ZERO-DIVISOR PLACEMENT, A CONDITION OF CAMILLO, AND THE MCCOY PROPERTY

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Dedicated to Vic Camillo for his 75th birthday

Abstract. The rings for which any polynomial with a nonzero right annihilator must have a nonzero constant right annihilator are called the right McCoy rings. This class of rings includes the duo, reversible, polynomially semicommutative, and Armendariz rings, among others. In this paper we introduce a new condition, strictly generalizing the reversible property, which still implies the McCoy condition. We call this new condition the outer McCoy property; it arises from guaranteeing annihilators in unexpected places.

This outer McCoy condition is further motivated by a property of 2-primal rings, which we call the Camillo property, first noticed by Victor Camillo and the fourth author. We study the relationships between the outer McCoy property, the Camillo property, and other standard ring-theoretic conditions, with many examples delimiting their connections. For instance, we show that any ring whose set of nilpotents is closed under multiplication must satisfy the Camillo property when restricted to linear polynomials.

1. Introduction

For a commutative ring R, McCoy [14, Theorem 2] proved:

(1.1) If $f, g \in R[x]$ and $fg = 0$ but $g \neq 0$, then there is some nonzero element $r \in R$ such that $fr = 0$.

This result may fail without the commutativity condition on R, in particular when R is a nonzero 2×2 matrix ring, as shown by Weiner [19]. Following [17], the rings which satisfy (1.1) are called the *right McCoy rings*. The *left McCoy rings* are defined in a left-right symmetric fashion, and rings which are both left and right McCoy are simply called *McCoy rings*.

There are multiple ways in which McCoy's theorem can be generalized by weakening the commutativity condition. For instance, all *reversible rings* are McCoy rings by [16, Theorem 2]. The reversible rings are those rings R where $ab = 0$ entails $ba = 0$ for any $a, b \in R$.

To describe another class of examples, recall that a ring is *right duo* if every right ideal is also a left ideal. The *left duo rings* are defined symmetrically, and rings with both conditions are called *duo rings*. By [3, Theorem 8.2], right duo rings are right McCoy, but it remains an open question whether right duo rings are left McCoy.

Encompassing both the reversible and one-sided duo rings are the *semicommutative rings*, also called zero insertive rings (ZI), the (S I) rings, and rings satisfying the insertion-of-factors-principle (IFP). They are defined by the following property:

$$\text{If } a, b \in R \text{ and } ab = 0, \text{ then } aRb = 0.$$

2010 *Mathematics Subject Classification.* Primary 16U80, Secondary 16S15, 16S20, 16S36.

Key words and phrases. Camillo ring, duo ring, (outer) McCoy ring, NI ring, reversible ring, semicommutative ring, 2-primal ring.

Surprisingly, some semicommutative rings are not McCoy, by [16, Section 3]. However, if $R[x]$ is semicommutative, then R is a McCoy ring, which is an easy corollary of another result of McCoy found in [15].

There are other natural conditions that imply the McCoy property, not related directly to commutativity. For instance, if R has no nonzero nilpotent elements (that is, R is a *reduced ring*), then R is reversible and $R[x]$ is semicommutative, hence R is a McCoy ring. Reduced rings satisfy the even stronger *Armendariz condition* on polynomials introduced by Rege and Chhawchharia [17]; namely, given

$$f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x],$$

then

$$fg = 0 \text{ implies } a_i b_j = 0 \text{ for all } i, j \geq 0.$$

Armendariz rings are clearly McCoy. It is also interesting to note that if R is semicommutative and Armendariz, then $R[x]$ is semicommutative; see [17, Proposition 4.6].

In this paper we study a new sufficient condition for the McCoy property, based on the placement of zero-divisors, which we call the *outer McCoy* condition. We introduce the definition in Section 2, and provide a number of examples showing its relationship to the properties discussed above. Then in Section 3 we study a weakening of the McCoy property, which we call the *Camillo property*, that—unlike the McCoy property—is satisfied by the semicommutative rings. In retrospect, it provides additional impetus for studying the outer McCoy rings.

All rings in this paper are associative, and they are unital unless we specify otherwise. However, many of our results hold for nonunital rings as well. Given any polynomial $f \in R[x]$ and any integer $i \in \mathbb{Z}$, we let $f[i]$ be the degree i coefficient (which is zero both for $i < 0$ and for $i > \deg(f)$). The terminology in this paper primarily follows that given in [10]. In particular, the definitions and basic facts for the types of rings discussed above, and others to be introduced later, can mostly be found in [10] or in [3]. By a *left zero-divisor* of R, we mean an element $r \in R$ such that $rs = 0$ for some nonzero element $s \in R$; this differs from the definition given in [10] in that we allow 0 to be a zero-divisor (except in the zero ring).

Many of our results and examples have a left-right symmetric analog, by passing to the opposite ring. For instance, the fact mentioned above that right duo rings are right McCoy automatically implies that left duo rings are left McCoy, and also that duo rings are McCoy. We leave such trivial observations unstated but implicitly understood.

2. Outer McCoy Rings

Let R be a ring, and let $f, g \in R[x]$ with $fg = 0$ and $g \neq 0$. The McCoy condition says that f is annihilated on the right by a nonzero constant polynomial. In a commutative ring, this further means that f is annihilated by a nonzero constant on the left.

Without the commutativity condition one may suspect that it is unlikely that f has any left annihilators, let alone a constant one. However, as we'll see in this and the next section, there are some very natural (noncommutative) situations where the *placement* of constant annihilators happens unexpectedly. Thus, it is worthwhile exploring the consequences of such placement.

We say that R is a *left outer McCoy ring* if the following condition holds:

$$\text{If } f, g \in R[x] \text{ and } fg = 0 \text{ but } g \neq 0 \text{, then there is some nonzero element } r \in R \text{ such that } rf = 0. \tag{2.1}$$

The "left" in "left outer McCoy" comes from the fact that the annihilator r occurs to the left of f. Similarly, the word "outer" comes from the fact that r is placed on the opposite side from g. We follow the usual left-right symmetry conventions in defining (*right*) *outer McCoy rings*.

Our first couple of results connect the outer McCoy conditions to some other standard conditions.

Proposition 2.2. *Reversible rings are outer McCoy. In particular, commutative and reduced rings are outer McCoy.*

Proof. By symmetry, it suffices to show that a reversible ring R is left outer McCoy. Let $f, g \in R[x]$ with $fg = 0$ and $g \neq 0$. As noted previously, reversible rings are McCoy by [16, Theorem 2], so we can fix some nonzero $r \in R$ such that $fr = 0$. In other words, r annihilates each coefficient of f from the right. The reversibility condition then entails that $rf = 0$. □

Proposition 2.3. *Outer McCoy rings are McCoy.*

Proof. By symmetry (once again), it suffices to show that an outer McCoy ring R is right McCoy. Let $f, g \in R[x]$ with $fg = 0$ and $g \neq 0$. By the left outer McCoy condition, there exists some nonzero constant polynomial $h \in R \subseteq R[x]$ such that $hf = 0$. Now, applying the right outer McCoy condition to this new zero-product, there exists some nonzero $r \in R$ such that $fr = 0$. □

Generalizing the terminology of [5], call R a *left eversible ring* if every left zero-divisor is a right zero-divisor. (These rings are also called right regular-duo in [9].) In other words, if an element $r \in R$ can be annihilated by a nonzero element from the right, then it can also be similarly annihilated nontrivially from the left.

Proposition 2.4. *Left outer McCoy rings are left eversible. Also, left eversible rings are Dedekind-finite, that is, every one-sided unit is a two-sided unit.*

Proof. The first sentence follows tautologically, by considering the left outer McCoy condition relative to constant polynomials.

The second sentence is proved in [9, Proposition 2.1(4)]; also see [5, Proposition 2.13]. We include the easy argument for completeness. Let R be a left eversible ring, and let $u, v \in R$ with $uv = 1$. Note that u cannot be a right zero-divisor since $xu = 0$ implies $x = xuv = 0$. Thus, u cannot be a left zero-divisor, by the contrapositive of the left eversible property. On the other hand $u(1 - vu) = 0$, and hence $1 - vu = 0$. Therefore, $1 = vu$. Thus, u and v are two-sided inverses of each other. □

The remainder of this section will be devoted to showing non-implications between the outer McCoy properties and other standard conditions in rings. We will pay special interest to semicommutative examples. By Proposition 2.2, we know that reversible rings are outer McCoy. The next example shows that many of the other conditions which imply the McCoy property do *not* imply the outer McCoy property.

Example 2.5. *There exists a duo, Armendariz ring that is right outer McCoy* (*and whose polynomial ring is semicommutative*)*, but the ring is not left eversible, and hence not left outer McCoy.*

Construction and proof. Our example is the ring $\overline{R}$ constructed in [20, Section 2]. We will assume the reader is familiar with that paper and its notations, which we quickly review here (the exact details take about three pages of work in [20], which we will not fully repeat).

Let G be the free abelian group, written multiplicatively, generated by the set $\{x_i : i \in \mathbb{Z}\}$ and let ψ be the endomorphism of G sending $x_i \mapsto x_{i+1}$. Further, define an order $\preceq$ on G by the rule that $g_1 \preceq g_2$ exactly when the largest subscript $k \in \mathbb{Z}$ such that x_k appears with a nonzero exponent in $g_1^{-1}g_2$ must appear with a positive exponent (if such a k exists).

Next, let $T \subseteq \mathbb{Z} \times G$ be the subset of ordered pairs (m, g) with either $m \geq 1$, or $m = 0$ and $1 \preceq g$. We make T a monoid, with multiplication given by

$$(m_1, g_1)(m_2, g_2) = (m_1 + m_2, \psi^{m_2}(g_1)g_2).$$

This monoid is ordered by $\leq$, where

$$(m_1, g_1) \leq (m_2, g_2) \text{ if and only if either } m_1 < m_2, \text{ or } m_1 = m_2 \text{ and } g_1 \preceq g_2.$$

With D a division ring, and $R = D[\![T]\!]$ the generalized power series ring over T with coefficients from D, then for $f \in R$ let $\pi(f)$ be the unique minimal element in the support of f. Finally, let

$$I = \{0\} \cup \{f \in R \setminus \{0\} : \pi(f) > (1, x_1^i x_2^j x_3) \text{ for any } i, j \in \mathbb{Z}\}$$

be the ideal of R given in the paper, and put $\overline{R} = R/I$.

The fact that $\overline{R}$ is a duo chain ring was proved directly in that paper. Chain rings are right distributive, and by [13] (or even by [12, Corollary 6.3]) we know such rings are Armendariz. The parenthetical statement, that $\overline{R}[x]$ is semicommutative, now follows from the fact that duo rings are semicommutative, and Armendariz semicommutative rings have semicommutative polynomial rings by [17, Proposition 4.6].

Next, we find

$$(0, x_2)(1, x_3) = (0 + 1, \psi^1(x_2)x_3) = (1, x_3^2) \in I$$

so $\overline{(0, x_2)}$ is a left zero-divisor in $\overline{R}$, since $(1, x_3) \notin I$. However, for any $m \in \mathbb{Z}$ and $g \in G$ with $(m, g) \notin I$, then either $m = 0$ or $g \preceq x_1^i x_2^j x_3$ for some $i, j \in \mathbb{Z}$. Hence

$$(m, g)(0, x_2) = (m + 0, \psi^0(g)x_2) = (m, gx_2) \notin I$$

because either $m = 0$ or $gx_2 \preceq x_1^i x_2^{j+1} x_3$. Thus, $\overline{(0, x_2)}$ is not a right zero-divisor in $\overline{R}$. This proves that R is not left eversible, and hence not left outer McCoy.

Finally, we prove that $\overline{R}$ is right outer McCoy. Fix $a, b \in R[x] \setminus \{0\}$ with $ab \in I[x]$ and $a \notin I[x]$. Fix some integer $\ell \geq 0$ so that $a[\ell] \notin I$. From the Armendariz property, we know $\pi(a[\ell])b \in I[x]$. Let $(m, g) \in T$ be the $\preceq$-smallest element in the support of any coefficient of b. Then $\pi(a[\ell])(m, g) \in I$.

It suffices to prove that there exists some $t \in T \setminus I$ with $(m, g)t \in I$, for then $bt \in I[x]$. In other words, we have reduced to showing that R is right eversible. This follows from the proof of [20, Claim 2.5], where it is shown (by replacing the words "neither right nor" with "not a" in the first sentence) that any element which is not a left zero-divisor of $\overline{R}$ is not a right zero-divisor. □

To capitalize further on Example 2.5, we need the following easy result.

Lemma 2.6. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. Then R is left outer McCoy if and only if R_j is left outer McCoy for each $j \in I$.*

Proof. ($\Rightarrow$): Fix $j \in I$ and suppose $f_j, g_j \in R_j[x]$ with $g_j \neq 0$ and $f_j g_j = 0$. Taking $f := (f_i)_{i \in I}$ where $f_i = 1$ if $i \neq j$, and taking $g := (g_i)_{i \in I}$ with $g_i = 0$ if $i \neq j$, then $fg = 0$ and $g \neq 0$. By the left outer McCoy property, there exists some nonzero $r = (r_i)_{i \in I} \in R$ with $rf = 0$. However, for $i \neq j$, we find

$$r_i = r_i \cdot 1 = r_i f_i = 0,$$

and hence $r_j \neq 0$. Further $r_j f_j = 0$, showing that R_j is left outer McCoy.

($\Leftarrow$): Let $f = (f_i)_{i \in I}, g = (g_i)_{i \in I} \in R$ with $fg = 0$ and $g \neq 0$. Fix an index $j \in I$ with $g_j \neq 0$. Then $f_j g_j = 0$ and so the left outer McCoy condition, applied in the jth coordinate ring, guarantees the existence of some nonzero $r_j \in R_j$ such that $r_j f_j = 0$. Taking $r = (r_i)_{i \in I}$ where $r_i = 0$ when $i \neq j$, then $rf = 0$, showing that R is left outer McCoy. □

Remark 2.7. The previous lemma is true after replacing "left outer McCoy" with "left McCoy" by [3, Lemma 4.1].

Corollary 2.8. *There exists a duo, Armendariz ring that is neither left nor right outer McCoy.*

Proof. Let $\overline{R}$ be the ring from Example 2.5, and let $\overline{R}^{\text{op}}$ be its opposite ring. We claim that $\overline{R} \times \overline{R}^{\text{op}}$ will give us the example we need. By Lemma 2.6, it is neither left nor right outer McCoy. The "Armendariz" and "duo" properties pass to finite direct products, so they both hold in $\overline{R} \times \overline{R}^{\text{op}}$. □

Our next example shows that one can get a strange mixing between the left and right McCoy and outer McCoy properties, where all guaranteed constant annihilators are with respect to the "right" polynomial g in a zero product $fg = 0$. This example will be a key step to proving, in Theorem 2.10, that there are essentially no implications between the one-sided McCoy and outer McCoy properties, except as implied by Proposition 2.3.

Example 2.9. *There exists a semicommutative ring that is left McCoy and right outer McCoy, but it is neither right McCoy nor left outer McCoy.*

Construction and proof. Our example is based on a remark in [3, p. 612–613]. Let S be the free unital $\mathbb{F}_2$-algebra generated by the six noncommuting variables a_0, a_1, a_2, a_3, b_0, and b_1, subject to the following relations:

- $a_0 b_0 = 0$, $a_0 b_1 = a_1 b_0$, $a_1 b_1 = a_2 b_0$, $a_3 b_0 = a_2 b_1$, and $a_3 b_1 = 0$. In other words, taking $f := a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $g := b_0 + b_1 x$, then $fg = 0$.
- $b_k m b_\ell = 0$ for any $k, \ell \in \{0, 1\}$ and any monomial $m \in S$. In other words, any monomial containing two of the b-variables is zero.
- $a_0 a_i m b_j = 0$ and $a_3 a_i m b_j = 0$ for any $i \in \{0, 1, 2, 3\}$, any $j \in \{0, 1\}$, and any monomial $m \in S$.
- $a_1 a_i m b_j = a_2 a_i m b_j$ for any $i \in \{0, 1, 2, 3\}$, any $j \in \{0, 1\}$, and any monomial $m \in S$.

We note that all the relations are homogeneous, which allows us to talk about grades of monomials in S (where a monomial is graded by the total number of variables it contains). As all the relations have grade two or greater, we see that f and g are nonzero polynomials. Also, every relation involves only monomials ending (on the right) in some b-variable. Thus f has no nonzero left annihilators in S, and so S is not left outer McCoy.

Let I be the ideal of S generated by b_0 and b_1, and note that

$$\overline{S} := S/I \cong \mathbb{F}_2\langle a_0, a_1, a_2, a_3\rangle =: S' \subseteq S,$$

which is a domain. Thus if $p, q \in S[x] \setminus \{0\}$ satisfy $pq = 0$, then either $\overline{p} = \overline{0}$ or $\overline{q} = \overline{0}$. Suppose, by way of contradiction, that $\overline{q} \neq \overline{0}$. Let $j \geq 0$ be the minimal index such that $\overline{q[j]} \neq 0$. Write $q[j] = q_1 + q_2$ where $q_1 \neq 0$ consists of monomials containing no b-variables while every monomial in q_2 has some b-variable (which are properties respected by all relations). Let $i \geq 0$ be the minimal index such that $p[i] \neq 0$. We then have that

$$0 = (pq)[i+j] = p[i]q[j] + \sum_{m \in \mathbb{Z}_{>0}} p[i+m]q[j-m] = p[i]q_1$$

since every coefficient of p belongs to I, as do the coefficients of q with index less than j, and $I^2 = 0$. But q_1 is not a right zero-divisor in S, as it contains no b-variables, giving us the needed contradiction. Thus $\overline{q} = 0$, and hence $b_0 q = q b_0 = 0$. As q was an arbitrary nonzero right zero-divisor in $S[x]$, this shows that S must be left McCoy and right outer McCoy.

Next, we claim that S is not right McCoy by showing that f is not annihilated on the right by a nonzero constant polynomial. It suffices to show that a_2 is not a left zero-divisor in S. Treating the relations in the bullet points above as a reduction system, by repeating replacing any instance of a monomial on the left side of an equality with the monomial on the right side, then Bergman's Diamond Lemma [2] applies, as a quick computation verifies that the hypotheses of his lemma are satisfied. Therefore, we see that multiplying by a_2 on the left of a nonzero reduced element merely appends a_2 to the left of every monomial in its support, which cannot be zero.

Finally we prove that S is semicommutative. Let $r, s \in S$ with $rs = 0$, and let $t \in S$. We need to show that $rts = 0$. It suffices to consider the case when t is one of the six variables. The claim is clear when $r = 0$ or $s = 0$, so we assume $r, s \neq 0$. By the argument from two paragraphs above, we know that $s \in I$, and thus each monomial in its support has exactly one b-variable. In other words, each such monomial is of the form mb_jm' for some $j \in \{0,1\}$ and monomials $m, m' \in S'$, so write

$$s = \sum_{j,m,m' \,:\, mb_jm' \in \mathrm{supp}(s)} mb_jm'.$$

From $rs = 0$, we get $\sum_{m'} \left(\sum_{j,m \,:\, mb_jm' \in \mathrm{supp}(s)} rmb_j\right) m' = 0$. Since none of the relations involve a-variables on the right, for each fixed m' we have $\sum_{j,m \,:\, mb_jm' \in \mathrm{supp}(s)} rmb_j = 0$. Thus, without loss of generality, we may assume $s \in S'b_0 + S'b_1$. We may also assume $t = a_i$ since $b_j s = 0$.

We now find it convenient to replace the reduction $a_3b_0 \mapsto a_2b_1$ with the reversed reduction $a_2b_1 \mapsto a_3b_0$. This allows us to assume that no monomial involving b_1 has an a-variable to its left. Thus, additionally using the relations in the third and fourth bullet points, we may write

$$s = \alpha_0 b_0 + \alpha_1 b_1 + g_1 a_1 b_0 + g_2 a_2 b_0 + g_3 a_3 b_0$$

for some constants $\alpha_0, \alpha_1 \in \mathbb{F}_2$ and polynomials $g_1, g_2, g_3 \in \mathbb{F}_2[a_2]$.

Note that 1 cannot belong to the support of r (by considering the product of the terms of minimal grade in $rs = 0$). Write

$$r = \sum_{i=0}^{3} f_i(a_0, a_1, a_2, a_3)a_i$$

where each $f_i \in S'$ is a noncommutative polynomial. Let $f_i' := f_i(0, a_2, a_2, 0)$ be the polynomial in the variable a_2 that we get by replacing each instance of a_0 and a_3 by 0, and each instance of a_1 by a_2. Computing the reduced form for rs, we find:

$$\begin{aligned} &(\alpha_1 f_0' + \alpha_0 f_1' + f_1' a_2 g_1 + f_2' a_2 g_1)a_1 b_0 \\ &+(\alpha_1 f_1' + f_1' a_2 g_2 + \alpha_0 f_2' + f_2' a_2 g_2)a_2 b_0 \\ &+(f_1' a_2 g_3 + \alpha_1 f_2' + f_2' a_2 g_3 + \alpha_0 f_3')a_3 b_0 \quad = 0. \end{aligned}$$

The rest of the proof is a simple case analysis.

Case 1: Suppose $\alpha_0 = \alpha_1 = 0$. Either $g_1 = g_2 = g_3 = 0$ or $f_1' + f_2' = 0$. The first option is impossible as $s \neq 0$, and under the second option we compute $rts = (f_1' + f_2')a_2 a_i s = 0$.

Case 2: Suppose $\alpha_0 = 1$ and $\alpha_1 = 0$. The first line of the displayed equation gives $f_1' = (f_1' + f_2')a_2 g_1$ while the second line gives $f_2' = (f_1' + f_2')a_2 g_2$. Adding the respective sides, we have $f_1' + f_2' = (f_1' + f_2')a_2(g_1 + g_2)$. First, if $f_1' + f_2' \neq 0$, then we have $1 = a_2(g_1 + g_2)$, which is clearly impossible (by considering grades). Thus, $f_1' + f_2' = 0$. Looking at the three lines of the displayed equations again, we then must have $f_1' = f_2' = f_3' = 0$. Then, we find $rts = f_0' a_0 a_i s = 0$.

Case 3: Suppose $\alpha_0 = 0$ and $\alpha_1 = 1$. This case is similar to Case 2, and we get $f_0' = f_1' = f_2' = 0$ and $rts = f_3' a_3 a_i s = 0$.

Case 4: Suppose $\alpha_0 = \alpha_1 = 1$. The middle line of the displayed equation says that $f_1' + f_2' = (f_1' + f_2')a_2 g_2$. As in Case 2, we must have $f_1' + f_2' = 0$, or in other words $f_1' = f_2'$. The first and third lines of the displayed equation then say that $f_0' = f_1' = f_2' = f_3'$. We find, using the relations in the third and fourth bullet points, that

$$rts = f_0'(a_0 + a_1 + a_2 + a_3)a_i s = f_0'(0 + a_2 + a_2 + 0)a_i s = 0. \qquad \square$$

Theorem 2.10. *Semicommutative rings can have, or fail to have, any combination of the left or right outer McCoy and left or right McCoy properties, except that if the ring is outer McCoy then it is also McCoy.*

Proof. Any commutative ring has all four of the McCoy and outer McCoy properties. The ring from Example 2.5, which we will call R (by a change of notation), and its opposite ring R^{op}, give examples for all the possibilities where exactly one of the four properties fails, as limited by Proposition 2.3.

Let S be the ring from Example 2.9. Let T be the graded semicommutative ring constructed in [16, Section 3], which was shown there to be left McCoy but not right McCoy. By an easy grading argument (as used previously), any coefficient of any left zero-divisor in $T[x]$ does not have 1 in its support, and is thus annihilated on the left by $b_0 \in T$ (using the relations defining T). Thus the ring T is left outer McCoy. It is not right outer McCoy by Proposition 2.3.

The rings $R \times R^{\mathrm{op}}$, S, S^{op}, T, and T^{op} give all the needed examples where exactly two of the properties fail. The rings $S \times T$, $S \times T^{\mathrm{op}}$, $S^{\mathrm{op}} \times T$, and $S^{\mathrm{op}} \times T^{\mathrm{op}}$ give the needed examples where exactly three of the properties fail. Finally, $S \times S^{\mathrm{op}} \times T \times T^{\mathrm{op}}$ has all four of the properties fail. In each case the ring is semicommutative. $\square$

It is well known that the set of nilpotents in a semicommutative (and hence reversible) ring is an ideal. In general, any ring where the set of nilpotents is an ideal is called an *NI* ring. Our next example shows that the outer McCoy rings do not need to be NI, and hence neither semicommutative nor reversible.

Example 2.11. *There exists an Armendariz, outer McCoy ring which is not an NI ring.*

Construction and proof. Let F be a field. The ring $S := F\langle a, b : b^4 = 0\rangle$ is Armendariz by [1, Theorem 4.7]. Let R be the unital subring of S generated by b and bab. Any subring of an Armendariz ring is still Armendariz. However, R is not NI since b is nilpotent but $b(bab)$ is not nilpotent. Note that S is a graded ring (with monomials graded by the total number of letters they contain).

Finally, we will show that R is left outer McCoy (and hence right outer McCoy, by symmetry). Let $f, g \in R[x]$ with $fg = 0$ and $g \neq 0$. Suppose, by way of contradiction, that some coefficient of f has 1 in its support. Let f_0 be the grade 0 component of f, and let g_i be the grade i component of g, with $i \geq 0$ chosen minimally so that $g_i \neq 0$. From $fg = 0$ we get $f_0 g_i = 0$. But this contradicts the fact that the leading term of f_0 is just a nonzero element of F, which is not a zero-divisor.

Thus, 1 does not belong the support of any coefficient of f, or in other words $f \in bS[x]$, and hence $b^3 f = 0$. This shows that R is left outer McCoy. □

For von Neumann regular rings, there is a long list of conditions which are equivalent to being reduced. The property of being left McCoy was recently added to this list, in [8, Theorem 20]. Here we show that left outer McCoy rings can also be added to the list. Note that eversibility cannot be added to this list, since any nontrivial matrix ring over a division ring is von Neumann regular, and eversible, but not reduced.

Proposition 2.12. *A von Neumann regular ring R is reduced if and only if it is left outer McCoy.*

Proof. The forward direction is clear, so we only deal with the reverse direction. Suppose, by way of contradiction, that $a^2 = 0$ for some $a \in R \setminus \{0\}$. By von Neumann regularity, there exists some $b \in R$ such that $aba = a$, and hence $(1 - ab)a = 0$.

Take $f := a + (1 - ab)x$, $g := a \in R[x]$. Clearly $fg = 0$ and $g \neq 0$. Thus, by the left outer McCoy property, there exists some nonzero $r \in R$ with $rf = 0$. Hence $ra = 0$ and $r(1 - ab) = 0$. Taken together, these equalities imply that $r = rab = 0$, yielding a contradiction. □

3. Camillo Rings

A ring in which the prime radical is exactly the set of nilpotents elements is called *2-primal*, and it is known that semicommutative rings are 2-primal (see, for instance, [11, Comment to Exercise 12.18]). The 2-primal rings are clearly NI. Surprisingly, 2-primal (and thus, semicommutative) rings have a property closely related to the McCoy and outer McCoy properties by [3, Theorem 9.2], namely:

(3.1) If $f, g \in R[x] \setminus \{0\}$ and $fg = 0$, then there is some nonzero element $r \in R$ such that $rf = 0$ or $rg = 0$.

What makes this condition especially interesting is that, *a priori*, we don't know that f has any nonzero left annihilators.

We call any ring satisfying (3.1) a *left Camillo ring*. The *right Camillo rings* are defined symmetrically, and rings which are both left and right Camillo will be called *Camillo rings*. Moreover, if we restrict the polynomials f and g in (3.1) to not have degree bigger than 1, we call that the *left linearly Camillo* condition.

Remark 3.2. As we mentioned above, the 2-primal rings are Camillo. It is also obvious that left McCoy and left outer McCoy rings are left Camillo. By Theorem 2.10, there exists a Camillo ring which is neither left nor right McCoy, and also neither left nor right outer McCoy.

The Camillo property does not pass to corner rings. However, it behaves well enough that we can classify exactly when a direct product of rings is left Camillo.

Proposition 3.3. *A direct product $R = \prod_{i\in I} R_i$ of rings is left Camillo if and only if either $R_i = 0$ for every $i \in I$, or $R_j \neq 0$ is left Camillo for some $j \in I$. In particular, a direct product of left Camillo rings is left Camillo. Moreover, those facts continue to hold if we replace "left Camillo" with "left linearly Camillo" everywhere.*

Proof. ($\Rightarrow$): Working contrapositively, assume, for each $i \in I$ where $R_i \neq 0$, that R_i is not left Camillo, and further assume that there is at least one such index. For those $i \in I$ where $R_i \neq 0$, let $f_i, g_i \in R_j[x] \setminus \{0\}$ be chosen so that $f_i g_i = 0$ but neither f_i nor g_i is annihilated from the left by a nonzero element of R_i. For those $i \in I$ where $R_i = 0$, take $f_i = g_i = 0$. Taking $f := (f_i)_{i\in I}, g := (g_i)_{i\in I} \in R[x] \setminus \{0\}$ we have $fg = 0$, but neither f nor g is annihilated on the left by a nonzero element of R.

($\Leftarrow$): If $R_i = 0$ for each $i \in I$, then R is the zero ring, which is vacuously Camillo. On the other hand, suppose there exists some index $j \in I$ such that $R_j \neq 0$ and R_j is left Camillo. Let $f, g \in R[x] \setminus \{0\}$ with $fg = 0$. If $f_j = 0$ or $g_j = 0$, take $r_j = 1$ and let $r \in R$ be the zero extension of r_j (so it is zero in all other coordinates). Either $rf = 0$ or $rg = 0$, and clearly $r \neq 0$.

On the other hand, if f_j and g_j are both nonzero, then fix some nonzero $r_j \in R_j$ such that either $r_j f_j = 0$ or $r_j g_j = 0$. Letting $r \in R \setminus \{0\}$ again be the zero extension of r_j, then either $rf = 0$ or $rg = 0$, as wanted.

The penultimate sentence of the Proposition is clear since 0 rings are Camillo. To prove the last sentence, just restrict the degrees of the polynomials in the proof above. $\square$

Proposition 3.3 makes it very easy to construct Camillo rings that fail to have other nice properties. For instance, the direct product of a nonzero commutative ring with a non-Dedekind-finite ring will fail to be Dedekind-finite, but it will be Camillo. Similarly, this proposition tells us that Proposition 2.12 cannot be improved by replacing the (outer) McCoy property with any of the Camillo properties; take the direct product of a field and any non-reduced von Neumann regular ring.

We'll see shortly that there exists a ring R which is left Camillo but not right Camillo. Thus, by Proposition 3.3, $R \times R^{\mathrm{op}}$ is a finite direct product of non-Camillo rings which is Camillo. Similarly, (central) corner rings, subrings, and factor rings of left Camillo rings do not need to be left Camillo. Now, on to the construction of an example of the left-right asymmetry of the Camillo property.

Example 3.4. *There exists a ring that is both left McCoy and left outer McCoy, but it is not right linearly Camillo. In particular, Dedekind-finite rings need not be right linearly Camillo.*

Construction and proof. Let R be the free unital $\mathbb{F}_2$-algebra generated by the five noncommuting variables a_0, a_1, b_0, b_1, and c, subject to the following relations:

- $a_0b_0 = 0$, $a_0b_1 = a_1b_0$, and $a_1b_1 = 0$. In other words, defining the polynomials $f := a_0 + a_1x$, $g := b_0 + b_1x \in R[x]$, then $fg = 0$.
- $cm = 0$ where m can be any of the five letters.

After checking that "overlaps resolve," Bergman's Diamond Lemma [2] tells us that we can put elements of R into reduced forms by repeatedly replacing monomials that appear on the left side of an equality (in the relations above) with the term on the right side.

Moreover, the relations are homogeneous (since each word that occurs has exactly two letters), so we can talk about grades of elements in R. Given $p, q \in R[x] \setminus \{0\}$ with $pq = 0$, by looking at the product of terms of minimal grade, we see that 1 cannot belong to the support of any coefficient in either p or q. Therefore $cp = cq = 0$. Hence R is left McCoy and left outer McCoy.

On the other hand, $fg = 0$ and $f, g \neq 0$. Note that g has no nonzero right annihilators since b_0 is not a left zero-divisor—it does not appear on the left in any reductions. Thus, to finish this example we will show that f does not have any nonzero constant right annihilators. Given any $r \in R$ we can write $r = r_1 + r_2c$ with r_1, r_2 in the subring generated by the a-variables and b-variables. If $fr = 0$ then $fr_1 = 0$ and $fr_2 = 0$, since c is not involved in any relations from the right (except $c^2 = 0$, which isn't relevant here). Thus, if f has a nonzero right annihilator in R, we may assume it contains no c's. But this would contradict [3, Proposition 6.5]. □

Having dealt with direct products, the next natural question is whether the (non-linearly) Camillo and outer McCoy properties pass to polynomial rings. The answer is yes. This is a straightforward exercise, which is probably more enlightening for readers to work out on their own.

The next few results will be devoted to studying how the Camillo property behaves under matrix ring extensions.

Theorem 3.5. *Any nontrivial matrix ring fails to be Camillo.*

Proof. We will show that such a ring is never left Camillo, the other case being similar. Let $E_{i,j}$, for $i, j \in \{1, \dots, n\}$ with $n \geq 2$, be the usual matrix unit with 1 in the (i, j)-entry and zeros elsewhere. Set

$$A := E_{1,n} \text{ and } B := E_{2,1} + E_{3,2} + \cdots + E_{n,n-1}.$$

For later use, we note that $B^iA = E_{i+1,n}$ and $B^iAB = E_{i+1,n-1}$, for any integer $0 \leq i \leq n-1$, which is established recursively.

Now, $A^2 = 0$ and $B^n = 0$. Putting $F := (I - Bx)^{-1}A(I - Bx)$, then $F^2 = 0$ also. Thus, it suffices to show that F has no nontrivial constant left zero-divisor. We find:

$$\begin{aligned} F &= (I + Bx + \cdots + B^{n-1}x^{n-1})A(I - Bx) \\ &= A + (BA - AB)x + \cdots + (B^{n-1}A - B^{n-2}AB)x^{n-1} + (-B^{n-1}AB)x^n \\ &= E_{1,n} + (E_{2,n} - E_{1,n-1})x + \cdots + (E_{n,n} - E_{n-1,n-1})x^{n-1} + (-E_{n,n-1})x^n. \end{aligned}$$

Let C be an arbitrary matrix. If $CF = 0$, then the constant term of this product is $CE_{1,n} = 0$. Therefore,

$$CE_{1,n-1} = CE_{1,n}E_{n,n-1} = 0.$$

Looking at the linear term of $CF = 0$, we then have $CE_{2,n} = 0$. Continuing in this fashion, we have $CE_{i,n}$ for all integers $0 \leq i \leq n$. Thus

$$\begin{aligned} C &= CE_{1,1} + \cdots + CE_{n,n} \\ &= CE_{1,n}E_{n,1} + CE_{2,n}E_{n,2} + \cdots + CE_{n-1,n}E_{n,n-1} + CE_{n,n} = 0 \end{aligned}$$

showing that F has no nonzero constant left annihilators. □

Notice that an easy consequence of the previous theorem is that eversible rings do not need to be Camillo. One might observe that when $n = 2$, the polynomial F constructed above had degree 2, which leaves open the possibility that some 2×2 matrix ring could "squeeze by" and be linearly Camillo. Surprisingly, this is the case. To prove it, we first need the following lemma, clarifying the relationship between eversibility and the linearly Camillo property.

Proposition 3.6. *Every left eversible ring is left linearly Camillo.*

Proof. Let $f, g \in R[x] \setminus \{0\}$ be linear polynomials with $fg = 0$. Write $f = a_0 + a_1 x$ and $g = b_0 + b_1 x$. If $b_1 = 0$ then $a_0 g = 0$ and $a_1 g = 0$, with at least one of a_0 or a_1 nonzero. So we may assume $b_1 \neq 0$.

Now, a_1 is a left zero-divisor, hence it must be a right zero-divisor. Fix $c \in R \setminus \{0\}$ such that $ca_1 = 0$. If $ca_0 = 0$, then $cf = 0$, and we are done. So suppose $ca_0 \neq 0$. We then have $ca_0 g = cfg = 0$. □

Corollary 3.7. *Any matrix ring over a division ring is linearly Camillo.*

Proof. Any such ring is eversible. □

By [3, Proposition 10.2], nonzero matrix rings are neither left nor right linearly McCoy, so Corollary 3.7 comes as even more of a surprise. The same result also says that nonzero upper triangular matrix rings are neither left nor right linearly McCoy. The situation completely reverses for the Camillo property.

Theorem 3.8. *Given a nonzero ring R and any integer $n \geq 1$, the ring of $n \times n$ upper triangular matrices, $\mathbb{T}_n(R)$, is left (linearly) Camillo if and only if R has the same property.*

Proof. ($\Rightarrow$): Let $f, g \in R[x] \setminus \{0\}$ with $fg = 0$. Let $F = \operatorname{diag}(f)$ and $G = \operatorname{diag}(g)$, the matrices whose diagonal entries are f and g respectively, with zeros off the main diagonal. Then $FG = 0$, so there exists some nonzero matrix A such that $AF = 0$ or $AG = 0$. Letting a be any nonzero entry of A, we then have $af = 0$ or $ag = 0$.

($\Leftarrow$): Let $F = (f_{i,j}), G = (g_{i,j}) \in \mathbb{T}_n(R)[x] \setminus \{0\}$ with $FG = 0$. If $f_{n,n} = 0$ then $E_{n,n}F = 0$ and $E_{n,n} \neq 0$ since R is nonzero. Similarly, if $g_{n,n} = 0$ then $E_{n,n}G = 0$. Thus, we may assume $f_{n,n}, g_{n,n} \neq 0$. As $FG = 0$ we get $f_{n,n}g_{n,n} = 0$. From the Camillo property for R, there exists some nonzero $r \in R$ with $rf_{n,n} = 0$ or $rg_{n,n} = 0$. Taking $A := rE_{n,n}$ then $AF = 0$ or $AG = 0$.

The proof for the linearly Camillo property is similar. □

For the final few results in this paper, we will partially extend [3, Theorem 9.2], which says that every 2-primal ring is Camillo. The 2-primal rings are exactly those rings where all the nilpotents belong to the prime radical. Nearly the same proof works for the larger class of rings whose nilpotents belong to the Levitzki radical, as follows.

Theorem 3.9. *If the set of nilpotents in a ring is a locally nilpotent ideal, then the ring is Camillo.*

Proof. As usual, we only need to show that such a ring is left Camillo. Let $f, g \in R[x] \setminus \{0\}$ with $fg = 0$. Let A be the set of coefficients of f and let B be the set of coefficients of g. By [1, Proposition 2.1], we know that AB is a set of nilpotent elements. From the locally nilpotent hypothesis, there exists some integer $k \geq 1$ such that $(AB)^k = 0 \neq (AB)^{k-1}$. Fix some nonzero $r \in (AB)^{k-1}$, where if $k = 1$ we take $r = 1$.

If $rA = 0$, then $rf = 0$ and we are done. Otherwise, fix some $a \in A$ such that $ra \neq 0$. Then $raB \subseteq (AB)^k = 0$, hence $rag = 0$. □

A natural question is whether this theorem generalizes further. We could weaken the hypotheses in two different ways, either by not assuming the nilpotents form an ideal, or by removing the locally nilpotent condition. Neither generalization works. In the paper [4], a ring R is constructed with its set of nilpotents, $\mathrm{Nil}(R)$, forming a nonunital subring of R satisfying $\mathrm{Nil}(R)^2 = 0$, and there exists a polynomial $f \in R[x] \setminus \mathrm{Nil}(R)[x]$ with $f^2 = 0$. Moreover, some of the coefficients of f are neither left nor right zero-divisors. Thus, R is not Camillo, even though the set of nilpotents is itself nilpotent of index 2, which is much stronger than being locally nilpotent.

Next, we will show that NI rings do not need to be Camillo. First, we need a combinatorial lemma.

Lemma 3.10. *The power series $f := (1 - 8x + 15x^2 + x^3 + x^4 + x^5 + \cdots)^{-1}$ has only positive coefficients.*

Proof. Let a be the smallest positive real solution of $a^3 - 15a^2 + 71a - 106 = 0$. Also let $b := -a^2 + 8a - 15$. Thus

$$a = 3.13919\ldots \quad \text{and} \quad b = 0.25901\ldots.$$

One may check directly that these constants satisfy the equalities

$$a = \frac{15 + b}{8 - a} \quad \text{and} \quad b = \frac{b + 1}{8 - a}. \tag{3.11}$$

Indeed, noting $a \neq 8$, we may replace b by $-a^2 + 8a - 15$ in both equations, and simplify.

Now, write $f = c_0 + c_1 x + c_2 x^2 + \cdots$. These coefficients satisfy $c_0 = 1$, $c_1 = 8$, and the recursion

$$c_n = 8c_{n-1} - 15c_{n-2} - \sum_{i=0}^{n-3} c_i, \quad \text{for } n \geq 2.$$

We claim that

$$c_n > ac_{n-1} + b \sum_{i=0}^{n-2} c_i \tag{3.12}$$

for each $n \geq 1$. When $n = 1$, this is immediate. Inductively, assume that (3.12) holds for some integer $k \geq 1$. Then using (3.11) we have

$$c_k > \frac{15 + b}{8 - a} c_{k-1} + \frac{b + 1}{8 - a} \sum_{i=0}^{k-2} c_i.$$

Clearing denominators, and then rearranging terms, we have

$$8c_k - 15c_{k-1} - \sum_{i=0}^{k-2} c_i > ac_k + b\sum_{i=0}^{k-1} c_i.$$

The left side is exactly c_{k+1}, by the recursive definition of the coefficients, which finishes our inductive argument.

Using (3.12), another easy induction proves that $c_n > 0$ for every $n \geq 0$. $\square$

Theorem 3.13. *There exists an NI ring which is neither left nor right Camillo.*

Proof. Let F be a countable field, and let $R_0 := F\{a_0, \ldots, a_7\}$ be the free nonunital F-algebra generated by eight noncommuting variables. This is a graded algebra in the usual way. We may enumerate the nonzero elements of R_0 as $\{r_n\}_{n\geq 1}$.

Let $f := \sum_{i=0}^{7} a_i x^i \in R_0[x]$. There are fifteen coefficients of f^2, each of them is homogeneous of grade 2. Let I_0 be the ideal generated by these coefficients.

Next, let $e_1 := 3$ and recursively define $e_n := e_{n-1} \cdot \mathrm{grade}(r_{n-1}) + 1$. For each integer $n \geq 1$, let I_n be the ideal generated by the homogeneous components of $r_n^{e_n}$. These homogeneous components all live in grades between e_n and $e_{n+1} - 1$, and there is at most one such component in each grade.

Let $I := \sum_{n\geq 0} I_n$, and let $R_1 := R_0/I$. By [6, Lemma 1], proven in [7], along with our combinatorial result Lemma 3.10, we know that R_1 is an infinite dimensional F-algebra. It is also clearly nil, since $r_n^{e_n} \in I$ for each $n \geq 1$. As R_1 is finitely generated, it is not locally nilpotent. Letting $P(R_1)$ be the prime radical of R_1, we thus have that $R_2 := R_1/P(R_1)$ is nonzero, nil, and prime.

Any constant left (or right) annihilators of $\overline{f} := (f + I[x]) + P(R_1)[x] \in R_2[x]$ must annihilate each of the generators of R_2, and thus annihilate all of R_2. In a prime ring, the only element that can do this is 0. But $f^2 \in I$. Thus, $\overline{f}^2 = \overline{0}$, but it has no nonzero constant annihilators. Thus R_2 is not Camillo. If the reader prefers a unital example, one can now take the Dorroh extension of R_2 by F. $\square$

While NI rings are not Camillo, it turns out that they are linearly Camillo. The same is true for a much larger class of ring; namely, those rings where the set of nilpotents is closed under multiplication. (For more information on this class of rings the reader is directed to [18].) The proof is as follows.

Theorem 3.14. *Every ring whose set of nilpotents is closed under multiplication is linearly Camillo.*

Proof. We will only show the left linearly Camillo property. Let

$$f := a_0 + a_1 x,\ g := b_0 + b_1 x \in R[x] \setminus \{0\}$$

with $fg = 0$. Thus $a_0 b_0 = 0$, $a_1 b_0 + a_0 b_1 = 0$, and $a_1 b_1 = 0$. Notice that $b_0 a_0$ and $b_1 a_1$ are nilpotent, of index ≤ 2. Therefore, our assumption on R tells us that $b_0 a_0 b_1 a_1$ is nilpotent, say with index $k \geq 1$. We claim that $a_1 b_0 = -a_0 b_1$ is also nilpotent. Indeed,

$$(a_1 b_0)^{2k+1} = a_1 (b_0 a_1 b_0 a_1)^k b_0 = (-1)^k a_1 (b_0 a_0 b_1 a_1)^k b_0 = 0.$$

Letting $A := \{a_0, a_1\}$ and $B := \{b_0, b_1\}$ then $AB = \{0, a_1 b_0, -a_1 b_0\}$, which is locally nilpotent. The same argument used to prove Theorem 3.9 now works here. $\square$

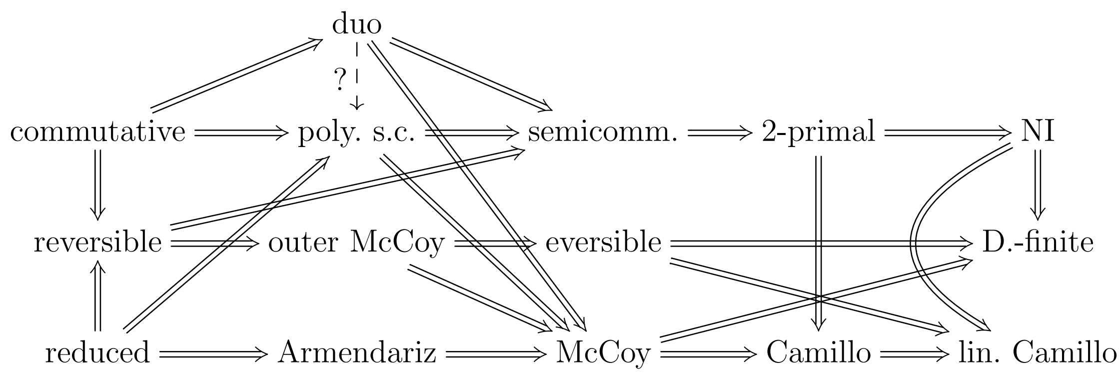
4. Final Remarks

The following diagram describes some of the implications among the properties studied in this paper. We avoided using the quantifiers "left" and "right" to keep the diagram simpler. All proven implications are drawn with a double lined arrow, and they follow from work in this paper or from the diagram at the end of [3].

There is one unknown implication; namely, whether duo rings must have semicommutative polynomial rings. This is marked with a single dashed arrow and a question mark. We state this question formally as:

Question 4.1. If R is a duo ring, is $R[x]$ semicommutative?

No other implications are possible (besides those that follow from transitivity) using known non-implications from [3] or using the examples constructed in this paper.



Acknowledgements

We thank David Cardon for comments which improved the paper. The second named author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2019R1F1A1057940).

References

1. Ramon Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140. MR 2408310
2. George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218. MR 506890
3. Victor Camillo and Pace P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615. MR 2365335
4. Thomas L. Draper, Pace P. Nielsen, and Janez Šter, *Nilpotent polynomials and nilpotent coefficients*, in preparation (2019).
5. E. Ghashghaei, M. Tamer Koşan, M. Namdari, and T. Yildirim, *Rings in which every left zero-divisor is also a right zero-divisor and conversely*, J. Algebra Appl. **18** (2019), no. 5, 1950096, 14. MR 3954649
6. E. S. Golod, *On nil-algebras and finitely approximable p-groups*, Amer. Math. Soc. Transl. Ser. 2 **48** (1965), 103–106. MR 0161878
7. E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), no. 2, 261–272. MR 0161852
8. Tai Keun Kwak and Yang Lee, *Rings over which coefficients of nilpotent polynomials are nilpotent*, Internat. J. Algebra Comput. **21** (2011), no. 5, 745–762. MR 2827201
9. Tai Keun Kwak, Yang Lee, and Young Joo Seo, *Some remarks on one-sided regularity*, Front. Math. China **13** (2018), no. 4, 833–847. MR 3835641

10. T. Y. Lam, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR 1838439
11. ______, *Exercises in classical ring theory*, second ed., Problem Books in Mathematics, Springer-Verlag, New York, 2003. MR 2003255
12. Greg Marks, Ryszard Mazurek, and Michał Ziembowski, *A unified approach to various generalizations of Armendariz rings*, Bull. Aust. Math. Soc. **81** (2010), no. 3, 361–397. MR 2639852
13. Ryszard Mazurek and Michał Ziembowski, *On a characterization of distributive rings via saturations and its applications to Armendariz and Gaussian rings*, Rev. Mat. Iberoam. **30** (2014), no. 3, 1073–1088. MR 3255001
14. N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), no. 5, 286–295. MR 6150
15. Neal H. McCoy, *Annihilators in polynomial rings*, Amer. Math. Monthly **64** (1957), no. 1.
16. Pace P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141. MR 2215121
17. M. B. Rege and Sima Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17. MR 1442245
18. Janez Šter, *Rings in which nilpotents form a subring*, Carpathian J. Math. **32** (2016), no. 2, 251–258. MR 3587893
19. Louis Weiner, *Concerning a theorem of McCoy: Solution to problem 4419*, Amer. Math. Monthly **59** (1952), no. 5, 336–337.
20. Michał Ziembowski, *On classical rings of quotients of duo rings*, J. Pure Appl. Algebra **218** (2014), no. 5, 919–924. MR 3149642

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