Perspectivity and von Neumann regularity

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Abstract. We investigate connections between von Neumann regularity of endomorphisms and perspectivity of direct summands in modules. This leads to a new classification of those rings whose regular elements are strongly regular, which turn out to be exactly the rings R whose idempotents are central modulo the Jacobson radical $J(R)$.

An important component of our work is an investigation of the left and right associate relations on idempotents, as well as chains of these relations. As applications we give new characterizations of strongly regular elements and of idempotents that are central modulo the Jacobson radical.

We also introduce a new class of regular elements that we call pc-regular elements, related to perspectivity in complement summands. These pc-regular elements are exactly the special clean elements. Generalizing the well-known fact that unit-regular rings are special clean, we then show that the unit-regular elements of any regular ring satisfying general comparability are special clean. Consequently, unit-regular endomorphisms of quasi-continuous modules are special clean, answering, in the positive, a conjecture of T. Y. Lam.

1. Introduction

In the study of von Neumann regular rings and elements, the inner inverses (also called quasi-inverses) play a central role. Letting R be a ring, then for any given element $a \in R$ we write its set of inner inverses as

$$\mathrm{I}(a) = \{b \in R : aba = a\}. \tag{1.1}$$

Thus, the set of (von Neumann) regular elements, which we denote as $\mathrm{reg}(R)$, consists of exactly those elements $a \in R$ with $\mathrm{I}(a) \neq \emptyset$.

By assuming additional conditions on $\mathrm{I}(a)$ one specializes the notion of regularity, and there are many important examples of such specializations. For instance, the set of unit-regular elements, $\mathrm{ureg}(R)$, consists of those element $a \in R$ such that $\mathrm{I}(a)$ contains a unit of R. Another noteworthy example, capturing much of the behavior of regular elements in commutative rings, is to assume that $\mathrm{I}(a)$ contains an element that commutes with a, in which case a is said to be strongly regular; the set of all such elements is denoted $\mathrm{sreg}(R)$. We freely use the well-known inclusions

$$\mathrm{sreg}(R) \subseteq \mathrm{ureg}(R) \subseteq \mathrm{reg}(R), \tag{1.2}$$

although a natural proof will appear later in the paper.

Conditions on the regular elements in a ring R have significant consequences on the behavior of finitely generated projective R-modules. For instance, as shown in the works of G. Ehrlich [5] and D. Handelman [8], the equality $\mathrm{reg}(R) = \mathrm{ureg}(R)$ holds if and only if

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isomorphic idempotents in R are conjugate, if and only if the finitely generated projective R-modules are cancellative in the sense that

$$A \oplus B \cong A \oplus C \text{ implies } B \cong C.$$

Due to this result, any ring R satisfying $\mathrm{reg}(R) = \mathrm{ureg}(R)$ is called an IC ring (short for "internal cancellation"). More information on IC rings is available in [9].

Rings satisfying the stronger equality

$$\mathrm{reg}(R) = \mathrm{sreg}(R)$$

are called strongly IC rings, but they have not been as well-studied. In this work we provide numerous new characterizations of these rings, many of them appearing in Theorem 3.13 below. For instance, these are exactly the rings where every idempotent of R is central modulo the Jacobson radical of R, yielding a nice generalization of [7, Theorem 3.5]. Also, a module-theoretic characterization of strongly IC rings appears in Theorem 3.17.

We also investigate a new natural class of regular elements, sitting between the strongly regular and unit-regular elements. These elements will be defined in terms of perspectivity of complement summands, and so we call them the *pc-regular* elements. Ultimately we prove that these regular elements are none other than the special clean elements; those $a \in R$ satisfying:

$$a = e + u,\ e \in \mathrm{idem}(R),\ u \in \mathrm{U}(R),\ aR \cap eR = (0).$$

This new way of looking at these elements leads to a proof that all unit-regular endomorphisms of a quasi-continuous module are special clean (see Theorem 4.11). Preliminary versions of these results were observed in a joint project with T. Y. Lam and J. Šter.

An outline of the paper is as follows. In Section 2 we recall some module-theoretic characterizations of the regular, unit-regular, and strongly regular elements. This naturally leads to the definition of the pc-regular elements, mentioned above, in Definition 2.2. Moreover, these module-theoretic conditions provide a link between the regular elements on one hand and pairs of isomorphic idempotents on the other. Similarly, the unit-regular elements are linked to pairs of conjugate idempotents.

In Section 3, we further link the strongly regular and pc-regular elements to another equivalence relation on pairs of idempotents, given by chains of associate idempotents. We define these chain conditions explicitly in Definitions 3.2 and 3.7; we also introduce some standard notation for chained idempotents. The 2-chained equivalence classes are given alternative descriptions in Theorem 3.10, and this guides us to the complete characterization of the strongly IC rings. Finally, in Section 4 we prove that the pc-regular elements are exactly the special clean elements. We also prove that the unit-regular elements in a regular ring satisfying general comparability are pc-regular. The result, mentioned above, about quasi-continuous modules is an immediate consequence.

Throughout the paper we let M be a right module over some ring k, and we put $R = \mathrm{End}(M)$. We will have no need to emphasize k, and thus will make no further reference to it, assuming implicitly that all modules are right k-modules, and module isomorphisms are right k-module isomorphisms. Note that any ring is isomorphic to an endomorphism ring of some module, so we lose no generality in our assumption that R is an endomorphism ring. We let $\mathrm{idem}(R)$ be the set of idempotents in R, and $\mathrm{U}(R)$ is the group of units. For standard results on unit-regular and strongly regular rings and elements, the reader is directed to [7]. Also see [11] for historical information and basic facts regarding special clean elements. We use the symbol $\oplus$ to denote only *internal* direct sums and summands.

2. Regularity and direct sum decompositions

In this section we recall some standard facts about regular elements that motivate the results in this paper. We begin with the following well-known proposition, which provides of a module-theoretic characterization of regular, unit-regular, and strongly regular elements.

Proposition 2.1. *Let M be a module and let $R = \mathrm{End}(M)$. For any $a \in R$, the following equivalences hold:*

(1) $a \in \mathrm{reg}(R)$ *if and only if*

$$M = \mathrm{im}(a) \oplus X = Y \oplus \ker(a) \text{ for some } X, Y \subseteq^{\oplus} M.$$

(2) $a \in \mathrm{ureg}(R)$ *if and only if*

$$M = \mathrm{im}(a) \oplus X = Y \oplus \ker(a) \text{ for some } X, Y \subseteq^{\oplus} M \text{ with } X \cong \ker(a).$$

(3) $a \in \mathrm{sreg}(R)$ *if and only if*

$$M = \mathrm{im}(a) \oplus \ker(a).$$

Proof. Part (1) is classical, and a proof is provided in the solution to [14, Exercise 4.14A$_1$]. Part (2) was first shown in the work of Ehrlich [5, Theorem 1]; the assumption there that $R = \mathrm{reg}(R)$ can be weakened, as we have done here, to $a \in \mathrm{reg}(R)$. Part (3) is also classical, and the reader is directed to [16, p. 3583] for a longer list of equivalent conditions. □

With this proposition in hand, the inclusions of (1.2) are tautological, as each inclusion is a consequence of assuming more stringent conditions on (direct summand) complements of $\mathrm{im}(a)$ and $\ker(a)$. The strongly regular elements are those where we may take $X = \ker(a)$ and $Y = \mathrm{im}(a)$.

Recall that two summands $P, Q \subseteq^{\oplus} M$ are perspective if they share a common complement, or in other words there exists some $N \subseteq^{\oplus} M$ with

$$M = P \oplus N = Q \oplus N.$$

Looking at the three module-theoretic conditions in Proposition 2.1, there is a conspicuously missing natural condition, related to perspectivity, which prompts us to make the following definition (first suggested by T. Y. Lam in 2016).

Definition 2.2. If M is a module with direct sum decompositions

$$M = A \oplus X = Y \oplus X = Y \oplus B,$$

for some submodules A, B, X, Y, then we say that A and B are *perspective in complements*. Equivalently, A has a complement that is perspective with B (and vice versa).

Given $a \in R = \mathrm{End}(M)$, if $\mathrm{im}(a)$ and $\ker(a)$ are perspective in complements, then we say that a is *pc-regular*. We write $\mathrm{pcreg}(R)$ for the set of all such elements in R.

Notice that the perspective in complements property implies $A \cong Y$ and $X \cong B$. Thus, the inclusions of (1.2) may be refined to the longer sequence

$$\mathrm{sreg}(R) \subseteq \mathrm{pcreg}(R) \subseteq \mathrm{ureg}(R) \subseteq \mathrm{reg}(R). \tag{2.3}$$

Later, we will characterize $a \in \mathrm{pcreg}(R)$ in terms of a condition on $\mathrm{I}(a)$, and thus see that the pc-regular elements are exactly the special clean elements.

We have seen that the different regularity conditions can be defined module-theoretically and in terms of assumptions on their inner inverses. There is a third way to think about regularity, in terms of idempotents. To help us see this connection, we introduce the following convenient nomenclature. Let us say that a pair of idempotents (e, f) and an element $a \in R$ are **compatible** with each other if

$$\operatorname{im}(a) = eM \text{ and } \ker(a) = (1 - f)M.$$

Compatibility is also definable in terms of purely ring-theoretic notions; it is equivalent to the couple of equalities $aR = eR$ and $r_R(a) = (1 - f)R$. By Proposition 2.1(1), the set of elements of R that are compatible with pairs of idempotents is exactly $\operatorname{reg}(R)$. Our goal is to reinterpret information about regularity assumptions as conditions on the compatible pair of idempotents.

First note that not every pair of idempotents is compatible with some element $a \in R$. Indeed, if $\operatorname{im}(a) = eM$ and $\ker(a) = (1 - f)M$, then by the First Isomorphism Theorem we must have $eM \cong fM$, for the map $a|_{fM} : fM \to \operatorname{im}(a) = eM$ is such an isomorphism. (Moreover, given $b \in \operatorname{End}(M)$, then $b \in \mathrm{I}(a)$ if and only if fbe is the inverse isomorphism.) We thus have that e and f are isomorphic idempotents. (More information about isomorphic idempotents can be found in [13, pp. 315–316].)

On the other hand, given $e, f \in \operatorname{idem}(R)$ with $e \cong f$, let $a : fM \to eM$ be an isomorphism. We can extend a to an endomorphism on M, by letting it be zero on $(1 - f)M$. Thus $\operatorname{im}(a) = eM$ and $\ker(a) = (1 - f)M$.

Putting this all together, we have shown that:

(2.4) A pair of idempotents (e, f) is compatible with some $a \in R$ if and only if $e \cong f$.

See [10, Proposition 2.4] for a strengthening of this fact.

If isomorphic idempotents are exactly those pairs that are compatible with regular elements, what pairs of idempotents are compatible with unit-regular elements? Putting the previous argument together with Proposition 2.1(2), these are the pairs (e, f) with

$$e \cong f \text{ and } (1 - e) \cong (1 - f).$$

By [14, Exercise 21.16(1)], this is equivalent to saying that the idempotents e and f are conjugate in R, and we write $e \sim f$.

In the next section we will similarly characterize the pairs of idempotents compatible with strongly regular and pc-regular elements, using another well-studied equivalence relation on idempotents.

3. Changing complements

Throughout this section let M be a fixed module and let $R = \operatorname{End}(M)$, as before. If some direct sum decomposition $M = P \oplus Q$ is generated by $e \in \operatorname{idem}(R)$, that is, $P = eM$ and $Q = (1 - e)M$, then we represent this situation with the diagram

$$P \overset{e}{\text{———}} Q.$$

Suppose that $M = P \oplus Q'$ is some other decomposition, generated by $e' \in \operatorname{idem}(R)$. We can capture this additional information by expanding the previous diagram, adding a new line corresponding to the new idempotent, as follows:

$$\begin{array}{ccl} P & \overset{e}{\text{———}} & Q \\ & \underset{e'}{\diagdown} & Q'. \end{array}$$

The summand on the left is unchanged in both decompositions, but the summand on the right varies. Thus, this picture captures the fact that Q and Q' are perspective.

The relationship between the idempotents e, e', as described by the previous diagram, is well-studied, and there are many equivalent formulations. The following lemma lists a few of these equivalent conditions; see [17, Lemma 4.2] for a fuller list and complete proofs.

Lemma 3.1. *For a module M and two idempotents $e, e' \in R = \mathrm{End}(M)$, the following are equivalent:*

(1) $eM = e'M$.
(2) $ee' = e'$ *and* $e'e = e$.
(3) $e' = e + er(1-e)$ *for some* $r \in R$.
(4) $e' = eu$ *for some* $u \in \mathrm{U}(R)$.
(5) $eR = e'R$.

We write $e \sim_r e'$ when these equivalent conditions hold.

Due to condition (4), we call $\sim_r$ the "right associate relation" on idempotents. This is an equivalence relation. There is correspondingly a left associate relation $\sim_\ell$. Note that any two left (or right) associate idempotents are isomorphic, as are their complements.

It happens that $e \sim_r e'$ if and only if $(1-e) \sim_\ell (1-e')$. This fact has an analogue in the diagrams we have been drawing; it corresponds to switching the left and right sides, and changing any idempotent attached to an edge to the complement idempotent.

$$\begin{tikzcd}[arrows=dash] P \arrow[r, "e"] \arrow[dr, "e'"'] & Q \\ & Q' \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd}[arrows=dash] Q \arrow[r, "1-e"] & P \arrow[dl, "1-e'"] \\ Q' & \end{tikzcd}$$

We can continue to expand our diagrams. For example, the diagram

$$\begin{tikzcd}[arrows=dash] P \arrow[r, "e"] \arrow[dr, "e'"'] & Q \\ P' \arrow[r, "e''"'] & Q' \end{tikzcd}$$

encapsulates the fact that $e \sim_r e' \sim_\ell e''$. To ease notation we make the following definitions.

Definition 3.2. Let R be a ring. If there exists idempotents $g_0, g_1, g_2, \ldots, g_n \in \mathrm{idem}(R)$, for some $n \in \mathbb{N}$, such that they are related in the alternating fashion

$$g_0 \sim_\ell g_1 \sim_r g_2 \sim_\ell \cdots g_n,$$

then we call this a *left n-chain* connecting g_0 to g_n. The *right n-chains* are defined similarly, by reversing the roles of $\sim_\ell$ and $\sim_r$.

We let $\approx$ denote the equivalence relation generated by the union of $\sim_\ell$ and $\sim_r$. Equivalently, $e \approx f$ holds if and only if e and f are connected by some left or right n-chain, for a sufficiently large value of $n \in \mathbb{N}$.

Note that n-chains can be lengthened to $(n+1)$-chains trivially, since any idempotent is both left and right associate to itself. This definition enables us to describe the strongly regular elements in new ways.

Theorem 3.3. *Let M be a module and let $R = \mathrm{End}(M)$. If $a \in R$, then the following conditions are all equivalent:*

(1) *$a \in \mathrm{sreg}(R)$.*
(2) *$a \in \mathrm{reg}(R)$ and any pair of idempotents compatible with $a \in R$ are connected by a right 2-chain.*
(3) *There is a pair of idempotents compatible with $a \in R$ that are connected by a 0-chain.*

Proof. In all three cases, there exists at least one pair of idempotents (e, f) compatible with $a \in \mathrm{reg}(R)$. Fix any such pair and consider the diagram

$$\begin{tikzcd} eM \arrow[r, dash, "e"] & (1-e)M \\ fM \arrow[r, dash, "f"] & (1-f)M. \end{tikzcd}$$

By Proposition 2.1, claiming that $a \in \mathrm{sreg}(R)$ is equivalent to asserting $M = eM \oplus (1-f)M$. This is equivalent to adding a line to the diagram, connecting the upper left and lower right corners by an idempotent g. Visually, this is equivalent to saying that there is a right 2-chain connecting e to f. This proves **(1)⇔(2)**.

Continuing the same notation, the pair of idempotents (g, g) is compatible with a, and g is connected to itself by a 0-chain. Thus **(2)⇒(3)**.

Any 0-chain goes from an idempotent to itself, so if (g, g) is compatible with a, we must have $gM = \mathrm{im}(a)$ and $(1-g)M = \ker(a)$. But then an application of Proposition 2.1(3) implies that $a \in \mathrm{sreg}(R)$. Hence **(3)⇒(1)**. □

A similar description of the pc-regular elements is possible, almost using the same proof.

Theorem 3.4. *Let M be a module and let $R = \mathrm{End}(M)$. If $a \in R$, then the following conditions are all equivalent:*

(1) *$a \in \mathrm{pcreg}(R)$.*
(2) *$a \in \mathrm{reg}(R)$ and pairs of idempotents compatible with $a \in R$ are connected by right 4-chains.*
(3) *Some pair of idempotents compatible with $a \in R$ is connected by a left 2-chain.*

Proof. **(1)⇒(2)**: Assuming $a \in \mathrm{pcreg}(R)$, then we can write

$$M = \mathrm{im}(a) \oplus X = X \oplus Y = Y \oplus \ker(a)$$

for some submodules $X, Y \subseteq^{\oplus} M$. For any pair of idempotents (e, f) compatible with a,

$$\begin{tikzcd} \mathrm{im}(a) \arrow[r, dash, "e"] \arrow[dr, dash, "g_1"'] & (1-e)M \\ Y \arrow[r, dash, "g_2"'] \arrow[dr, dash, "g_3"'] & X \\ fM \arrow[r, dash, "f"'] & \ker(a) \end{tikzcd}$$

which demonstrates that (2) holds.

(2)⇒(3): As $a \in \operatorname{reg}(R)$ we may fix a pair of idempotents (e, f) compatible with a. By hypothesis, they are connected by a right 4-chain, so there is a diagram as above. The pair (g_1, g_3) is also compatible with a, and they are connected by a left 2-chain.

(3)⇒(1): If $g_1 \sim_\ell g_2 \sim_r g_3$ is a left 2-chain, and a is compatible with (g_1, g_3), then taking $X = (1 - g_1)M$ and $Y = g_3M$, we have $M = \operatorname{im}(a) \oplus X = X \oplus Y = Y \oplus \ker(a)$. □

Remark 3.5. The diagrams we have been drawing are only a useful pneumonic, reminding us how changing complements affects the idempotents that generate the decompositions. Everything we have done can be redone without any diagrams.

The previous two theorems assert that a regular element $a \in \operatorname{reg}(R)$ is strongly regular (or pc-regular) exactly when any two isomorphic idempotents $e \cong f$ compatible with a are connected by a right 2-chain (respectively, 4-chain). We can in turn ask the broader question: For what rings is it true that any two isomorphic idempotents are connected by some right n-chain? The following lemma tells us that the word 'right' is sometimes irrelevant.

Lemma 3.6. *If every pair of $\approx$-related idempotents are connected by a left n-chain, then they are connected by a right n-chain, and conversely.*

Proof. By symmetry considerations, it suffices to verify the first claim. Let R be a ring and suppose that $e \approx f$ for some $e, f \in \operatorname{idem}(R)$. By hypothesis, e and f are connected by a left n-chain. By passing to complements, we see that $1 - e$ and $1 - f$ are connected by a right n-chain, and in particular $(1 - e) \approx (1 - f)$. Applying our hypothesis again, $1 - e$ is connected to $1 - f$ by a left n-chain. Passing to complements once more, this becomes a right n-chain from e to f. □

In light of the previous lemma, let us make the following definitions.

Definition 3.7. Let R be a ring and let $S \subseteq R$ be a subset. If every two idempotents $e, f \in S$ with $e \approx f$ are connected by a right n-chain, we will say that S is *right n-chained.* The *left n-chained* subsets are defined symmetrically. If S is both left and right n-chained, we will say S is *strongly n-chained.* On the other hand, if any two idempotents $e, f \in S$ are connected by either a left or a right n-chain, we will say that S is *weakly n-chained.*

Clearly, strongly n-chained subsets are both left and right n-chained, which are in turn weakly n-chained. In the converse direction, Lemma 3.6 asserts that if an entire ring is left n-chained or right n-chained, then it is strongly n-chained. However, it is possible for a proper subset of a ring to be left n-chained without being right n-chained, and simple examples abound. For instance, see Example 3.9 below for a weakly 1-chained ring that is not strongly 1-chained.

Next, we recall some additional facts about n-chains, for very small values of $n \geq 0$. Most of these facts appear in [4].

Proposition 3.8. *Let R be a ring, and let S be the $\approx$-equivalence class of some fixed element $e \in \operatorname{idem}(R)$. The following hold:*

(1) *e is central in R iff S is strongly 0-chained, iff S is strongly 1-chained.*
(2) *e is left semicentral in R* (meaning $(1 - e)Re = 0$) *iff S is right 1-chained.*
(3) *If S is weakly 1-chained, it is left 1-chained or right 1-chained.*
(4) *Idempotents in S are perspective (meaning $f_1R, f_2R \subseteq^{\oplus} R_R$ are perspective, for any $f_1, f_2 \in S$) iff S is strongly 3-chained.*

Proof. It is easy to show that idempotents that are both left and right associate are equal. The rest of (1) follows by making minor changes to the proof of Proposition 6.5 in [4]. Part (2) follows similarly from the same proof, and part (4) follows from modifying Theorem 6.7 of that paper.

Finally, to prove part (3) assume by way of contradiction that there exist some idempotents $f_1, f_2 \in S - \{e\}$ with $e \sim_\ell f_1$ and $e \sim_r f_2$. Since f_1 and f_2 are connected by some 1-chain, without loss of generally we may assume $f_1 \sim_\ell f_2$. As $\sim_\ell$ is a transitive relation, we have $e \sim_\ell f_2$. Since $e \sim_r f_2$ also holds, we get $e = f_2$ (by the first sentence of the proof), contradicting our assumption. □

Example 3.9. Proposition 3.8 says that the strongly 1-chained rings are exactly the abelian rings. If F is a field, and $R = \mathbb{T}_2(F)$ is the ring of 2×2 upper-triangular matrices over F, then R is not abelian, and hence not strongly 1-chained. The idempotents of the form $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$, for some $\alpha \in F$, are left semicentral. These idempotents, their complements, and the trivial idempotents $\{0, 1\}$ comprise all of $\mathrm{idem}(R)$. Thus by Proposition 3.8(2), the ring R is weakly 1-chained.

Missing from Proposition 3.8 is a characterization of the 2-chained $\approx$-equivalence classes. We give a very strong characterization of such classes that is apparently entirely new, which also has far reaching consequences.

Theorem 3.10. *Let R be a ring, and let S be the $\approx$-equivalence class of some fixed element $e \in \mathrm{idem}(R)$. The following are equivalent:*

(1) *e is central modulo $\mathrm{J}(R)$.*
(2) *S is strongly 2-chained.*
(3) *S is weakly 2-chained.*

Proof. **(1)⇒(2)**: Fix $f \in S$, so $e \approx f$. Use "bar notation" to denote passing to the factor ring $\overline{R} = R/\mathrm{J}(R)$. It is straightforward to check that associate idempotents remain associates in every factor ring, and so we have $\overline{e} \approx \overline{f}$ in $\overline{R}$. Since $\overline{e}$ is central, Proposition 3.8(1) implies that $\overline{e} = \overline{f}$, since they are connected by a 0-chain. It is a little known fact that idempotents of R that are equal modulo $\mathrm{J}(R)$ are connected by a left 2-chain (and connected by a right 2-chain too); [3, Proposition 2.4] has the proof.

(2)⇒(3): This is a tautological weakening.

(3)⇒(1): By the left-right analog of Lemma 3.1, if $e \sim_\ell g$ for some $g \in \mathrm{idem}(R)$, then $g = e + r$ for some $r \in (1-e)Re$. We will write elements of R in the Peirce matrix decomposition, relative to the identification

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Thus,

$$e = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} e & 0 \\ r & 0 \end{pmatrix}.$$

If $g \sim_r f$ for some $f \in \mathrm{idem}(R)$, then Lemma 3.1 also gives $f = g + gs(1-g)$ for some $s \in R$. Since $eg = e$ we also have $(1-e)(1-g) = (1-g)$. Also as $ge = g$, we lose no generality assuming $s \in eR(1-e)$. We calculate that

$$f = g + gs(1-g) = (e+r) + (e+r)s(1-(e+r)) = e + r + s + rs - sr - rsr,$$

or as a Peirce matrix

$$f = \begin{pmatrix} e - sr & s \\ r - rsr & rs \end{pmatrix}.$$

Thus, if e is connected to an idempotent f by a left 2-chain, then f has this form for some $r \in (1-e)Re$ and $s \in eR(1-e)$. Conversely, any two such r and s determine such an idempotent f.

Now, fix $r \in (1-e)Re$ and $s \in eR(1-e)$. Equivalently, we have fixed idempotents $f, g \in S$ with $e \sim_\ell g \sim_r f$, as above. We will consider two cases.

Case 1: Assume e is also connected to f by a right 2-chain. Thus, there exists some $h \in \mathrm{idem}(R)$ with $e \sim_r h$ and $h \sim_\ell f$. As before, we can write h in its Peirce matrix form as

$$h = \begin{pmatrix} e & t \\ 0 & 0 \end{pmatrix}$$

for some $t \in eR(1-e)$. Now, as $h \sim_\ell f$, we have $fh = f$ and $hf = h$.

Comparing the upper right corners of fh and f, we must have $t - srt = s$. Multiplying on the right by r, and adding e to both sides, we obtain $e + tr - srtr = e + sr$. Moving sr to the left and factoring, we obtain

$$(e - sr)(e + tr) = e.$$

Next, comparing the upper left corners of hf and h, we must have $e - sr + tr - trsr = e$. Factoring, we obtain

$$(e + tr)(e - sr) = e.$$

Thus $e - sr \in \mathrm{U}(eRe)$.

Case 2: Assume e is not connected to f by a right 2-chain. In this case, take

$$h = e - s = \begin{pmatrix} e & -s \\ 0 & 0 \end{pmatrix} \sim_r e.$$

Since $h, f \in S$, there must either be a left 2-chain or a right 2-chain from h to f. The latter option is impossible, for if $h \sim_r h' \sim_\ell f$, then $e \sim_r h' \sim_\ell f$.

Thus, there is some $h' \in S$ with $h \sim_\ell h' \sim_r f \sim_r g$. In particular, $h'g = g$ and $gh' = h'$. Using previous computations and symmetry, we can write the Peirce matrix of h' as

$$h' = \begin{pmatrix} e + st & -s - sts \\ t & -ts \end{pmatrix}$$

for some $t \in (1-e)Re$. Comparing the upper left corners of $h'g$ and g, we obtain the equality $e + st - sr - stsr = e$. Factoring we have

$$(e + st)(e - sr) = e.$$

Next, comparing the lower left corners of gh' and h', we have $r + rst = t$. Multiplying on the left by $-s$, adding e to both sides, rearranging, and factoring (as before), we get

$$(e - sr)(e + st) = e.$$

Thus $e - sr \in \mathrm{U}(eRe)$.

We have thus shown that in every case we have

$$e - sr \in \mathrm{U}(eRe) \text{ for each } r \in (1-e)Re \text{ and } s \in eR(1-e). \tag{3.11}$$

Given an arbitrary element $p \in R$, we then find that

$$1 - sp = \begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix} - \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} epe & ep(1-e) \\ (1-e)pe & (1-e)p(1-e) \end{pmatrix} = \begin{pmatrix} e - spe & -sp(1-e) \\ 0 & (1-e) \end{pmatrix}.$$

The upper left entry is a unit, in the corner ring eRe, by (3.11) with $r = (1-e)pe$. Also, the lower right entry is the identity of the corner ring $(1-e)R(1-e)$. Thus $1 - sp \in \mathrm{U}(R)$ for each $p \in R$, and therefore $s \in \mathrm{J}(R)$. As $s \in eR(1-e)$ is arbitrary, we get $eR(1-e) \subseteq \mathrm{J}(R)$. Symmetrically, $(1-e)Re \subseteq \mathrm{J}(R)$.

It now suffices to assume $eR(1-e) = 0 = (1-e)Re$ and show that e is central. Given any $x \in R$ we have $ex(1-e) = 0$ and so $ex = exe$. Similarly $exe = xe$, and so $ex = xe$. $\square$

Remark 3.12. The ring R of Example 3.9 has $\approx$-equivalence classes that are left 1-chained but not right 1-chained (and vice versa). Thus, it is surprising that the weakly 2-chained $\approx$-equivalence classes must be strongly 2-chained. In joint work with X. Mary, we have shown that weakly 3-chained classes are similarly strongly 3-chained. When $n \geq 4$, we were unable to discover if something similar happens for the weakly n-chained $\approx$-equivalence classes.

By work of Ehrlich [5] and Handelman [8], it is well-known that $\mathrm{reg}(R) = \mathrm{ureg}(R)$ holds in a ring R if and only if R_R satisfies a form of internal cancelation, and so such rings are called internal cancelation rings, or IC rings for short. (See [9] for additional information on these rings.) These are equivalently the rings where isomorphic idempotents are conjugate.

It is similarly an interesting problem to classify those rings with the stronger condition $\mathrm{reg}(R) = \mathrm{sreg}(R)$. These rings have appeared previously in the literature, going by the name "strongly IC rings" in [12]. Theorem 5.4 of that paper gives some provocative equivalent conditions for such rings. Below, we give many new characterizations of the strongly IC rings. Just as for IC rings, we can characterize the strongly IC rings in terms of an assumption on the idempotents of R.

Theorem 3.13. *For a ring R, the following are equivalent:*

(1) *Isomorphic idempotents are strongly 2-chained.*
(2) $\mathrm{reg}(R) = \mathrm{sreg}(R)$.
(3) $\mathrm{ureg}(R) = \mathrm{sreg}(R)$.
(4) $\mathrm{pcreg}(R) = \mathrm{sreg}(R)$.
(5) *If $x \in \mathrm{reg}(R)$, then $x \in Rx^2$.*
(6) *If $x \in \mathrm{ureg}(R)$, then $x \in Rx^2$.*
(7) *If $x \in \mathrm{pcreg}(R)$, then $x \in Rx^2$.*
(8) *Idempotents of R are central modulo the Jacobson radical.*
(9) *The $\approx$-equivalence classes of R are weakly 2-chained.*

Proof. **(1)⇒(2)**: We show the inclusion $\mathrm{reg}(R) \subseteq \mathrm{sreg}(R)$, as the reverse inclusion always holds. Fix $a \in \mathrm{reg}(R)$. Let (e, f) be any pair of compatible (isomorphic) idempotents. By hypothesis, e is connected to f by a right 2-chain, and so Theorem 3.3 entails that $a \in \mathrm{sreg}(R)$.

(2)⇒(3): This implication is clear, as $\mathrm{sreg}(R) \subseteq \mathrm{ureg}(R) \subseteq \mathrm{reg}(R)$. The implications **(3)⇒(4)**, **(2)⇒(5)**, **(3)⇒(6)**, **(4)⇒(7)**, **(5)⇒(6)**, and **(6)⇒(7)** follow similarly, noting that $x \in Rx^2$ is a weakening of the strongly regular condition.

(7)⇒(8): Let $e \in \mathrm{idem}(R)$, and let $r, s \in R$ be arbitrary. Note that $v = 1 + (1-e)re$ and $w = 1 + es(1-e)$ are units, and hence so is $u = wv$. We have $v(1-e) = (1-e)$ and $we = e$, so in particular we have $wv(1-e)R = w(1-e)R$ and $weR = eR$. Thus, treating elements of R as endomorphisms of R_R, we can set $A = \operatorname{im}(ue) = ueR$, $X = u(1-e)R = w(1-e)R$, $Y = weR = eR$, and $B = (1-e)R = \ker(ue)$. With these choices, we have

$$R_R = A \oplus X = Y \oplus X = Y \oplus B.$$

This shows that ue is pc-regular.

By hypothesis, $ue = yueue$ for some $y \in R$. Multiplying on the left by u^{-1}, and then by e, we have $e = (eu^{-1}yue)(eue)$. Thus eue is left invertible in eRe. The Peirce matrix decomposition of $1 + s(1-e)re$, with respect to the complement idempotents e and $1-e$, is

$$\begin{pmatrix} eue & 0 \\ (1-e)s(1-e)re & 1-e. \end{pmatrix}$$

The entries along the diagonal are left invertible, and the matrix is lower triangular, so $1 + s(1-e)re$ is left invertible in R. As $s \in R$ is arbitrary, then by [13, Lemma 4.1] we have $(1-e)re \in \mathrm{J}(R)$. By a symmetric computation, replacing e with $1-e$, we get that $er(1-e) \in \mathrm{J}(R)$. Thus $er - re = er(1-e) - (1-e)re \in \mathrm{J}(R)$. Since $r \in R$ is arbitrary, we see that e is central modulo $\mathrm{J}(R)$.

(8)⇔(9): This follows from Theorem 3.10.

(8)⇒(1): Denote passage to $\overline{R} = R/\mathrm{J}(R)$ using bar notation. Let $e, f \in \operatorname{idem}(R)$ with $e \cong f$. Thus $e = pq$ and $f = qp$ for some $p, q \in R$. Now, $\overline{e} = \overline{pq}$ and $\overline{f} = \overline{qp}$ are central idempotents of $\overline{R}$. Thus

$$\overline{e} = \overline{p(qp)q} = \overline{qppq} = \overline{q(pq)p} = \overline{f}.$$

As mentioned previously, [3, Proposition 2.4] shows that e and f are strongly 2-chained. □

Remark 3.14. There is an example due to G. Bergman of a regular ring R that is not unit-regular, where the $\sim$ and $\approx$ relations on $\operatorname{idem}(R)$ are different, and yet perspectivity is transitive so the $\approx$-equivalence classes are strongly 3-chained. This example was first mentioned in [8], and appears in [4, Theorem 6.11]. Thus, for regular rings, the three inclusions in (2.3) can be (simultaneously) proper.

Moreover, this shows that Theorem 3.13 is quite surprising, for when we pass from considering strongly 3-chained rings to considering weakly 2-chained rings, then these different notions suddenly collapse together. The inclusions in (2.3) are now all equalities, and the $\cong$, $\sim$, and $\approx$ relations on idempotents are all the same.

The following corollary is now immediate, and it provides another way to view some of the results of Sections 4 and 5 of [12].

Corollary 3.15. *Assume that idempotents lift modulo the Jacobson radical of a ring R. Then* $\operatorname{reg}(R) = \operatorname{sreg}(R)$ *if and only if $R/\mathrm{J}(R)$ is abelian.*

If $R/\mathrm{J}(R)$ is abelian, then $\operatorname{reg}(R) = \operatorname{sreg}(R)$ holds without any additional assumptions. However, the lifting assumption in the previous corollary is needed for the converse direction. In other words, condition (8) of Theorem 3.13 cannot, in general, be strengthened to say that $R/\mathrm{J}(R)$ is abelian, due to the following example.

Example 3.16. Let $S = \mathbb{Z}_{(p)}$ be the ring of integers localized at the maximal ideal $p\mathbb{Z}$, with $p \in \mathbb{Z}$ an odd prime. Let $R = \mathbb{H}(S)$ be the ring of quaternions over S, or in symbols

$$R = \{a + bi + cj + dk \,:\, a, b, c, d \in \mathbb{Z}_{(p)},\ i^2 = j^2 = -1,\ k = ij = -ji\}.$$

Clearly R is a domain (being a subring of the division ring $\mathbb{H}(\mathbb{Q})$), so its two (trivial) idempotents are central, and hence central modulo $\mathrm{J}(R)$.

Now $R/pR = \mathbb{H}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{M}_2(\mathbb{Z}/p\mathbb{Z})$ has noncentral idempotents. As any matrix ring over a finite field is Jacobson semisimple, we also now know that $\mathrm{J}(R) \subseteq pR$. On the other hand, the inverse of $x = 1 - p(a + bi + cj + dk)$ in $\mathbb{H}(\mathbb{Q})$ is

$$x^{-1} = \frac{1}{(1-pa)^2 + (pb)^2 + (pc)^2 + (pd)^2)}((1-pa) + pbi + pcj + pdk) \in R.$$

Thus $1 - pR \subseteq \mathrm{U}(R)$, and so $p \in \mathrm{J}(R)$. Hence $\mathrm{J}(R) = pR$, which shows that $R/\mathrm{J}(R)$ has noncentral idempotents.

While Theorem 3.13 gives multiple ring-theoretic characterizations for 2-chained rings, there is also a nice module-theoretic equivalence.

Theorem 3.17. *If M is a module, then $R = \mathrm{End}(M)$ is weakly 2-chained if and only if isomorphic summands of M share all of their complements.*

Proof sketch. Draw the same diagram as in the proof of Theorem 3.3, and assume $e \cong f$. Sharing complements would mean that $(1-f)M$ is a complement to eM, giving a right 2-chain from e to f. Conversely, any right 2-chain must force $(1-f)M$ to be a complement to eM. □

Some parts of Theorem 3.13 have analogs for longer chains. For instance, the rings where every pair of isomorphic idempotents is strongly 3-chained are exactly the perspective rings of [6]. (An endomorphism ring $R = \mathrm{End}(M)$ is *perspective* if any two isomorphic summands of M share at least one common complement.) The proof of this characterization is easy to see, by repeating previous ideas and drawing the appropriate diagrams, as follows. Let $e \cong f$ be arbitrary isomorphic idempotents. There exists a common complement of eM and fM if and only if there is a diagram

$$\begin{tikzcd}[arrows=dash]
eM \arrow[r, "e"] \arrow[dr, "g_1"'] & (1-e)M \\
 & X \\
fM \arrow[r, "f"'] \arrow[ur, "g_2"] & (1-f)M.
\end{tikzcd}$$

This is equivalent to the existence of a right 3-chain from e to f.

Similarly, by adding one more edge to this diagram, we have:

Proposition 3.18. *Let R be a ring.*

(1) $\mathrm{reg}(R) = \mathrm{pcreg}(R)$ *iff isomorphic idempotents are connected by right 4-chains.*
(2) $\mathrm{ureg}(R) = \mathrm{pcreg}(R)$ *iff conjugate idempotents are connected by right 4-chains.*

Proof. Theorem 3.4 implies that for any two idempotents $e, f \in \mathrm{idem}(R)$, they are connected by a right 4-chain if and only if all elements $a \in R$ that they are compatible with belong to $\mathrm{pcreg}(R)$. But the (unit-)regular elements are exactly the elements of R compatible with pairs of isomorphic (respectively, conjugate) idempotents. □

The rings where conjugate idempotents are connected by right 3-chains are studied in [2] and called weakly perspective. Our work with general n-chains generalizes some of the work in that paper.

4. Special clean elements

Let M be a module and take $R = \mathrm{End}(M)$, as usual. Fixing $a \in \mathrm{pcreg}(R)$, we have a diagram of the form

$$\begin{tikzcd} \mathrm{im}(a) \arrow[r, dash, "g_1"] & X \arrow[dl, dash, "g_2"] \\ Y \arrow[r, dash, "g_3"'] & \ker(a) \end{tikzcd}$$

for some submodules $X, Y \subseteq^{\oplus} M$ and idempotents $g_1, g_2, g_3 \in \mathrm{idem}(R)$. We know that $a|_Y \colon Y \to \mathrm{im}(a)$ is an isomorphism. Similarly, $(1-g_2)|_{\ker(a)} \colon \ker(a) \to X$ is an isomorphism. Since $a|_{\ker(a)} = 0$ and $(1-g_2)|_Y = 0$, we see that

$$u = a - (1-g_2) = a|_Y - (1-g_2)|_{\ker(a)}$$

maps $M = Y \oplus \ker(a)$ to $M = \mathrm{im}(a) \oplus X$ isomorphically on summands, and thus u is an automorphism of M. Writing $e = 1 - g_2 \in \mathrm{idem}(R)$, we have $a = e + u$. Additionally,

$$aR \cap eR = g_1R \cap (1-g_2)R = g_1R \cap (1-g_1)R = 0.$$

We have thus written $a = e + u$, for some $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$ with $aR \cap eR = 0$, which is the defining characterization of special clean elements.

Conversely, suppose that for some $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$, we have $a = e + u$ and $aR \cap eR = 0$. By [11, Lemma 2.3] it happens that $au^{-1}a = a$. Clearly, $au^{-1}, u^{-1}a \in \mathrm{idem}(R)$. Further,

$$au^{-1}(1-e) = au^{-1}(1-a+u) = au^{-1} - au^{-1}a + a = au^{-1}$$

as well as

$$(1-e)au^{-1} = (1-e)(e+u)u^{-1} = (1-e).$$

Thus, by Lemma 3.1, $au^{-1} \sim_\ell (1-e)$. Symmetrically, $u^{-1}a \sim_r (1-e)$. In other words, au^{-1} and $u^{-1}a$ are connected by a left 2-chain.

Also notice that $au^{-1}M = aM = \mathrm{im}(a)$. Since $a(1-u^{-1}a) = a - au^{-1}a = 0$, we also have $(1-u^{-1}a)M \subseteq \ker(a)$; the reverse inclusion also holds, since a takes the complement summand $u^{-1}aM$ isomorphically to aM (with the inverse isomorphism given by $u^{-1}|_{aM}$). Theorem 3.4 tells us that $a \in \mathrm{pcreg}(R)$. We have thus shown the following.

Theorem 4.1. *For any ring R, the pc-regular elements are exactly the special clean elements.*

The question remains whether or not there are natural examples of modules where the pc-regular (i.e., special clean) elements abound. One of the most natural cases to consider is that of vector spaces. If M is an infinite dimensional vector space, then M is isomorphic to a proper subspace of itself. Thus, unlike many of the situations in the previous section, isomorphic idempotents in $\mathrm{End}(M)$ are not necessarily conjugate, let alone $\approx$-connected.

Moreover, conjugate idempotents do not need to be perspective. For instance, suppose M is countable dimensional, say with a basis $\{m_n : n \in \mathbb{N}\}$. Let $e \in \mathrm{End}(M)$ be the idempotent with image $A = \mathrm{Span}(m_0, m_2, m_4, \ldots)$ and kernel $B = \mathrm{Span}(m_1, m_3, m_5, \ldots)$. Similarly, let f be the idempotent with image $A' = \mathrm{Span}(m_2, m_4, m_6, \ldots)$ and kernel $B' = \mathrm{Span}(m_0) \oplus B$. Each of these spaces is countable dimensional, and so

$$M \cong A \cong A' \cong B \cong B'$$

and in particular $e \sim f$. However, A cannot have a common complement with A', nor B with B'. Thus, e and f are not connected by either a left or a right 3-chain.

Thus, the best we could hope for is that conjugate idempotents are connected by right 4-chains (and hence, by Lemma 3.6, also left 4-chains). Equivalently, we would need to show that all unit-regular endomorphisms are special clean. This was shown by T. Y. Lam in a private note in 2016, where he further conjectured that the result should be true for the larger class of continuous modules. We will prove Lam's conjecture for the even larger class of quasi-continuous modules. Our proof uses many of the same ideas as in Lam's proof for the vector space case. Note that Handelman also raises similar questions, using the more coarse notion of n-perspectivity in [8], obtaining similar results. Indeed, to begin our investigations we need the following lemma found in [8].

Lemma 4.2. *Let M be a module, and let $P, Q \subseteq M$. If $P \cap Q = 0$ and $P \cong Q$, then P and Q are perspective in $P + Q$. If moreover $P + Q \subseteq^{\oplus} M$, then P and Q are perspective in M.*

Proof. The first part is result (iv) on page 2 of [8]. Writing $P + Q = P \oplus N = Q \oplus N$, if $M = (P + Q) \oplus N'$, then $N \oplus N'$ is a common complement to both P and Q in M, so the last sentence follows. □

Given two modules M and N, we write $M \lesssim N$ to mean that M is isomorphic to a submodule of N. We also recall the following definition.

Definition 4.3. Let M be a module, and let $P, Q \subseteq M$. We say that P and Q satisfy *general comparability* if there exists a central idempotent $e \in \operatorname{End}(M)$ such that $eP \lesssim eQ$ and $(1 - e)Q \lesssim (1 - e)P$.

This general comparability axiom is the key to distinguishing a large class of modules with the property we desire, as described in the following theorem. Even in the case when M is a vector space, it appears that this theorem does not appear in the literature, but was first observed by T. Y. Lam in a private correspondence.

Theorem 4.4. *Let M be a module such that its direct summands are closed under pairwise intersection and addition (so they form a sublattice of the lattice of all submodules of M) and satisfy general comparability. If $A, B \subseteq^{\oplus} M$ with $M/A \cong B$ and $M/B \cong A$, then A and B are perspective in complements.*

Proof. Let $C = A \cap B$, which is a direct summand of M by hypothesis. Thus, we can write $A = A' \oplus C$ and $B = B' \oplus C$ for some submodules A', B'. Moreover,

$$A' \oplus C \oplus B' = A + B \subseteq^{\oplus} M.$$

Let D be a direct summand complement to $A + B$ in M.

From general comparability, fix some central idempotent $e \in \operatorname{End}(M)$ such that $eA' \lesssim eB'$ and $(1 - e)B' \lesssim (1 - e)A'$. It suffices to show that

(1) eA and eB are perspective in complements in eM, and
(2) $(1 - e)A$ and $(1 - e)B$ are perspective in complements in $(1 - e)M$.

Indeed, if we have

$$\begin{aligned} eM &= eA \oplus X_1 = Y_1 \oplus X_1 = Y_1 \oplus eB, \text{ and} \\ (1-e)M &= (1-e)A \oplus X_2 = Y_2 \oplus X_2 = Y_2 \oplus (1-e)B, \end{aligned}$$

then taking $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$, we have

$$M = A \oplus X = Y \oplus X = Y \oplus B$$

as desired. The proofs of (1) and (2) turn out to be equivalent by symmetry considerations, so we will only prove (1).

For notational simplicity we may take $e = 1$, and thus assume $A' \lesssim B'$. Fix an injection $\varphi\colon A' \to B'$, and take

$$P = \{a' + \varphi(a') \,:\, a' \in A'\} \subseteq A' \oplus B' \subseteq M,$$

which essentially acts like the graph of φ. As $M = A' \oplus C \oplus B' \oplus D$, by a straightforward computation we also have $M = P \oplus C \oplus B' \oplus D$. In particular $P \oplus D$ is a complement to $C \oplus B' = B$. It now suffices to show that $P \oplus D$ and A have a common complement.

We have a string of isomorphisms

$$P \oplus D \cong M/(C \oplus B') = M/B \cong A.$$

Thus, by Lemma 4.2, $P \oplus D$ and A will have a common complement if we can show that $(P + D) \cap A = 0$. Given $a \in A$, we can write $a = a' + c$ for some $a' \in A$ and $c \in C$. Thus

$$a = (a' + \varphi(a')) + c + (-\varphi(a')) \in P \oplus C \oplus B' \subseteq P \oplus C \oplus B' \oplus D.$$

If it were the case that $a \in P \oplus D$, then $c = 0$ and $-\varphi(a') = 0$. But as φ is an injection, we get $a' = 0$, and hence $a = a' + c = 0$. $\square$

We should mention that a few of the methods in the previous theorem are also reminiscent of those used in [1, Theorem 3.1].

Corollary 4.5. *If R is a regular ring with R_R satisfying the general comparability axiom, then* $\mathrm{ureg}(R) = \mathrm{pcreg}(R)$*. In particular, this equality holds when R is a right self-injective regular ring.*

Proof. Let M be the right R-module R_R. As first observed by von Neumann, the direct summands of M are closed under pairwise intersection and addition; for a modern proof see [7, Theorem 2.3]. The corollary now follows by Proposition 3.18(2) and Theorem 4.4. Right self-injective regular rings satisfy general comparability by [7, Corollary 9.15]. $\square$

Before answering Lam's conjecture we need two more lemmas. The first is a key ingredient in passing from a direct product of rings to each component.

Lemma 4.6. *Suppose R_1 and R_2 are rings and $a = (a_1, a_2) \in R_1 \times R_2$. Then $a \in \mathrm{reg}(R)$ iff $a_1 \in \mathrm{reg}(R_1)$ and $a_2 \in \mathrm{reg}(R_2)$. The same statement holds replacing the set of regular elements in a ring with the set of unit-regular, strongly regular, or pc-regular elements.*

Proof sketch. Let $b = (b_1, b_2) \in R$. An easy computation shows that $b \in \mathrm{I}(a)$ iff $b_1 \in \mathrm{I}(a_1)$ and $b_2 \in I(a_2)$. One similarly checks that b is a unit (or commutes with a, or is a unit such that $a - b^{-1}$ is idempotent) in $R_1 \times R_2$ if and only if b_i is a unit (respectively, commutes with a_i, or is a unit such that $a_i - b_i^{-1}$ is idempotent) in R_i for each $i \in \{1, 2\}$. $\square$

Second, we need to handle lifting chains of idempotents modulo a certain ideal. Given a module M, we set

$$\Delta = \{\varphi \in \mathrm{End}(M) \,:\, \ker(\varphi) \text{ is essential in } M\}.$$

This is an ideal of $\mathrm{End}(M)$, and the reader is directed to [15, Section 3] for more detailed information about this ideal and its connection to continuous and quasi-continuous modules.

Lemma 4.7. *Let M be a quasi-continuous module, let $R = \mathrm{End}(M)$, and write $\overline{R} = R/\Delta$. If $e, f \in \mathrm{idem}(R)$ with $\overline{e} \sim_r \overline{f}$, then there exists some $g \in \mathrm{idem}(R)$ with $e \sim_r g \sim_\ell f$.*

Proof. Since $\overline{e} \sim_r \overline{f}$, then $\overline{e}\overline{R} = \overline{f}\overline{R}$. Thus

$$\overline{e}\overline{R} \oplus \overline{(1-f)}\overline{R} = \overline{R}. \tag{4.8}$$

By [15, Lemma 3.8], there exist orthogonal idempotents $g_1, g_2 \in \mathrm{idem}(R)$ such that

$$eR = g_1R \text{ and } (1-f)R = g_2R. \tag{4.9}$$

Now, from orthogonality we obtain $R = g_1R \oplus g_2R \oplus (1-g_1-g_2)R$. Idempotents remain orthogonal modulo ideals, and hence

$$\overline{R} = \overline{g_1}\overline{R} \oplus \overline{g_2}\overline{R} \oplus \overline{(1-g_1-g_2)}\overline{R}.$$

Combining this with (4.8) and (4.9), we have $\overline{(1-g_1-g_2)}\overline{R} = 0$, or thinking of this in another way, $1-g_1-g_2 \in \Delta$. The only idempotent with an essential kernel is 0, and so $1-g_1-g_2 = 0$, hence $g_2 = 1-g_1$.

Now (4.9) yields $e \sim_r g_1$ and $(1-f) \sim_r g_2 = (1-g_1)$. The latter relation is equivalent to $f \sim_\ell g_1$. Taking $g = g_1$, we are done. □

Corollary 4.10. *Letting M be a quasi-continuous module, take $R = \mathrm{End}(M)$ and $\overline{R} = R/\Delta$. Given $e, f \in \mathrm{idem}(R)$, if $\overline{e}$ and $\overline{f}$ are connected by a right n-chain, for some $n \geq 2$, then so are the idempotents e and f.*

Proof. Fix idempotents $\overline{g_0}, \overline{g_1}, \overline{g_2}, \ldots, \overline{g_n} \in \mathrm{idem}(\overline{R})$ such that $g_0 = e$ and $g_n = f$, with

$$\overline{g_0} \sim_r \overline{g_1} \sim_\ell \overline{g_2} \sim_r \cdots \overline{g_n}.$$

If $\overline{g_{n-1}} \sim_\ell \overline{g_n} = \overline{f}$, then by Lemma 3.1 $\overline{g_{n-1}} = \overline{f + (1-f)rf}$ for some $r \in R$. Thus, we might as well take $g_{n-1} = f + (1-f)rf \in \mathrm{idem}(R)$, which them implies $g_{n-1} \sim_\ell f$. Similarly, if $\overline{g_{n-1}} \sim_r \overline{g_n}$ then we may assume $g_{n-1} \in \mathrm{idem}(R)$ with $g_{n-1} \sim_r f$.

Repeating this process, we thus have $g_1, g_2, \ldots, g_n \in \mathrm{idem}(R)$ with

$$g_1 \sim_\ell g_2 \sim_r \cdots g_n = f.$$

Further, $\overline{e} \sim_r \overline{g_1}$, and so by the previous lemma there exists some idempotent $g_1' \in \mathrm{idem}(R)$ with $e \sim_r g_1' \sim_\ell g_1$. As $g_1 \sim_\ell g_2$ we also have $g_1' \sim_\ell g_2$. (This is the one place where we need the lower bound $2 \leq n$.) Thus

$$e \sim_r g_1' \sim_\ell g_2 \sim_r g_3 \sim_\ell \cdots g_n = f,$$

which is a right n-chain connecting e to f. □

Using known structure theorems for quasi-continuous modules, we are now able to prove:

Theorem 4.11. *If M is a quasi-continuous module, then taking $R = \mathrm{End}(M)$ we have* $\mathrm{ureg}(R) = \mathrm{pcreg}(R)$.

Proof. According to [15, Corollary 3.13], $\overline{R} = R_1 \times R_2$ where R_1 is right self-injective and regular and where R_2 is reduced. By Corollary 4.5, $\mathrm{ureg}(R_1) = \mathrm{pcreg}(R_1)$. Also reduced rings have only central idempotents, and hence in the ring R_2 we have the much stronger equality $\mathrm{reg}(R_2) = \mathrm{sreg}(R_2)$. By Lemma 4.6 we obtain

$$\mathrm{ureg}(\overline{R}) = \mathrm{pcreg}(\overline{R}). \tag{4.12}$$

Now, suppose that $e, f \in \mathrm{idem}(R)$ are conjugate idempotents. Then $\overline{e}, \overline{f} \in \mathrm{idem}(\overline{R})$ are conjugate in $\overline{R}$. By (4.12) and Proposition 3.18, $\overline{e}$ and $\overline{f}$ are connected by a right 4-chain. Thus, the previous corollary implies that the same holds for e and f. By another application of Proposition 3.18, this time to the ring R rather than $\overline{R}$, we obtain $\mathrm{ureg}(R) = \mathrm{pcreg}(R)$. □

Note that Theorem 4.11 fails for the larger class of CS modules. For instance, take $M = \mathbb{Z} \times \mathbb{Z}$, as a $\mathbb{Z}$-module. It is well-known that there is no upper bound on the lengths of chains of idempotents in $\mathrm{End}(M) \cong \mathbb{M}_2(\mathbb{Z})$; see [4, Proposition 6.9].

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