Periodic elements and lifting connections

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Abstract. It is well-known that idempotents lift modulo any nil one-sided ideal. While this is not true for periodic elements, it does hold true in many special cases. We investigate the connections between these special cases, as well as limitations. We also answer three questions from the literature. For example, we construct a nilpotent ideal where torsion-units lift, but periodic elements do not lift, modulo that ideal.

1. Periodic elements

Let M be a monoid. The main objects of study in this paper are the *periodic* elements $x \in M$, which satisfy an equation of the form $x^{i+j} = x^j$ for some integers $i \geq 1$ and $j \geq 0$. This equation merely asserts that among the nonnegative powers $1, x, x^2, \ldots$, there must be at least one repetition. (Note that we take $x^0 = 1$.)

Let x^{m+n} be the first power that equals a previous power x^n, for some integers $m \geq 1$ and $n \geq 0$. Following the literature, we call m the *period* of x, and we call n the *index* of x. The integers m and n are minimal in other ways, as follows. Notice that $1, x, x^2, \ldots, x^{m+n-1}$ are all distinct. An inductive argument shows that for any $i \geq m$, we have $x^{i+n} = x^{i'+n}$ where i' is the unique integer $0 \leq i' < m$ such that $i \equiv i' \pmod{m}$. Putting these computations together, we have the following well-known result:

Proposition 1.1. *Let x be a periodic element of a monoid, with period m and index n. Then*

$$x^{i+j} = x^j \text{ for some integers } i \geq 1 \text{ and } j \geq 0 \text{ if and only if } m \mid i \text{ and } n \leq j.$$

If we fix an integer $k \geq n$ such that m divides k, then by Proposition 1.1 we have $x^{2k} = x^k$. In other words, the element $e := x^k$ is an idempotent that commutes with x. Conversely, if $x^{2k} = x^k$ for some integer $k \geq 1$, then Proposition 1.1 requires that $k \geq n$ and that $m \mid k$. Consequently, there is exactly one idempotent that is equal to a positive power of x.

Moreover, if x is an element of a ring R, rather than merely a monoid, we can say more. We have a decomposition $x = ex + (1-e)x \in eRe + (1-e)R(1-e)$ into complementary corner rings. Further, $(ex)^k = e$ and $((1-e)x)^k = 0$; the decomposition splits x into a unit and a nilpotent, in the respective corners.

Note that $(ex)^t = e$ if and only if $x^{k+t} = x^k$, and this happens if and only if $m \mid t$ by Proposition 1.1. Thus, ex is a unit in eRe of order exactly m, which is the period of x.

Also, $((1-e)x)^\ell = 0$ if and only if $x^{k+\ell} = x^\ell$. Since $m \mid k$, we know by Proposition 1.1 that the previous equality holds if and only if $\ell \geq n$. Thus, the nilpotence index of $(1-e)x$ is n, which is exactly the (periodic) index of x. (This fact holds true when $n = 0$, by defining the nilpotence index of the zero element of the zero ring to be 0. Of course, the zero element in all nonzero rings has nilpotence index 1.)

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In the special case when $n \leq 1$, so that $(1-e)x = 0$, we say that x is *potent* (some texts call these the *group elements of finite order*). Moreover, when $n = 0$, or in other words when $e = 1$ so that $(1-e)R(1-e)$ is the zero ring and $R = eRe$, then x is a *torsion-unit* (or a *unit of finite order*). We thus have the following string of easy implications among these properties:

$$\text{torsion-unit} \Rightarrow \text{potent} \Rightarrow \text{periodic}. \tag{1.2}$$

In the paper [4], the behavior of these properties when lifting them modulo ideals—especially modulo nil ideals—was studied, as a generalization of the classical notion of lifting idempotents. There is a strong connection to idempotent lifting since, as we mentioned before, periodic elements are exactly the elements with some positive power that is idempotent. Of particular note, in this paper we are able to show that in many situations the period (but not the index) can be preserved when periodic elements lift.

Our work here forms a companion study to the results in [4]. We will thus assume that readers are familiar with the results and proofs there. We answer three questions left open there, as well as develop some extensions of the theory. As in that paper, our rings are associative and unital, but not necessarily commutative.

2. Lifting terminology

As in the previous section, let R be a ring. Let $\mathscr{P}$ be a property of elements in rings. Given a two-sided ideal I of R, and an element $x \in R$, one says that x has $\mathscr{P}$ *modulo* I if $x + I \in R/I$ has $\mathscr{P}$ (as an element of the ring R/I).

When I is only a one-sided ideal (or, even more generally, only a subset of R), then there is no factor ring R/I, and so it may not make sense to assert that x has $\mathscr{P}$ modulo I. However, when $\mathscr{P}$ is a property that can be described equationally, this obstacle can be overcome. To illustrate the idea we will consider idempotence. Following the literature one says that x is idempotent modulo I if $x^2 - x \in I$; this is because the equation $x^2 - x = 0$ captures idempotence. Note that these two definitions of "idempotent modulo I" agree when I is a two-sided ideal. Similarly, we say that x is

$$\left.\begin{matrix}\text{unit-torsion}\\ \text{potent}\\ \text{periodic}\end{matrix}\right\} \text{modulo } I \text{ if } \left\{\begin{array}{l} x^m - 1 \in I \text{ for some integer } m \geq 1,\\ x^{m+1} - x \in I \text{ for some integer } m \geq 1, \text{ and}\\ x^{m+n} - x^n \in I \text{ for some integers } m \geq 1 \text{ and } n \geq 0.\end{array}\right.$$

Readers should be cautious, because $\mathscr{P}$ could have different equational characterizations that give rise to different conditions modulo I. As I loses more of the structure of a two-sided ideal, this issue worsens. For instance, if I is merely a subset of R, then $x^3 - x \in I$ does not necessarily imply that $x^{2k} - x^k \in I$ for any integer $k \geq 1$; and so periodic elements modulo I need not have idempotent powers modulo I. That said, when I is a one-sided ideal, the implications of (1.2) still hold "modulo I", and the monoid-theoretic argument used to define the period and index can easily be modified to apply modulo I (since that argument can be done using multiplications only on the left, or on the right, as desired).

Given $x \in R$, then a *lift* of x (modulo I) is any element $y \in R$ with $y - x \in I$. (We will also say that y is a *lift* of $x + I$.) Let $\mathscr{Q}$ be any property of elements in rings, possibly different than $\mathscr{P}$. We say that $x \in R$ "lifts to an element with $\mathscr{Q}$" when there exists some lift of x satisfying $\mathscr{Q}$. We say that "$\mathscr{P}$ lifts to $\mathscr{Q}$" whenever any element $x \in R$ that has $\mathscr{P}$ modulo I must have a lift to an element with $\mathscr{Q}$. When $\mathscr{P}$ and $\mathscr{Q}$ are the same property, we will say more shortly that "$\mathscr{P}$ lifts" in that case.

It is well-known that idempotents lift modulo nil one-sided ideals; see [4, Theorem 2] for a recent proof. This is the first step of the following general result requiring only the invertibility of the period.

Theorem 2.1. *Periodic elements lift modulo nil one-sided ideals to periodic elements of the same period, if the period is a unit in the ring.*

Proof. We follow the ideas used in [4, Theorem 8]. Let R be a ring, and let I be a nil one-sided ideal of R. Assume $x \in R$ with $x^{m+n} - x^n \in I$ for some integers $m \geq 1$ and $n \geq 0$, with m (and n) minimal. Also assume $m \cdot 1_R$ is a unit of R. Let S be the commutative, unital subring of R generated by x and the center of R, so $m \cdot 1_S$ is a unit in S. Also, $J := (x^{m+n} - x^n)S \subseteq I$ is a nil two-sided ideal of S. It suffices to show that x lifts modulo J to a periodic element of S, of the appropriate period.

For some integer $k \geq n$, we know that $x^{2k} - x^k \in J$. As idempotents lift modulo nil ideals, fix some idempotent $e \in S$ with $e - x^k \in J$. Now, eJ is a nil ideal of eS, and $(ex)^m - e \in eJ$. By [4, Lemma 7], there exists a torsion-unit $v \in eS$, lifting ex, with $v^m = e$.

On the other hand $(1-e)J$ is nil, and $[(1-e)x]^k \in (1-e)J$, so $(1-e)x$ is nilpotent, say of nilpotence index n'. Thus, $y := ev + (1-e)x$ is periodic in S, with $y^{m+n'} = y^{n'}$. Moreover,

$$y = ev + (1-e)x \equiv ex + (1-e)x = x \pmod{J}.$$

If y has period m', then $y^{m'+n'} - y^{n'} \in J$, and hence $x^{m'+n'} - x^{n'} \in J$. By the minimality of m, we see that $m' = m$. □

Under the assumptions of Theorem 2.1, the period of lifts can be preserved. This raises the question of whether the index can also be preserved, which was asked and then answered in [4]. When $n = 0$ or $n = 1$, the index and period can be preserved together (as shown by Lemma 7 and Theorem 8 of [4]), but when $n = 2$ there is an explicit example on page 5 of [4] showing that the index may not be preserved in any lift. (That example can easily be generalized to any index $n \geq 2$.)

The following proposition gives another situation where periodic elements lift, although now neither the period nor the index are necessarily preserved.

Proposition 2.2 (cf. [4, Theorem 5])**.** *Periodic elements lift modulo nil one-sided ideals, in rings with finite characteristic.*

The previous two results combine to give us:

Corollary 2.3. *Let R be a ring such that for every integer $n \geq 1$ there exists a direct product decomposition $R = R_{1,n} \times R_{2,n}$ where n is invertible in $R_{1,n}$ and $R_{2,n}$ has finite characteristic. Periodic elements lift modulo nil one-sided ideals of R.*

Recall that a ring R is π-regular if every element $x \in R$ has a positive power that is (von Neumann) regular. The first paragraph of the proof of [4, Theorem 13] shows that π-regular rings satisfy the conditions of the previous corollary. Thus, we obtain:

Corollary 2.4. *If R is π-regular, then periodic elements lift modulo nil one-sided ideals.*

The previous corollary answers [4, Question 17] in the positive.

3. Connections among lifting properties

Given the connections between the torsion-unit, potent, and periodic properties, as encapsulated in (1.2), it is natural to ask whether lifting for one property implies lifting for another property. In this section we will show that, indeed, there are some natural relationships among the lifting properties. We begin with the following result going from periodic lifting to potent lifting.

Theorem 3.1. *If periodic elements lift modulo a two-sided ideal, then potent elements also lift.*

Proof. Let R be a ring, and let I be a two-sided ideal of R. Assume $x \in R$ and that $x^{m+1} - x \in I$ for some integer $m \geq 1$. Thus, x is periodic modulo I, and so from the lifting hypothesis there exists some $y \in R$ such that $y^{i+j} = y^j$ (for some integers $i \geq 1$ and $j \geq 0$) and $y - x \in I$. By the usual shenanigans, we can assume that $m = i = j$, after increasing each of them as necessary. Thus $y^m = y^{2m}$. Then using this equality repeatedly we have

$$y^{m+1} = y^{2m+1} = \cdots = y^{(m+2)m+1} = (y^{m+1})^{m+1}.$$

Also $y^{m+1} \equiv x^{m+1} \equiv x$ (this is the only place we use the fact I is a two-sided ideal, rather than a one-sided ideal). Thus, x lifts to the potent element y^{m+1}. □

Theorem 3.1 fails for one-sided ideals. Indeed, take $R = \mathbb{F}_2\langle x, y : y^2 = 0\rangle$ and let L be the left ideal of R generated by $x + x^2$ and $x + y$. Every element of R is uniquely equivalent to either 0, 1, y, or $1 + y$ modulo L. Thus, every element lifts to a periodic element. However, let us show that x does not lift to a potent element z. If the constant term of z is 0, then by degree considerations we see that $z = 0$; but since $x \notin L$ this cannot be a lift of x. If the constant term of z is 1, then we see directly that $x \not\equiv z \pmod{L}$. Note that the idempotents in this ring are trivial, so R is abelian.

We now prove our second result, which goes in the other direction, from potent lifting to periodic lifting.

Theorem 3.2. *If potent elements lift modulo a nil two-sided ideal, then periodic elements also lift.*

Proof. Let R be a ring, and let I be a nil two-sided ideal of R. Assume $x^{m+n} - x^n \in I$ for some integers $m, n \geq 1$ and some $x \in R$. We may as well assume $m = n$, so we have $x^{2m} - x^m \in I$. Using this repeatedly, we have (as in the displayed equation of the previous proof)

$$(x^{m+1})^{m+1} - x^{m+1} \in I.$$

From the lifting hypothesis, there exists some $y \in R$ such that $y^{k+1} = y$ for some integer $k \geq 1$, and $y - x^{m+1} \in I$. We may as well assume $m = k$, increasing them both as necessary. Now, $e := y^m$ is an idempotent. We see that

$$e = y^m \equiv x^{m(m+1)} \pmod{I}$$

and so e commutes with x modulo I. Next, note that

$$0 = (1 - e)y \equiv ((1 - e)x)^{m+1} \pmod{I}$$

so $(1 - e)x(1 - e)$ is nilpotent, say of nilpotence index less than m (after increasing m as necessary). On the other hand,

$$ex = y^m x = y(y^{m-1}x) \equiv yx^{m^2} \equiv yx^{m^2+m} \equiv ey \pmod{I}.$$

Consider the element $z := eye + (1-e)x(1-e)$. It is a lift of x modulo I, and $z^m = e = z^{2m}$, so it is a periodic lift of x. □

Theorem 3.2 fails if we drop the nil hypothesis. For instance, working modulo $4\mathbb{Z}$ in $\mathbb{Z}$, the three potent elements ($0 + 4\mathbb{Z}$, $1 + 4\mathbb{Z}$, and $3 + 4\mathbb{Z}$) each lift to potent elements (namely, 0, 1, and -1, respectively), but the periodic element $2 + 4\mathbb{Z}$ does not lift to a periodic element.

In the previous two theorems, we can weaken the lifting hypothesis to "potent elements lift to periodic elements" and the same proofs work with only minor modifications. Notice that this new lifting hypothesis is a simultaneous weakening of potent lifting as well as periodic lifting.

In the next theorem, we weaken the lifting hypothesis even more.

Theorem 3.3. *If torsion-unit elements lift to periodic elements modulo* (*any subset of*) *the Jacobson radical, then they lift to torsion-unit elements.*

Proof. Let R be a ring and let $I \subseteq J(R)$. Suppose $x^m - 1 \in I$ for some integer $m \geq 1$ and some $x \in R$. By the lifting hypothesis, there exists some $y \in R$ such that $y^{k+\ell} = y^{\ell}$ (for some integers $k \geq 1$ and $\ell \geq 0$) and $y - x \in I$. Since x is a unit modulo $J(R)$, we see that $y \in \mathrm{U}(R)$. Thus $y^k = 1$. □

Recall that a ring is called *abelian* if all idempotents are central. Theorem 4 of [4] tells us:

Proposition 3.4. *For a nil two-sided ideal I of an abelian ring R, the following are equivalent:*

(1) *Torsion-units lift modulo I.*
(2) *Potent elements lift modulo I.*
(3) *Periodic elements lift modulo I.*

Our work above shows that without the abelian hypothesis, we have

$$(3) \Leftrightarrow (2) \Rightarrow (1).$$

Question 16 from [4] asks whether the last arrow is reversible for a general nonabelian ring. We will shortly construct an example showing that this is not the case. To that end, we start with a useful lemma concerning the ring $R_0 := \mathbb{Z}[w : w^2 = 1]$. Note that R_0 is not a domain, since $(1-w)(1+w) = 0$ while $w \neq \pm 1$.

Lemma 3.5. *The set of torsion-units in R_0 is $\{1, -1, w, -w\}$.*

Proof. An element $x \in R_0$ can be written uniquely in the reduced form $x = i + jw$ for some $i, j \in \mathbb{Z}$. Define a map $N\colon R_0 \to \mathbb{Z}$ by the rule $N(x) = i^2 - j^2$. This is a multiplicative map, since given $y \in R_0$, and writing $y = k + \ell w$ for some $k, \ell \in \mathbb{Z}$, we find

$$N(xy) = N((ik + j\ell) + (i\ell + jk)w) = (ik + j\ell)^2 - (i\ell + jk)^2 = (i^2 - j^2)(k^2 - \ell^2) = N(x)N(y).$$

Notice that $N(1) = 1$, and so if x is a unit, then $N(x) = \pm 1$. The only solutions of the diophantine equation $i^2 - j^2 = \pm 1$ are $(i, j) \in \{(\pm 1, 0), (0, \pm 1)\}$. Thus, the only possible units are the four elements listed in the statement of the lemma, and they are each easily checked to be torsion-units. □

We are now ready to construct:

Example 3.6. There is a ring R with a nilpotent ideal I of nilpotence index 3, such that torsion-units lift modulo I, but not all potent elements lift to periodic elements modulo I.

Construction. The ring R we will construct is a (unital, noncommutative) ring generated by letters e, a, b, x, y, z, v, subject to certain relations. First, the letter e is an idempotent. Thus, we can view R as a 2×2 matrix ring with respect to the Peirce decomposition

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

We will assume the other letters belong to certain corners, so we want to force the memberships $a, y \in eR(1-e)$, and $b, z \in (1-e)Re$, and $x, v \in eRe$. Thus, this gives rise to the following twenty-nine obvious relations:

$$\begin{array}{llllllll} ea = a, & ae = 0, & ey = y, & ye = 0, & eb = 0, & be = b, & ez = 0, & ze = z, \\ ax = 0, & yx = 0, & xb = 0, & xz = 0, & av = 0, & yv = 0, & vb = 0, & vz = 0, \\ a^2 = 0, & ay = 0, & ya = 0, & y^2 = 0, & b^2 = 0, & bz = 0, & zb = 0, & z^2 = 0, \\ & e^2 = e, & ex = x, & xe = x, & ev = v, & ve = v. & & \end{array}$$

We will want the set $S = \{a, b, x, y, z\}$ to generate a nilpotent ideal. It would be convenient to assume that $S^2 = 0$, but that assumption is just too strong. After some computations, we discovered that we can take every product of two letters in S to be zero except ab and yz. Thus, this gives rise to another eleven relations:

$$az = 0,\ za = 0,\ ba = 0,\ yb = 0,\ by = 0,\ zy = 0,\ xa = 0,\ xy = 0,\ bx = 0,\ zx = 0,\ x^2 = 0.$$

We will want the matrix $P := \left(\begin{smallmatrix} v & a \\ b & 1-e \end{smallmatrix}\right)$ to be a torsion-unit of period 2, and so this gives rise to the following three relations

$$v^2 = e - ab,\ va = -a,\ bv = -b.$$

Finally, we will also want the matrix $Q := \left(\begin{smallmatrix} v+x & y \\ z & -(1-e) \end{smallmatrix}\right)$ to be a torsion-unit of period 2, and so this gives rise to the following three relations

$$vx = -xv - yz + ab,\ vy = y,\ zv = z.$$

Subject to these forty-six relations, we see that all products of two letters reduce except xv, ab, and yz. Thus,

$$R = \begin{pmatrix} \mathbb{Z}e + \mathbb{Z}v + \mathbb{Z}x + \mathbb{Z}xv + \mathbb{Z}ab + \mathbb{Z}yz & \mathbb{Z}a + \mathbb{Z}y \\ \mathbb{Z}b + \mathbb{Z}z & \mathbb{Z}(1-e) \end{pmatrix}.$$

To see that there are no additional relations among the remaining monomials one can apply Bergman's diamond lemma [3]. It suffices to perform the easy check that all overlaps resolve (since there are no inclusions). We checked this via a computer algebra package. The computations are unenlightening, so we do not include them except for the important case $(v^2)x = v(vx)$. On one hand, we find that

$$(v^2)x = (e - ab)x = x,$$

and on the other hand

$$\begin{aligned} v(vx) = v(-xv - yz + ab) &= -(vx)v - yz - ab = (xv + yz - ab)v - yz - ab \\ &= xv^2 + yz + ab - yz - ab = x(e - ab) = x. \end{aligned}$$

Most of the other overlaps are completely trivial to resolve.

Note that this ring is, besides the action of v, essentially a Morita context with commutative corners. The letter v acts on a and b essentially like -1, while on y and z it acts like 1, and lastly on x it acts as a strange combination of the two.

Let $I = \mathbb{Z}a+\mathbb{Z}b+\mathbb{Z}ab+\mathbb{Z}x+\mathbb{Z}xv+\mathbb{Z}y+\mathbb{Z}z+\mathbb{Z}yz$, which is the two-sided ideal generated by S. We see that $I^3 = 0$. Moreover, $R/I \cong R_0 \times \mathbb{Z}$. By Lemma 3.5, the only torsion-units in the direct product are the eight elements $(\pm 1, \pm 1)$ and $(\pm w, \pm 1)$. The corresponding elements of R/I lift, modulo I, to the torsion-units ± 1, $\pm(1-2e)$, $\pm P$, and $\pm Q$. To finish, we will show that v does not lift modulo I to a periodic element, even though $v^3 - v \in I$.

To that end, let

$$r := \begin{pmatrix} v + m_1x + m_2xv + m_3ab + m_4yz & m_5a + m_6y \\ m_7b + m_8z & 0 \end{pmatrix}$$

be an arbitrary lift of v modulo I. It suffices to show that no power of r^2 is idempotent. We compute that r^2 equals

$$\begin{pmatrix} e + (-1 + m_1 - m_2 - 2m_3 + m_5m_7)ab + (-m_1 - m_2 + 2m_4 + m_6m_8)yz & -m_5a + m_6y \\ -m_7b + m_8z & 0 \end{pmatrix}.$$

Setting $\alpha := -1 + m_1 - m_2 - 2m_3$ and $\beta := -m_1 - m_2 + 2m_4$, then by induction on $k \geq 1$, we have

$$r^{2k} = \begin{pmatrix} e + (k\alpha + (2k-1)m_5m_7)ab + (k\beta + (2k-1)m_6m_8)yz & -m_5a + m_6y \\ -m_7b + m_8z & 0 \end{pmatrix}.$$

If $r^{2k} = r^{4k}$, then looking at the coefficients of ab we would have

$$k\alpha + (2k-1)m_5m_7 = 2k\alpha + (4k-1)m_5m_7,$$

or in other words

$$k\alpha = -2km_5m_7.$$

After cancelling the k (which is not a zero-divisor), and solving for m_1, we see that m_1 is an odd number plus m_2. Performing this same computation, but using the coefficient yz, we get that m_1 is even number plus m_2, giving us our contradiction. □

Remark 3.7. Question 15 from [4] asks the following: If I is a nil ideal of a ring R, and if torsion-units lift modulo I, then given an idempotent $e \in R$ do torsion-units lift modulo eIe in the corner ring eRe? The answer is no. To see this, take R, I, and e as in the previous example, and consider $v = eve \in eRe$. We have $v^2 = e - ab \equiv e \pmod{eIe}$, since $ab \in eIe$. Thus, v is a torsion-unit modulo the nilpotent ideal eIe.

Let $w = v + m_1x + m_2xv + m_3ab + m_4yz$ be an arbitrary lift of v modulo eIe, for some $m_1, m_2, m_3, m_4 \in \mathbb{Z}$. This is a special case of the element r constructed in Example 3.6. The computation there shows that no power of w is an idempotent, hence no lift of v is a torsion-unit in eRe.

While the implication (1) ⇒ (2) of Proposition 3.4 does not hold without the abelian condition, it is interesting to consider whether we can weaken the abelian hypothesis slightly and still obtain that implication. To that end, consider the following situation. Suppose R is a ring with a nil two-sided ideal I and that $x \in R$ is potent modulo I. Thus some power x^k is an idempotent modulo I. Since I is nil, we can lift x^k to an idempotent $e \in R$. Since

$$exe \equiv x \pmod{I},$$

we may as well replace x by exe, and so e commutes with x in R.

Now, $x + (1 - e)$ is a torsion-unit modulo I. Suppose it lifts to a torsion-unit $v \in R$, say of period $m \geq 1$. The Peirce decomposition of v is of the form

$$v = \begin{pmatrix} eve & ev(1-e) \\ (1-e)ve & (1-e)v(1-e) \end{pmatrix}$$

while the Peirce decomposition of $x + (1 - e)$ is, simply enough, $\mathrm{diag}(x, 1 - e)$. Thus $eve \equiv x \pmod{I}$, and $ev(1 - e), (1 - e)ve \in I$.

Note that if $I^2 = 0$, then by induction we have, for each integer $k \geq 1$, that

$$v^k = \begin{pmatrix} (eve)^k & * \\ * & ((1-e)v(1-e))^k \end{pmatrix} \tag{3.8}$$

where the off-diagonal entries belong to I. Since $v^m = \mathrm{diag}(e, 1 - e)$, we see that $(eve)^m = e$, and hence eve is a potent lift of x. The ideal in Example 3.6 has nilpotence index 3, and thus that nilpotence index cannot be any smaller without breaking the example.

There are two more consequences of the calculations above. First, this also shows that if $I \subseteq R$ is a nilpotent ideal of nilpotence index 2 for which torsion-units lift modulo I, and if $e \in R$ is an idempotent, then torsion-units lift modulo eIe in eRe. In other, less formal, words: torsion-unit lifting passes to corner rings when working modulo nilpotent ideals of nilpotence index 2.

Second, consider the following list of subsequently weaker conditions (each assumed to hold for all idempotents e in a ring R):

(0) $e = 0$.
(0)′ $e = 0$ or $1 - e = 0$.
(1) $eR(1 - e) = 0$.
(1)′ $eR(1 - e) = 0$ or $(1 - e)Re = 0$.
(2) $eR(1 - e)Re = 0$.
(2)′ $eR(1 - e)Re = 0$ or $(1 - e)ReR(1 - e) = 0$.
⋮

Condition (0) means that R is the zero ring, while condition (0)′ means that R has only the trivial idempotents. A ring R is abelian exactly when condition (1) holds, and rings satisfying condition (1)′ are called *semiabelian* in the literature. Likewise, let us call rings satisfying condition (n) the *n-abelian rings*, and those satisfying $(n)'$ the *semi-n-abelian rings*.

In rings that are 2-abelian, the diagonal entries of a power of a Peirce matrix are computed by taking powers of those entries. For a proof of that fact, see [1, Lemma 3.4(5)]. (Also, we recommend the paper [2], where idempotents satisfying condition (2) are called *inner Peirce trivial*. Thus, 2-abelian rings could also be called Peirce trivial rings. They also go by the name "quarter-abelian"; see [5].) Now, (3.8) holds in such a ring. Therefore, the argument above shows that Proposition 3.4 holds if we change the "abelian" hypothesis to "2-abelian". However, the ring constructed in Example 3.6 is semi-2-abelian, so the hypothesis cannot be weakened any further along these lines.

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