TRANSITIVITY OF PERSPECTIVITY

DINESH KHURANA AND PACE P. NIELSEN

Abstract. We study modules in which perspectivity of summands is transitive. Generalizing a 1977 result of Handelman and a 2014 result of Garg, Grover, and Khurana, we prove that for any ring R, perspectivity is transitive in $\mathbb{M}_2(R)$ if and only if R has stable range one. Also generalizing a 2019 result of Amini, Amini, and Momtahan we prove that a quasi-continuous module in which perspectivity is transitive is perspective.

1. Introduction

All rings in this article are assumed to be with identity and modules over them are unital. Throughout

$$\begin{aligned} \mathrm{idem}(R) &= \text{ set of idempotents in } R\text{, and} \\ \mathrm{U}(R) &= \text{ set of units in } R. \end{aligned}$$

By M^n we shall denote the direct product of n copies of a module M, for some $n \in \mathbb{N}$.

Two direct summands A and B of a module M are called *perspective*, which is denoted by $A \sim_P B$, when

$$M = A \oplus X = B \oplus X \text{ for some submodule } X \subseteq M.$$

In other words, A and B have a common (direct summand) complement in M. It is clear that perspective summands are isomorphic, and so are their complements.

Following [4], we call a single module *perspective* if any two isomorphic summands have a common complement. It was proved in [4, Theorem 3.3] that R_R is perspective if and only if ${}_RR$ is perspective, for any ring R, and so rings satisfying these equivalent properties are called *perspective rings*. In [4, Theorem 3.4] it was proved that any module M is perspective if and only if $\mathrm{End}(M)$ is a perspective ring. Thus, one is free to think about this property from either a module-theoretic or ring-theoretic point of view.

A ring R is said to have *stable range one* when, for any $c, d \in R$,

$$cR + dR = R \Rightarrow cz + d \in \mathrm{U}(R) \text{ for some } z \in R.$$

It is known, as proved originally by Vasershtein [11], that this is a left-right symmetric notion. As proved by Fuchs [3] and Warfield [12], a module M has the substitution property if and only if $\mathrm{End}(M)$ has stable range one. Substitution is a strengthening of the cancellation property, and it thus follows that if $\mathrm{End}(M)$ has stable range one, then M^n is perspective for every $n \in \mathbb{N}$. For a "crash course" on these and related results, see [9].

As proved by Garg, Grover, and Khurana [4, Corollary 5.14], if M^2 is perspective, then $\mathrm{End}(M)$ has stable range one. However, it is possible that $\mathrm{End}(M)$ may not have stable range one when M is perspective, for instance take $M = \mathbb{Z}_{\mathbb{Z}}$.

2020 *Mathematics Subject Classification.* Primary 16D70, Secondary 16E50, 16U99.

Key words and phrases. Dedekind-finite, quasi-continuous modules, stable range one, (transitivity of) perspectivity, von Neumann regular rings.

We will say that a module M has *transitive perspectivity*, or that *perspectivity is transitive* in M, when

$$A \sim_P B \sim_P C \text{ implies } A \sim_P C,$$

for any summands A, B, $C \subseteq^{\oplus} M$. It is clear that perspectivity is transitive in a perspective module, but there exist multiple examples due to Bergman of (von Neumann) regular rings R such that R_R is not perspective but does have transitive perspectivity (see [6, Page 13] and [2, Theorem 6.11]). We will see in the beginning of Section 2 that perspectivity is transitive in R_R if and only if it is transitive in ${}_RR$, and so we say that R is a ring where *perspectivity is transitive*. We will also see in Section 2 that perspectivity is transitive in a module M if and only if it is transitive in $\mathrm{End}(M)$. Thus, for a ring R it happens that

$$R \text{ has stable range one} \Rightarrow R \text{ is perspective} \Rightarrow R \text{ has transitive perspectivity,} \tag{1.1}$$

and none of the implications is reversible.

Handelman in [6, Theorem 15] proved that a regular ring R is unit-regular if perspectivity is transitive in $\mathbb{M}_2(R)$. As a regular ring is unit-regular if and only if it has stable range one [5, Proposition 4.12], Handelman's result can be rephrased as saying that for a regular ring R, if perspectivity is transitive in $\mathbb{M}_2(R)$, then R has stable range one. Recently Garg, Grover, and the first author in [4, Theorem 5.12] proved, for any ring R, that if $\mathbb{M}_2(R)$ is perspective, then R has stable range one. Generalizing both of these results we prove in Section 2, for an arbitrary ring R, that

$$\mathbb{M}_2(R) \text{ has transitive perspectivity} \Leftrightarrow R \text{ has stable range one.}$$

As a corollary it follows that if perspectivity is transitive in M^2, for some module M, then M has the substitution property, and conversely.

In [1] a module is called *weakly perspective* if any two isomorphic summands with isomorphic direct complements are perspective. In that paper they showed that for any module M, taking $R = \mathrm{End}(M)$, then M is weakly perspective if and only if R_R is weakly perspective, or equivalently ${}_RR$ is weakly perspective. Thus, we can define weak perspectivity for rings, and this property sits between perspectivity and transitivity of perspectivity in (1.1). The two examples of Bergman mentioned above demonstrate that all implications in the extended chain are proper.

One of the main result of [1] is Theorem 16, which says that a quasi-injective, weakly perspective module is perspective. In Section 3 we generalize this result by proving that a quasi-continuous module with transitive perspectivity is perspective.

2. When perspectivity is transitive in $\mathbb{M}_2(R)$

The main result of this section is that if perspectivity is transitive in $\mathbb{M}_2(R)$, then R has stable range one. To prove this we will need several straightforward results from the literature, which we list below for convenience of the reader.

Lemma 2.1. (*See* [4, Proposition 3.2] *and* [4, Corollary 5.2])

1. *Let R be a ring. For any $e, f \in \mathrm{idem}(R)$, then $eR \sim_P fR$ in R_R if and only if $R(1-e) \sim_P R(1-f)$ in ${}_RR$.*
2. *Let M be a module and let $R = \mathrm{End}(M)$. For any $e, f \in \mathrm{idem}(R)$, then $eR \sim_P fR$ in R_R if and only if $eM \sim_P fM$ in M.*

The following is immediate from Lemma 2.1.

Corollary 2.2. *The following both hold:*

1. *For any ring R, perspectivity is transitive in R_R if and only if it is transitive in ${}_RR$.*
2. *For any module M and taking $R = \mathrm{End}(M)$, perspectivity is transitive in M if and only if it is transitive in R_R.*

The next lemma gives criteria for when two elements of R^2 can be generators for complement direct summands.

Lemma 2.3. *Let R be a ring and let $a, b, c, d \in R$.*

1. *$R(a,b) + R(c,d) = R^2$ if and only if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is left invertible in $\mathbb{M}_2(R)$.*
2. *If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible in $\mathbb{M}_2(R)$, then $R(a,b) \oplus R(c,d) = R^2$.*

Proof. (1) $R(a,b) + R(c,d) = R^2$ if and only if there exist $x, y, z, t \in R$ such that

$$x(a,b) + y(c,d) = (1,0) \text{ and } z(a,b) + t(c,d) = (0,1).$$

This pair of equations is equivalent to $\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = I$.

(2) Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible. By (1) we have $R(a,b) + R(c,d) = R^2$. To show that the sum is direct, suppose $x(a,b) + y(c,d) = (0,0)$ for some $x, y \in R$. This implies that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

But as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ invertible, then $x = y = 0$. □

Recall that a ring is *Dedekind-finite* when one-sided units are two-sided. The following lemma can be viewed as a stepping stone to our main result of this section.

Lemma 2.4. *Let R be a ring. If perspectivity is transitive in $\mathbb{M}_2(R)$, then R is Dedekind-finite.*

Proof. Suppose $ab = 1$, for some $a, b \in R$. A straightforward calculation shows that the matrix $\left(\begin{smallmatrix} a & 0 \\ 1 & b \end{smallmatrix}\right)$ is invertible, with inverse $\left(\begin{smallmatrix} b & 1-ba \\ -1 & a \end{smallmatrix}\right)$. It is even easier to see that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & b \end{smallmatrix}\right)$ is invertible. By Lemma 2.3(2), the modules $R(a,0)$ and $R(0,1)$ are perspective in R^2, with common complement $R(1,b)$.

Similarly, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b & 1-ab \\ -1 & a \end{smallmatrix}\right)$ are invertible. By another application of Lemma 2.3(2), the summands $R(0,1)$ and $R(b, 1-ba)$ are perspective in R^2. As perspectivity is transitive in ${}_RR^2$ (by Corollary 2.2), we get that $R(a,0) \sim_P R(b, 1-ba)$.

Write $R^2 = R(a,0) \oplus X = R(b, 1-ba) \oplus X$ for some $X \subseteq R^2$. Then $(1,0) = r(a,0) + (x,y)$ for some $r \in R$ and some $(x,y) \in X$. So $x = 1 - ra$ and $y = 0$, entailing that

$$(x,y) = (1-ra, 0) = (1-ra)a(b, 1-ba) \in R(b, 1-ba) \cap X = 0.$$

This implies that $ra = 1$. □

We can now prove our main result of this section.

Theorem 2.5. *Let R be a ring. If perspectivity is transitive in $\mathbb{M}_2(R)$, then R has stable range one.*

Proof. Suppose $a + bx$ is a unit in R, for some $a, b, x \in R$. We have to show that $ay + b$ is a unit for some $y \in R$. (That this is equivalent to the definition of stable range one, given in the introduction, is established by a straightforward calculation.)

As $a+bx$ is a unit, the matrix

$$\begin{pmatrix} a & b \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a+bx & 0 \\ -x & 1 \end{pmatrix}$$

is invertible, being a product of units. Similarly,

$$\begin{pmatrix} a-(1-b)x & 1 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a+bx & 0 \\ -x & 1 \end{pmatrix}$$

is invertible. By Lemma 2.3(2) we have $R(a,b) \sim_P R(a-(1-b)x,1)$.

Next, both $\left(\begin{smallmatrix} a-(1-b)x & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are invertible, so the same argument as in the previous paragraph yields $R(a-(1-b)x,1) \sim_P R((0,1)$. As perspectivity is transitive in R^2, then $R(a,b) \sim_P R(0,1)$.

Write $R^2 = R(a,b) \oplus X = R(0,1) \oplus X$ for some $X \subseteq R^2$. As $X \cong R^2/R(0,1) \cong R \times 0$, we know that X is cyclic. Thus, we may write $X = R(c,d)$ for some $c,d \in R$. Then by Lemma 2.3(1), both $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ c & d \end{smallmatrix}\right)$ are left invertible. Thus c is left invertible, and hence it is invertible by Lemma 2.4. Now as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is left invertible and $\left(\begin{smallmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ is invertible, we have

$$\begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b-ac^{-1}d \\ c & d \end{pmatrix}$$

is left invertible. Write

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & b-ac^{-1}d \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some $\alpha, \beta, \gamma, \delta \in R$. Then $\delta c = 0$ implies that $\delta = 0$, as c is invertible. So $\gamma(b-ac^{-1}d) = 1$ implying that $b-ac^{-1}d$ is left invertible. But as R is Dedekind-finite by Lemma 2.4, this means $b-ac^{-1}d$ is invertible. Taking $y = -c^{-1}d$, we are done. □

The converse of Theorem 2.5 is true. A quick proof is as follows. Assuming R has stable range one, then $\mathbb{M}_2(R)$ does as well, since the stable range one property is Morita invariant (see [9, Theorem 5.6]). Therefore $\mathbb{M}_2(R)$ has transitive perspectivity by (1.1).

Theorem 2.5 also has a module-theoretic formulation.

Corollary 2.6. *For any module M, perspectivity is transitive in M^2 if and only if M has the substitution property.*

Proof sketch. Translate Theorem 2.5 from the ring-theoretic setting to the module-theoretic setting, using Corollary 2.2. □

3. Quasi-continuous modules with transitive perspectivity

As mentioned in the introduction, a module is called weakly perspective if any two isomorphic summands with isomorphic direct complements are perspective. It was proved in [1, Theorem 16] that a quasi-injective, weakly perspective module is perspective. In this section we generalize this result by proving that a quasi-continuous module with perspectivity transitive is perspective. As the class of quasi-injective modules is a proper subclass of quasi-continuous modules, and the class of weakly perspective modules is a proper subclass of those with transitive perspectivity, our result is a double generalization.

For definitions and properties of quasi-continuous and quasi-injective modules we refer the reader to [10]. We will also need the following concepts defined on idempotents.

Definition 3.1. Let R be a ring. For $e, f \in \mathrm{idem}(R)$ we write $e \sim_r f$ (respectively $e \sim_\ell f$) to denote $eR = fR$ (respectively $Re = Rf$).

Given $n \in \mathbb{N}$ and given idempotents $e_0, \ldots, e_n \in \mathrm{idem}(R)$, any chain of the form

$$e_0 \sim_r e_1 \sim_\ell e_2 \sim_r \cdots e_n$$

will be called a *right n-chain* connecting e_0 to e_n. (It is a *right* n-chain because it starts with the relation $\sim_r$.) Left n-chains are defined in a symmetric manner.

Chains are associated to perspectivity in a fundamental way.

Lemma 3.2 ([2, Lemma 6.3])**.** *Let R be a ring. For $e, f \in \mathrm{idem}(R)$, then $eR \sim_P fR$ in R_R if and only if there is a right 3-chain connecting e to f.*

For any module M we let

$$\Delta_M = \{\phi \in \mathrm{End}(M) \, : \, \ker(\phi) \text{ is an essential submodule of } M\}.$$

Then Δ_M is an ideal of $\mathrm{End}(M)$, by [10, Lemma 3.2]. Recall that a ring is called *reduced* if 0 is its only nilpotent element. These concepts are connected by the following results in the literature.

Lemma 3.3. *Let M be a quasi-continuous module, let $R = \mathrm{End}(M)$, let $I = \Delta_M$, and put $\overline{R} = R/I$. Then*

(1) *Idempotents lift modulo I in R.*
(2) *$\overline{R} = R_1 \times R_2$, where R_1 is a regular, right self-injective ring and R_2 is a reduced ring.*
(3) *For any e, $f \in \mathrm{idem}(R)$, if $\overline{e}$ and $\overline{f}$ are connected by a right n-chain, for some $n \geq 2$, then so are the idempotents e and f.*

Proof. Parts (1) and (2) are Lemma 3.7 and Corollary 3.13 in [10], respectively. Part (3) is [8, Corollary 4.10]. □

We will need one more result from the literature.

Lemma 3.4 ([5, Theorem 9.17], [6, Theorem 2])**.** *Any Dedekind-finite, regular, right self-injective ring is perspective.*

Also note that a reduced ring is perspective (see [4, page 3] for a more general class of perspective rings). We are now ready to prove the main result of this section.

Theorem 3.5. *Any quasi-continuous module with perspectivity transitive is perspective.*

Proof. Let M be a quasi-continuous module with perspectivity transitive, let $R = \mathrm{End}(M)$, let $I = \Delta_M$, and put $\overline{R} = R/I$. Fix idempotents $e, f \in \mathrm{idem}(R)$ with $eR \cong fR$. We will show that eR and fR have a common complement.

Working modulo I, we have that $\overline{e}\overline{R}$ and $\overline{f}\overline{R}$ are isomorphic summands. Suppose, for a moment, that $\overline{e}\overline{R} \sim_P \overline{f}\overline{R}$. Then by Lemma 3.2 there is a right 3-chain connecting $\overline{e}$ to $\overline{f}$. Thus, by Lemma 3.3(3) there is also a right 3-chain connecting e to f, and so by Lemma 3.2 again, $eR \sim_P fR$. Therefore, it suffices to show that $\overline{R}$ is perspective.

Let $\overline{x}, \overline{y}, \overline{z} \in \mathrm{idem}(\overline{R})$ and assume

$$\overline{x}\overline{R} \sim_P \overline{y}\overline{R} \sim_P \overline{z}\overline{R}.$$

By Lemma 3.3(1), we may assume $x, y, z \in \mathrm{idem}(R)$. Further, by the argument given in the previous paragraph, we have $xR \sim_P yR \sim_P zR$. As perspectivity is transitive in M, by Corollary 2.2 the same is true for R, and so $xR \sim_P zR$, say with common complement $X \subseteq R$. Working modulo I, we then have $\overline{x}R$ and $\overline{z}R$ have common complement $\overline{X}$. Thus perspectivity is transitive in $\overline{R}$.

Next, by Lemma 3.3(2) we have a decomposition $\overline{R} = R_1 \times R_2$, where R_1 is a regular, right self-injective ring and R_2 is a reduced ring. As R_2 is perspective, we only need to show that R_1 is perspective. By [5, Proposition 10.21] there is a further decomposition $R_1 = R_3 \times R_4$, where R_3 is purely infinite and R_4 is Dedekind-finite. As R_4 is perspective by Lemma 3.4, we only have to show that R_3 is perspective. By [5, Theorem 10.16], we have $R_3 \cong R_3^2$ as (right R_3) modules, and hence $R_3 \cong \mathbb{M}_2(R_3)$ as rings. The second isomorphism, in conjunction with Lemma 2.4, says that R_3 is Dedekind-finite. That fact, in conjunction with the first isomorphism, entails that $R_3 = 0$, and we are done. □

4. Future work

If R is a regular ring, then R is unit-regular if and only if it has stable range one. Unit-regularity is also equivalent (for regular rings) to the weaker property of being an internal cancellation ring, or IC ring for short (see [7]). This raises the question of whether or not Handelman's result could be generalized further. For instance, if R is an IC ring with perspectivity transitive, is R perspective? The answer is no, and we plan to address this fact in future joint work with Xavier Mary.

5. Acknowledgements

We thank Xavier Mary for comments on our paper, as well as sending us an alternative proof of Theorem 2.5.

References

[1] Babak Amini, Afshin Amini, and Ehsan Momtahan, *Weakly perspective rings and modules*, J. Algebra Appl. **18** (2019), no. 1, 1950014, 9 pages. MR 3910667

[2] Alexander J. Diesl, Samuel J. Dittmer, and Pace P. Nielsen, *Idempotent lifting and ring extensions*, J. Algebra Appl. **15** (2016), no. 6, 1650112, 16 pages. MR 3479816

[3] L. Fuchs, *On a substitution property of modules*, Monatsh. Math. **75** (1971), 198–204. MR 296096

[4] Shelly Garg, Harpreet K. Grover, and Dinesh Khurana, *Perspective rings*, J. Algebra **415** (2014), 1–12. MR 3229504

[5] K. R. Goodearl, *von Neumann regular rings*, second ed., Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991. MR 1150975

[6] David Handelman, *Perspectivity and cancellation in regular rings*, J. Algebra **48** (1977), no. 1, 1–16. MR 447329

[7] Dinesh Khurana and T. Y. Lam, *Rings with internal cancellation*, J. Algebra **284** (2005), no. 1, 203–235. MR 2115012

[8] Dinesh Khurana and Pace P. Nielsen, *Perspectivity and von Neumann regularity*, submitted (2020), 17 pages.

[9] T. Y. Lam, *A crash course on stable range, cancellation, substitution and exchange*, J. Algebra Appl. **3** (2004), no. 3, 301–343. MR 2096452

[10] Saad H. Mohamed and Bruno J. Müller, *Continuous and discrete modules*, London Mathematical Society Lecture Note Series, vol. 147, Cambridge University Press, Cambridge, 1990. MR 1084376

[11] L. N. Vasershtein, *Stable rank of rings and dimensionality of topological spaces*, Funktsional. Anal. i Prilozhen. **5** (1971), no. 2, 17–27, [English Translation: *Funct. Anal. Appl.* **5** (1971), no. 2, 102–110]. MR 0284476

[12] R. B. Warfield, Jr., *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91** (1980), no. 2, 457–485. MR 615693

Department of Mathematics, Panjab University, Chandigarh-160014, India
Email address: dkhurana@pu.ac.in

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
Email address, Corresponding author: pace@math.byu.edu