_1)e \in \mathrm{U}(eRe),$$

as desired. □

One can think of the previous theorem, when put in the context of the Peirce decompositions with respect to $e$, as identifying association chains with certain (standard) division chains. To motivate this identification, first recall that given a pair of elements $(x, d)$ from a ring $S$, thought of as a "numerator" and "denominator" respectively, then a *right division chain* consists of a sequence of "remainders" $r_1, r_2, \ldots \in S$ satisfying the recurrence

$$\begin{aligned} x &= dq_0 + r_1, \\ d &= r_1 q_1 + r_2, \\ r_1 &= r_2 q_2 + r_3, \\ &\ \vdots \end{aligned}$$

for some (right) "quotients" $q_0, q_1, \ldots \in S$. (This definition of a right division chain is similar to the definition given in [2, page 483].) Alternatively, we can think of the division chain giving rise to the sequence of pairs

$$(x, d) \mapsto (d, r_1) \mapsto (r_1, r_2) \mapsto \cdots .$$

Equivalently, if we set $x = r_{-1}$ and $d = r_0$, then we have the recurrence

$$(r_{n+1}, r_n) = (r_n, r_{n-1}) \begin{pmatrix} -q_n & 1 \\ 1 & 0 \end{pmatrix}.$$

If $r_n = 0$ for some $n \in \mathbb{N}$, then we say the division chain is *terminating*; the least such $n$ among all such chains is called the *termination length* of right division chains for $(x, d)$.

If $(x, d)$ is *right unimodular*, meaning $xS + dS = S$, and there exists some terminating division chain of termination length $n$, then $r_{n-1}$ is right invertible in $S$, and so we can force $r_{n+2} = 1$. Thus, we can identify terminating division chains for unimodular pairs either by exhibiting a zero remainder or a unit remainder. When $S$ is commutative, or even merely Dedekind-finite, such a unit remainder can be forced to occur exactly one step before the termination length of the division chain.

There are two issues with these standard division chains, arising from the form that the matrix $M_n = \left(\begin{smallmatrix} -q_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$ takes. This matrix doesn't make sense in general Morita contexts because, firstly, quotients cannot always be chosen to belong to the upper left entry. A second issue is that the nondiagonal corners of a Morita context might not possess identity elements. Thus, we will next describe (nonstandard) division chains in arbitrary contexts, using "elementary" matrices instead of the $M_n$. (Note that $M_n M_{n+1} = \left(\begin{smallmatrix} 1 & -q_n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -q_{n+1} & 1 \end{smallmatrix}\right)$.)

Assume we have some Morita context

$$R = \begin{pmatrix} S & P \\ Q & T \end{pmatrix}.$$

If $x \in S$ and $p \in P$, then an *even* division chain for the pair $(x, p)$ is of the form

$$\begin{aligned} x &= pq_0 + s_1, \\ p &= s_1 p_1 + p_2, \\ s_1 &= p_2 q_2 + s_3, \\ &\vdots \end{aligned}$$

for some $p_1, p_2, \ldots \in P$, $q_0, q_2, \ldots \in Q$, and $s_1, s_3, \ldots \in S$. In terms of pairs from $S \times P$, we have

$$(x, p) \mapsto (s_1, p) \mapsto (s_1, p_2) \mapsto (s_3, p_2) \mapsto \cdots .$$

Setting $p_0 = p$ and $s_{-1} = x$, this chain is described by the following recurrence using "elementary" matrices:

$$(s_{2n+1}, p_{2n}) = (s_{2n-1}, p_{2n}) \begin{pmatrix} 1 & 0 \\ -q_{2n} & 1 \end{pmatrix} \text{ and } (s_{2n+1}, p_{2n+2}) = (s_{2n+1}, p_{2n}) \begin{pmatrix} 1 & -p_{2n+1} \\ 0 & 1 \end{pmatrix}.$$

Notice that in this more general situation, each step of the division chain alternatively modifies the entry in $S$ or in $P$, and we must keep track of which is which. Also, we suppress calling this a *right* division chain; all "quotients" occur on the right because of the structure of the Morita context and the fact that our pairs come from $S \times P$.

An *odd* division chain for the pair $(x,p)$ is of the form

$$\begin{aligned} p &= xp_0 + p_1, \\ x &= p_1q_1 + s_2, \\ p_1 &= s_2p_2 + p_3, \\ &\ \vdots \end{aligned}$$

for some $p_0, p_1, \ldots \in P$, $q_1, q_3, \ldots \in Q$, and $s_2, s_4, \ldots \in S$. In terms of pairs from $S \times P$, we have

$$(x,p) \mapsto (x,p_1) \mapsto (s_2,p_1) \mapsto (s_2,p_3) \mapsto \cdots.$$

The reader is invited to similarly describe this chain using elementary matrices.

An even (respectively, odd) division chain is *terminating* if $p_k = 0$ for some even (respectively, odd) index $k$. The smallest such index is the *ordered termination length*.

Given any pair $(x,p) \in S \times P$, we say it is *right unimodular* if $xS + pQ = S$. For a right unimodular pair over a Dedekind-finite ring, having a zero remainder in $P$ is equivalent to saying that the previous remainder in $S$ is a unit.

We may now rephrase Theorem 3.1. Suppose we have reflexive inverses $a, b \in R$, and put $e = ab$ and $f = ba$, so that $e \cong f$. In the Peirce decomposition with respect to $e$, the matrix $a$ has the form $\left(\begin{smallmatrix} x & p \\ 0 & 0 \end{smallmatrix}\right)$ for some right unimodular pair $(x,p)$. To say that $f$ is connected by a left $(n+2)$-chain to $e$ (for some $n \geq 0$) means that we can reach a unit remainder in $S$ (after $n$ steps), described (in reverse order) using the elementary matrices

$$\begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix}, \ldots.$$

Thus the chain has ordered termination length at most $n+1$. If $S = eRe$ is Dedekind-finite, then, conversely, when $(x,p)$ has a division chain of ordered termination length $n+1$, the (negated) "quotients" take the place of the $z$'s in Theorem 3.1, showing that $f$ is connected to $e$ by a left $(n+2)$-chain.

Using the notations from the previous paragraph, the Peirce decomposition for $b$ has the form $\left(\begin{smallmatrix} y & 0 \\ q & 0 \end{smallmatrix}\right)$ and the pair $(y,q)$ is *left* unimodular. When $(x,p)$ has a terminating (right) division chain, then $(y,q)$ has a terminating (left) division chain, of ordered termination length at most one different. The ordered termination lengths can be different; indeed, this happens regularly in $\mathbb{M}_2(\mathbb{Z})$, for instance this occurs for the pairs $(x,p) = (2,3)$ and $(y,q) = (-1,1)$.

Putting this all together we obtain the following:

**Theorem 3.2.** *Let $n \in \mathbb{N}$, and let $S$ be a Dedekind-finite ring such that every nontrivial idempotent in $R = \mathbb{M}_2(S)$ is isomorphic to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Every left and every right unimodular pair from $S$ has a division chain of ordered termination length at most $n+1$ if and only if $R$ satisfies $\mathscr{P}(n+2)$.*

**Corollary 3.3.** *Let $K$ be a number field, let $X$ be a finite set of valuations on $K$ including the archimedean valuations, and let*

$$\mathscr{O}_X = \{x \in K : x = 0 \text{ or } \nu(x) \geq 0 \text{ for all } \nu \notin X\}$$

*be the ring of $X$-integers in $K$. If $\mathscr{O}_X$ has infinitely many units, then the ring $R = \mathbb{M}_2(\mathscr{O}_X)$ satisfies $\mathscr{P}(9)$, and under a generalized Riemann hypothesis it satisfies $\mathscr{P}(6)$.*

*Proof.* The ring $\mathscr{O}_X$ is a commutative domain, so it is Dedekind-finite, and the nontrivial idempotents in $R$ are all rank 1, hence isomorphic to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Thus, by Theorem 3.2, it suffices to find a bound of 8 (and 5 under GRH) on ordered termination lengths.

Let $(x,p) \in \mathscr{O}_X^2$ be any unimodular pair. If we know that $(p,x)$ has a standard division chain with odd (unordered) termination length, then this easily translates into an odd (nonstandard) division chain for $(x,p)$ with equal ordered termination length. Similarly, if $(x,p)$ has a standard division chain with even termination length, there is an even (nonstandard) division chain for $(x,p)$ with the same ordered termination length.

Thus, it suffices to find the appropriate bound on (unordered) termination lengths of standard division chains for all (ordered) unimodular pairs. Under GRH this follows from [2, Theorem 2.2]. Without GRH, we know from [20, Theorem 1.1] that every matrix in $\mathrm{SL}_2(\mathscr{O}_X)$ is a product of at most 9 elementary matrices. Thus, every unimodular row is of the form

$$\begin{pmatrix}x & p\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix} U_1 U_2 \cdots U_9$$

for some elementary matrices $U_1, U_2, \ldots, U_9$. Moreover, as noted in [7], the proof given for [20, Theorem 1.1] actually allows us to choose $U_1$ to be lower triangular, and so we have $\begin{pmatrix}1 & 0\end{pmatrix} U_1 = \begin{pmatrix}1 & 0\end{pmatrix}$. Thus $\begin{pmatrix}x & p\end{pmatrix} U_9^{-1} U_8^{-1} \cdots U_2^{-1} = \begin{pmatrix}1 & 0\end{pmatrix}$. This corresponds to a division chain for $(x,p)$ that has termination length at most 8. □

**Remark 3.4.** Example 2.12 in [2] shows that there is a ring $\mathscr{O}_X$ as in Corollary 3.3 that does not satisfy $\mathscr{P}(5)$.

**Corollary 3.5.** *Let $S$ be a nontrivial localization of $\mathbb{Z}$. Then $R = \mathbb{M}_2(S)$ satisfies $\mathscr{P}(5)$, and it satisfies $\mathscr{P}(4)$ under GRH.*

*Proof.* If $S$ is of the form $\mathbb{Z}[1/q]$ for some *prime* $q > 0$, then [24, Theorem 1.1 and Remark 1.1] tells us that every matrix in $\mathrm{SL}_2(S)$ is a product of (at most) five elementary matrices, with the first lower triangular. Thus, by the same argument used in the previous proof, every unimodular pair has a terminating division chain of length 4. If we assume GRH, then the result [2, Theorem 2.14] applies instead.

More generally, suppose that $S$ is any nontrivial localization. Then $S$ is a localization of $S' = \mathbb{Z}[1/q]$ for some prime $q > 0$. Thus, any unimodular pair $(x,p) \in S^2$ can be written in the form $(x'u^{-1}, p'u^{-1})$ for some unimodular pair $(x',p') \in S'^2$ and some $u \in \mathrm{U}(S)$. Since there is a terminating division chain for $(x',p')$ in $S'$, of the appropriate length (by the work in the previous paragraph), the same holds true for $(x'u^{-1}, p'u^{-1})$ in $S$, using exactly the same quotients from $S'$. □

It is not difficult to find explicit rings $S$ such that $R = \mathbb{M}_2(S)$ satisfies $\mathscr{P}(n)$, but not $\mathscr{P}(n-1)$, for each of $n = 4, 5, 6$. We have been unable to find a single example satisfying $\mathscr{P}(n)$, but not $\mathscr{P}(6)$, for any $n > 6$. We expect GRH to hold true, and so by Corollary 3.3 we then cannot expect any such example to be a $2 \times 2$ matrix ring over a (special type of) localization of a ring of integers over a number field. However, we still expect examples of another kind to exist. Indeed, we raise the following:

**Conjecture 3.6.** For each integer $n > 6$, there exists a ring $R$ satisfying $\mathscr{P}(n)$ but not $\mathscr{P}(n-1)$.

Any universal bound on the lengths of division chains, for unimodular pairs over a commutative ring $S$, gives a bound on the number of elementary matrices needed to generate

$\mathrm{SL}_2(S)$. By the proof of [2, Corollary 2.3], the latter bound is at most 4 greater, which was improved in [7, Theorem 3.6] to at most 2 greater.

In the proofs of the previous two corollaries, we used, to great effect, the extra condition that the first matrix in a product of elementary matrices could be chosen to be lower triangular. Via transposition, this extra condition roughly corresponds to uniform bounds on left division chains translating to right division chains, and vice versa. Surprisingly, there are some noncommutative situations where a universal bound on the lengths of right division chains implies the same bound on the lengths of left division chains. To prove this, we start with the following general lemma:

**Lemma 3.7.** *Let $e, f \in \mathrm{idem}(R)$. If $e \sim_{\ell r\ell} g := f + fe(1-f)$, then $e \sim_{\ell r\ell} f$ and $e \sim_{r\ell r} f$.*

*Proof.* Assume that $e \sim_{\ell r\ell} g$. Since $g \sim_r f$, we have $e \approx f$. In particular, $e$ and $f$ are isomorphic. Fix reflexive inverses $a, b \in R$ with $ab = e$ and $ba = f$.

Write the Peirce decompositions of $a$ and $b$, with respect to $e$, as

$$a = \begin{pmatrix} x & p \\ 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} y & 0 \\ q & 0 \end{pmatrix}.$$

Note that $ab = e$ is equivalent to

$$xy + pq = e.$$

We find that

$$g = f + fe(1-f) = ba + bae(1-ba) = b(a + ae(1-ba)).$$

Set $a' := a + ae(1-ba)$. A quick matrix computation shows us that

$$a' = \begin{pmatrix} x & p \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x & p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e - yx & -yp \\ -qx & (1-e) - qp \end{pmatrix} = \begin{pmatrix} 2x - xyx & pqp \\ 0 & 0 \end{pmatrix},$$

using the fact that $p - xyp = pqp$, which follows from $xy + pq = e$.

Since $bab = b$, we see that

$$a'b = ab + ae(1-ba)b = ab = e.$$

Also, since $a'$ has a zero second row, we have $a' = ea' = a'ba'$. On the other hand $ba'b = be = b$, and so we see that $a'$ and $b$ are a pair of reflexive inverses.

By Theorem 3.1, applied to $a'$ and $b$, there exists some $z \in (1-e)Re$ such that $ea'(1+z)e \in \mathrm{U}(eRe)$. Equivalently, $(2x - xyx) + pqpz \in \mathrm{U}(eRe)$. In other words

$$x + (e - xy)x + pqpz = x + pqx + pqpz = x + pqv \in \mathrm{U}(eRe)$$

with $v = x + pz \in eRe$. Applying [15, (1.25)], there exists some $w \in eRe$ such that $y + wpq \in \mathrm{U}(eRe)$. Taking $z' = wp$ we have that $e(1+z')be \in \mathrm{U}(eRe)$, so applying (the left-right dual of) Theorem 3.1 we get $e \sim_{r\ell r} f$.

Similarly, since $qx + qpz \in (1-e)Re$ and

$$ea(1 + (qx + qpz))e = x + p(qx + qpz) \in \mathrm{U}(eRe),$$

then another application of Theorem 3.1 yields $e \sim_{\ell r\ell} f$. □

This lemma has some important consequences.

**Theorem 3.8.** *Let $R$ be a ring, let $e \in \mathrm{idem}(R)$, and let $X$ denote the $\approx$-equivalence class of $e$. If every element of $X$ is connected to $e$ either by a left or a right 3-chain, then every pair of elements in $X$ is connected by both a left and a right 3-chain.*

*Proof.* Let $f \in X$ be arbitrary. Fixing $g := f + fe(1-f)$, then since $g \sim_r f$, we have $g \in X$. By the 3-chaining hypothesis, we have either $e \sim_{r\ell r} g$ or $e \sim_{\ell r\ell} g$. In the first case, since $g \sim_r f$, we can combine the chains to get $e \sim_{r\ell r} f$. In the second case, Lemma 3.7 immediately yields $e \sim_{r\ell r} f$.

Thus, in all cases, $e \sim_{r\ell r} f$. By symmetry considerations, we also have $e \sim_{\ell r\ell} f$. We have thus shown that $e$ is connected to every other element of $X$ by both a left and a right 3-chain. Since $e$ is conjugate to every other element of $X$, and since conjugation is an isomorphism on $R$ that preserves $\approx$-equivalence classes, this chaining property holds for every other element of $X$. □

Given $e, f \in \mathrm{idem}(R)$, recall that the summands $eR$ and $fR$ are *perspective* if they share a common complement. Equivalently, by [4, Lemma 6.3], $e$ and $f$ are connected by a right 3-chain. Consequently, perspective summands must be isomorphic.

On the level of rings, we say that $R$ is *perspective* when any two isomorphic summands of $R_R$ are perspective. This is a left-right symmetric property by [5, Theorem 3.3]. In terms of association chains, by Proposition 2.1(3) a ring $R$ is perspective if and only if any pair of isomorphic idempotents are connected by both a left and a right 3-chain. Theorem 3.8 allows us to simplify this criterion, as follows.

**Corollary 3.9.** *A ring $R$ is perspective if and only if any two isomorphic idempotents are connected by either a left or a right 3-chain.*

We were unable to answer the question of whether or not this result generalizes to longer chains, and so we ask to following:

**Question 3.10.** Let $R$ be a ring. If any two isomorphic (or merely $\approx$-related) idempotents of $R$ are either left or right 4-chained, must they be both left and right 4-chained?

If we replace 3-chains by 2-chains in Theorem 3.8, then the result remains true; this follows from [11, Theorem 3.10]. The proof given there is quite complicated, but using the methods above we can give an alternative, simpler argument.

**Lemma 3.11.** *Let $e, f \in \mathrm{idem}(R)$. If $e \sim_{r\ell} g := f + f(1-e)(1-f)$, then $e \sim_{\ell r} f$ and $e \sim_{r\ell} f$.*

*Proof.* Assume $e \sim_{r\ell} g$. As in the proof of Lemma 3.7, fix reflexive inverses $a$ and $b$ with $e = ab$ and $f = ba$, and write their Peirce decompositions as before. Now, since

$$g = f + f(1-e)(1-f) = b(a + a(1-e)(1-ba))$$

we can set $a' := a + a(1-e)(1-ba)$, and a quick matrix computation shows us that

$$a' = \begin{pmatrix} xyx & 2p - pqp \\ 0 & 0 \end{pmatrix}.$$

By calculations that are only slightly different than those done in Lemma 3.7, we get that $a'$ and $b$ are reflexive inverses with $e = a'b$ and $g = ba'$.

By Theorem 3.1, applied to $a'$ and $b$, then $ea'e \in \mathrm{U}(eRe)$. Equivalently, $xyx \in \mathrm{U}(eRe)$. Thus, $x$ is both left and right invertible in $eRe$, so $eae = x \in \mathrm{U}(eRe)$. By Theorem 3.1 (applied to $a$ and $b$), this says that $e \sim_{r\ell} f$. On the other hand, $y = x^{-1}(xyx)x^{-1} \in \mathrm{U}(eRe)$, or in other words $ebe \in \mathrm{U}(eRe)$. Applying Theorem 3.1 again, we get $e \sim_{\ell r} f$. □

**Theorem 3.12.** *Let $R$ be a ring, let $e \in \mathrm{idem}(R)$, and let $X$ denote the $\approx$-equivalence class of $e$. If every element of $X$ is connected to $e$ either by a left or a right 2-chain, then every idempotent that is isomorphic to $e$ is in $X$, and every pair of elements in $X$ is connected by both a left and a right 2-chain.*

*Proof.* First, we claim that any element of $X$ is connected by both a left and right 2-chain to $e$. This follows, *mutatis mutandis*, from the proof of Theorem 3.8.

With this claim established, using the proof of [11, Theorem 3.10], specifically the implication (3) $\Rightarrow$ (1) restricted to Case 1, we conclude that $e$ is central modulo the Jacobson radical of $R$. Let $f \in \mathrm{idem}(R)$ be any idempotent isomorphic to $e$. By the argument used to prove the implication (1) $\Rightarrow$ (2) in [11, Theorem 3.10], we know that $f$ is connected to $e$ by both a left and right 2-chain. Thus, $X$ contains all idempotents isomorphic to $e$, and they are all left and right 2-chained to $e$.

The same facts hold for any other element of $X$, by conjugating as necessary. $\square$

Theorem 3.12 has a stronger conclusion than Theorem 3.8, as it includes idempotents isomorphic to $e$, not merely the association chained idempotents. However, Theorem 3.8 cannot be improved to have a similar conclusion; by [19, Theorem 1.1] there exists a ring where perspectivity is transitive (so the association chained idempotents are left and right 3-chained by [4, Theorem 6.7]), but isomorphic (and even conjugate) idempotents are not always perspective.

If $R$ is the endomorphism ring of a module $M$, then by Proposition 2.1(2), the conclusion of Theorem 3.12 can be rephrased as saying that isomorphic summands of $M$ share all their complements. Such modules are called *strongly perspective* [6] and also 1/2-*perspective* [18]. In answer to a question posed in personal communication by T. Y. Lam, we find that Lemma 3.11 reveals the following interesting module-theoretic result, which also shows that the condition defining strongly perspective modules may be weakened.

**Proposition 3.13.** *Let $M$ be a module, and let $A$ and $B$ be direct summands of $M$. If all of the complements of $A$ are also complements of $B$, then the reverse is true.*

*Proof.* Let $R = \mathrm{End}(M)$, and fix direct sum decompositions $M = A \oplus X = B \oplus Y$. It suffices to show that $Y$ is a complement to $A$.

Idempotents are right associate exactly when their images are the same, and they are left associate exactly when their kernels are the same, by [22, Lemma 4.2]. We will use these facts freely, and repeatedly.

Let $f \in \mathrm{idem}(R)$ be the idempotent whose image is $A$ and whose kernel is $X$, and similarly define $e \in \mathrm{idem}(R)$ as the idempotent whose image is $B$ and whose kernel is $Y$. Construct the idempotent $g \in \mathrm{idem}(R)$ as in the proof of Lemma 3.11. In particular, $g \sim_r f$ and so $g(M) = f(M) = A$. Thus, setting $X' = (1-g)(M)$, our hypothesis tells us that $X'$ is a complement to $B$. Letting $h'$ be the idempotent whose image is $B$ and whose kernel is $X'$, we then have $e \sim_r h' \sim_\ell g$. Thus, Lemma 3.11 yields $e \sim_{\ell r} f$; so we can fix an idempotent $h''$ such that $e \sim_\ell h'' \sim_r f$. We then have $h''(M) = f(M) = A$ and $(1-h'')(M) = (1-e)(M) = Y$. This shows that $Y$ is a complement to $A$, as desired. $\square$

Readers who are interested in connecting the work in this section with continuant polynomials are directed to Section 2.7 in [1].

## 4. Elementary matrices

For this section, let $S$ be a ring, and let $k$ and $n$ be positive integers. All matrices will be understood as matrices with entries in $S$. Let $A$ be a $k\times(k+n)$ matrix. Write $A=\begin{pmatrix}A' & A''\end{pmatrix}$ where $A'$ is a $k\times(k+n-1)$ matrix and $A''$ is a $k\times 1$ single column matrix.

We say that $A$ is (right) *unimodular* if there exists some $(k+n)\times k$ matrix $B$ such that $AB=I_k$ (the $k\times k$ identity matrix). When $A$ is unimodular, we further say that the columns of $A$ are *reducible* if there exists some $1\times(k+n-1)$ single row matrix $Z$ such that $A'+A''Z$ is unimodular.

Recall, that a ring $S$ has *$n$ in its stable range* if every (right) unimodular matrix of size $1\times(1+n)$ is reducible. By the left-right analog of [23, Theorem 3′] this is equivalent to saying that every unimodular matrix of size $k\times(k+n)$ is reducible (for each $k\geq 1$).

It is well known that if $S$ has $n$ in its stable range, and $m\gg n$, then $\mathrm{SL}_m(S)$ is generated by a bounded number of elementary matrices. For a good overview of this topic see [3]. In particular, the ring $\mathbb{Z}$ has stable range 2, and $\mathrm{SL}_m(\mathbb{Z})$ has bounded generation for each $m\geq 3$. We approach a similar question, but rather than focus on bounded generation by elementary matrices, we focus instead on small bounds for association chains. We first need a couple of lemmas.

**Lemma 4.1.** *Let $k,m,n$ be positive integers with $m\geq n+2k-1$, and let $S$ be a ring with $n$ in its stable range. Let $E,F\in\mathrm{idem}(\mathbb{M}_m(S))$, with $E=\left(\begin{smallmatrix}I_k & 0\\ 0 & 0\end{smallmatrix}\right)$. If $E\cong F$, then $E$ is both left and right 4-chained to $F$.*

*Proof.* By symmetry considerations (noting that the condition of having stable range $n$ is left-right symmetric by [23, Theorem 2]), it suffices to show $E\sim_{r\ell r\ell}F$. Fix a pair of reflexive inverses $A,B\in\mathbb{M}_m(S)$ with $AB=E$ and $BA=F$. We may write

$$A=\begin{pmatrix}A' & A''\\ 0 & 0\end{pmatrix},$$

where $A'$ is $k\times k$ and $A''$ is $k\times(m-k)$. Further, since $AB=E$, we see that $\begin{pmatrix}A' & A''\end{pmatrix}$ is unimodular. Applying the stable range condition $k$ times, there exists some $k\times(m-k)$ matrix $X$ such that $A'X+A''$ is unimodular. Thus, we can fix some $(m-k)\times k$ matrix $Y$ such that $(A'X+A'')Y=I_k-A'$. In particular, $A'+(A'X+A'')Y\in\mathrm{U}(\mathbb{M}_m(S))$. By Theorem 3.1, taking $z_1=\left(\begin{smallmatrix}0 & 0\\ Y & 0\end{smallmatrix}\right)$ and $z_2=\left(\begin{smallmatrix}0 & X\\ 0 & 0\end{smallmatrix}\right)$, we have what we needed. $\square$

We now play with complement idempotents. We require an additional assumption that our ring is projective-free, meaning that it has the invariant basis property, or IBN for short (see [13, Section 1]) and that all finitely generated projective modules are free. This assumption will guarantee that the isomorphism type of an idempotent in a matrix ring is uniquely determined by its rank.

**Lemma 4.2.** *Let $m,n$ be positive integers with $m\geq 2n-2$ and $n\geq 2$, and let $S$ be a projective-free ring with $n$ in its stable range. If $E\in\mathbb{M}_m(S)$ is an idempotent of rank $2n-2$ or larger, then it is both left and right 4-chained to any other idempotent of the same rank.*

*Proof.* We will handle the case where the rank is exactly $2n-2$, leaving it to the reader to check that the proof can be easily modified to work for any larger rank. The projective-free condition has, as part of its definition, the hypothesis that $S$ has IBN. Hence, it is easy to then see that the monoid of isomorphism classes of finitely generated projective modules is cancellative. So every idempotent of a given rank is conjugate to every other. Thus, it suffices to take $E, F \in \mathrm{idem}(\mathbb{M}_m(S))$ with $E = \left(\begin{smallmatrix} I_{2n-2} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $E \cong F$. Fix a pair of reflexive inverses $A, B \in \mathbb{M}_m(S)$ with $AB = E$ and $BA = F$.

If $m \leq 3n-3$, then the complement of a rank $2n-2$ idempotent has rank $k = m-2n+2$. We compute $n + 2k - 1 = 2m - 3n + 3 \leq m$, so Lemma 4.1 says that the complements of $E$ and $F$ are both left and right 4-chained. Hence, so are $E$ and $F$, by applying Lemma 6.2(3d) of [4] repeatedly.

Proceeding by way of induction, we may now suppose that $m > 3n-3$ and that the claim holds true for rank $2n-2$ idempotents in $\mathbb{M}_{m-1}(S)$. Write

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \end{pmatrix}$$

with $A_1$ of size $(2n-2) \times (2n-2)$, with $A_2$ of size $(2n-2) \times (m-2n+1)$, and $A_3$ of size $(2n-2) \times 1$. Since $\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$ is unimodular, the stable range condition lets us fix a $1 \times (2n-2)$ matrix $Z_1$ and a $1 \times (m-2n+1)$ matrix $Z_2$ such that $A' = \begin{pmatrix} A_1 + A_3 Z_1 & A_2 + A_3 Z_2 \end{pmatrix}$ is unimodular, say $A'B' = I_{2n-2}$ for some $(m-1) \times (2n-2)$ matrix $B'$. Thus, after extending by 0's, the matrices $A', B'$ give inverse isomorphisms between idempotents of rank $2n-2$ in $\mathbb{M}_{m-1}(S)$. Thus, by the inductive hypothesis, there exist matrices $X$ and $Y$ (of the appropriate sizes) so that $(A_1 + A_3 Z_1) + ((A_1 + A_3 Z_1)X + (A_2 + A_3 Z_2))Y \in \mathrm{U}(\mathbb{M}_{2n-2}(S))$. This can be rewritten as

$$A_1 + (A_1 X + A_2)Y + (A_1 0 + A_3)Z \in \mathrm{U}(M_m(S))$$

with $Z = Z_1 + Z_1 XY + Z_2 Y$. Applying Theorem 3.1 with

$$z_1 = \begin{pmatrix} 0 & 0 \\ Y & 0 \\ Z & 0 \end{pmatrix} \text{ and } z_2 = \begin{pmatrix} 0_{(2n-2)\times(2n-2)} & X & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(where the 0 matrices have the appropriate sizes to guarantee $z_1, z_2 \in \mathbb{M}_m(S)$) yields a left 4-chain from $F$ to $E$. A right 4-chain exists, similarly, by symmetry considerations. □

**Theorem 4.3.** *Let $n \geq 2$ be an integer, and assume that $S$ is a projective-free ring with $n$ in its stable range. If $m \geq 4n-5$, then $R = \mathbb{M}_m(S)$ satisfies $\mathscr{P}(4)$. If $S$ does not have* 1 *in its stable range, then $\mathscr{P}(3)$ fails (i.e., $R$ is not a perspective ring).*

*Proof.* Let $E, F \in \mathrm{idem}(R)$ with $E \cong F$. If the rank of $E$ is at least $2n-2$, then $E$ and $F$ are connected by left and right 4-chains by Lemma 4.2. If the rank of $E$ is smaller than $2n-2$, the complement idempotents $I_m - E$ and $I_m - F$ have the same rank, which is at least $m - (2n-3) \geq 2n-2$. Thus $I_m - E$ and $I_m - F$ are connected by left and right 4 chains by Lemma 4.2. This in turn implies that $E$ and $F$ are so connected, by applying Lemma 6.2(3d) of [4] repeatedly.

The statement about stable range 1 is a quick consequence of [12, Theorem 2.5] (or alternatively [19, Proposition 4.1]) and the fact that $\mathscr{P}(3)$ is equivalent to perspectivity. □

Suppose $S$ is a projective-free ring with stable range $n = 2$. This just leaves the $m = 1$ and $m = 2$ cases undecided. The $m = 1$ case is trivial (there are only the trivial idempotents). When $S = \mathbb{Z}$, which is a ring of stable range 2, we know that there is no finite bound on association chains in $\mathbb{M}_2(\mathbb{Z})$. Thus, in this case we have an interesting dichotomy, as described in the following corollary. For that result, recall that a ring element $a \in R$ is *special clean* when we can write $a = e + u$ for some idempotent $e \in \mathrm{idem}(R)$ and some unit

$u \in U(R)$ with $au^{-1}a = a$. Thus, this is a common strengthening of both the (unit-)regular and cleanness conditions. The theory of such elements is developed in [10] and [11].

**Corollary 4.4.** *Given $m \in \mathbb{N}$, all regular elements in $\mathbb{M}_m(\mathbb{Z})$ are special clean if and only if $m \neq 2$.*

*Proof.* When $m = 2$, there are many unit-regular matrices that are not clean, as shown in [8]. The other direction follows from Theorem 4.3 and Proposition 2.1(4). □

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