SPECIAL CLEAN ELEMENTS IN RINGS

DINESH KHURANA, T. Y. LAM, PACE P. NIELSEN, AND JANEZ ŠTER

ABSTRACT. A clean decomposition $a = e + u$ in a ring R (with idempotent e and unit u) is said to be *special* if $aR \cap eR = 0$. We show that this is a left-right symmetric condition. Special clean elements (with such decompositions) exist in abundance, and are generally quite accessible to computations. Besides being both clean and unit-regular, they have many remarkable properties with respect to element-wise operations in rings. Several characterizations of special clean elements are obtained in terms of exchange equations, Bott-Duffin invertibility, and unit-regular factorizations. Such characterizations lead to some interesting constructions of families of special clean elements. Decompositions that are both special clean and strongly clean are precisely spectral decompositions of the group invertible elements. The paper also introduces a natural involution structure on the set of special clean decompositions, and describes the fixed point set of this involution.

1. Introduction

For any unital ring R, the definition of the set of special clean elements in R is motivated by the earlier introduction of the following three well known sets:

$$\operatorname{sreg}(R) \subseteq \operatorname{ureg}(R) \subseteq \operatorname{reg}(R).$$

To recall the definition of these sets, let

$$\mathrm{I}(a) = \{r \in R : a = ara\}$$

denote the set of "inner inverses" for a (as in [28]). Using this notation, the set of regular elements $\operatorname{reg}(R)$ consists of $a \in R$ for which $\mathrm{I}(a)$ is nonempty, the set of unit-regular elements $\operatorname{ureg}(R)$ consists of $a \in R$ for which $\mathrm{I}(a)$ contains a unit, and the set of strongly regular elements $\operatorname{sreg}(R)$ consists of $a \in R$ for which $\mathrm{I}(a)$ contains an element commuting with a. The alternative characterization of $\operatorname{sreg}(a)$ as the set

$$\{a \in R : a \in a^2R \cap Ra^2\}$$

(as well as the aforementioned inclusion relation $\operatorname{sreg}(R) \subseteq \operatorname{ureg}(R)$) will be assumed in this paper; see [33, p. 3583] for additional characterizations. By definition, R is a *regular ring* (resp. *unit-regular ring, strongly regular ring*) if $R = \operatorname{reg}(R)$ (resp. $R = \operatorname{ureg}(R)$, $R = \operatorname{sreg}(R)$).

In connection with the study of Warfield's exchange rings in [40], Nicholson introduced a new class of clean rings in [31]. An element $a \in R$ is said to be *clean* if $a = e + u$ where $e \in \operatorname{idem}(R)$ (the set of idempotents in R), and $u \in \mathrm{U}(R)$ (the group of units in R). The set of clean elements in R is denoted by $\operatorname{cn}(R)$, and R is said to be a *clean ring* if $R = \operatorname{cn}(R)$. In [31] and [8], Nicholson and Camillo-Yu showed that clean rings constitute a proper subclass of exchange rings.

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While the existence of a canonical *spectral decomposition* of every strongly regular element shows that $\mathrm{sreg}\,(R) \subseteq \mathrm{cn}\,(R)$, it is well known (after [19]) that there is in general no inclusion relation between the two sets $\mathrm{ureg}\,(R)$ and $\mathrm{cn}\,(R)$. On the other hand, Camillo and Khurana have shown earlier in [7] (correcting an oversight in [8]) that unit-regular rings are clean. More significantly, they showed that *a ring R is unit-regular iff every element $a \in R$ has a clean decomposition $a = e + u$ such that $aR \cap eR = 0$.* This result brings us to the concept that will play a central role in this paper:

Definition 1.1. An element $a \in R$ is **special clean**, which we will write as $a \in \text{sp-cn}\,(R)$, if $a = e + u$ for some idempotent $e \in \mathrm{idem}\,(R)$ and unit $u \in \mathrm{U}\,(R)$ such that $aR \cap eR = 0$. A ring R is special clean if $R = \text{sp-cn}\,(R)$.

We will often refer to the decomposition $a = e + u$ in this definition as a special clean decomposition.

The definition above is originally due to Abrams and Rangaswamy [1]; a more detailed account on the early history of the introduction of the set $\text{sp-cn}\,(R)$ can be found in §2.

Special clean elements are a remarkable "common refinement" of both clean elements and unit-regular elements. They exist in abundance; they are quite well behaved, and they have very nice properties. In this paper, we lay the foundations for a systematic study of the set $\text{sp-cn}\,(R)$ in any ring R. Some highlights in this study are as follows.

(1.2) *The definition for membership in the set $\text{sp-cn}\,(R)$ is left-right symmetric.* This is proved early in §2 via a general study of any pair $a, b \in R$ satisfying $u := a - b \in \mathrm{U}\,(R)$, for which we show that $aR \cap bR = 0$ iff $Ra \cap Rb = 0$, iff $au^{-1}b = 0$, iff $au^{-1}a = a$. Applying these facts to the case $b \in \mathrm{idem}\,(R)$ gives us a number of simple characterizations for a clean decomposition $a = e + u$ to be special clean, showing in particular that $\text{sp-cn}\,(R) \subseteq \mathrm{ureg}\,(R) \cap \mathrm{cn}\,(R)$; see Theorem 2.3. An example in §4 shows, however, that this inclusion is in general not an equality.

(1.3) In §2, we also characterize special clean elements via strongly regular elements and Bott-Duffin invertible elements, as defined in (2.12). Recall that a "reflexive inverse" for an element $a \in R$ is an element $b \in \mathrm{I}\,(a)$ such that $a \in \mathrm{I}\,(b)$. The main result in this direction is that $a \in \text{sp-cn}\,(R)$ iff it has a reflexive inverse that is strongly regular, iff some element in $\mathrm{I}\,(a)$ is a Bott-Duffin inverse of a relative to some idempotent in R; see Theorem 2.13. These characterizations for $a \in \text{sp-cn}\,(R)$ point to the interesting fact that the notion of a special clean element can be defined using *only* the structure of the multiplicative semigroup $(R, \times)$, and that, as such, this notion would have made sense in the context of the theory of semigroups.

(1.4) While there is in general no inclusion relation between $\mathrm{ureg}\,(R)$ and $\mathrm{cn}\,(R)$, we can still think of the set $\text{sp-cn}\,(R)$ as a suitable kind of "common refinement" of $\mathrm{ureg}\,(R)$ and $\mathrm{cn}\,(R)$. In §3, we define an element $a \in R$ to be *special unit-regular* if $a = fu$ for some $f \in \mathrm{idem}\,(R)$ and $u \in \mathrm{U}\,(R)$ such that $-u \in \mathrm{I}\,(1-f)$. In Theorem 3.5, we show that, in any ring, *special unit-regular elements are precisely the special clean elements.* This result enables us to reinterpret the Camillo-Khurana theorem in [7] as saying that *a ring is unit-regular iff all its elements are special unit-regular.*

(1.5) Specializing the exchange equations approach in [20], Theorem 3.1 shows that $a \in \text{sp-cn}\,(R)$ *iff there exist* $x \in \mathrm{U}\,(R)$ *and* $f \in \text{idem}\,(R)$ *such that* $af = 0$ *and* $xa - fx = 1$. Following the lead of Zhang and Camillo [43, Lemma 8], we also use the exchange equations method in [20] to give an expanded description (Theorem 3.2) of the special clean elements, in a way that is similar to Nicholson's definition of "suitable elements" in [31, 32].

(1.6) In §4, examples are constructed to show that the set $\text{sp-cn}\,(R)$ does not satisfy "Jacobson's Lemma", in the sense that $ab \in \text{sp-cn}\,(R)$ does not imply $ba \in \text{sp-cn}\,(R)$, and also, $1 - ab \in \text{sp-cn}\,(R)$ does not imply $1 - ba \in \text{sp-cn}\,(R)$.

(1.7) In §5, various sufficient conditions are given for membership in the set $\text{sp-cn}\,(R)$. For instance, Theorem 5.1 shows that $a \in \text{sreg}\,(R)$ *iff* a *has a special clean decomposition* $a = e + u$ *such that* $eu = ue$. (The latter property defines a *strongly clean decomposition* in the sense of [33].) The second half of §5 also offers an ad hoc proof for the fact that any square-zero regular element is special clean.

(1.8) Another testimony to the rich structure of the set of special clean decompositions is that it comes with a natural duality (in Theorem 3.7), which sends $a = e + u$ to its "dual" $(1 - e)\,u^{-1} = (-eu^{-1}) + u^{-1}$. (Note that this duality is *undefined* on the set of clean decompositions, since in general $e \in \text{idem}\,(R)$ does not entail $-eu^{-1} \in \text{idem}\,(R)$.) With respect to this duality, the "self-dual" decompositions $a = e + u$ are analyzed in Theorem 5.3, where it is shown that a must be a tripotent (that is, $a^3 = a$).

(1.9) The paper concludes with a relatively detailed study in §6 on the behavior of special clean elements under the passage to corner rings. A somewhat surprising result here is that, for any ring S, $a \in \text{reg}\,(S)$ iff $\text{diag}\,(a, 0, \ldots, 0)$ is special clean in the matrix ring $\mathbb{M}_n(S)$ for any fixed integer $n \geq 2$.

The terminology and notations introduced so far in this section will be used freely throughout the paper. Other standard facts in ring theory needed for our proofs can largely be found in the references [31], [15], [24], and [26].

2. Special Clean Elements: Examples and Characterizations

The main goal of this section is to introduce the idea of special clean decompositions and special clean elements in rings, and to give a number of basic characterizations for these notions. To make this relatively new theory easier to understand, various explicit examples and properties of special clean decompositions and special clean elements will be given in the text.

Instead of working with an equation $a = e + u$ with $e \in \text{idem}\,(R)$ and $u \in \mathrm{U}\,(R)$, we start more generally with an equation $a = b + u$ with $a, b \in R$ and $u \in \mathrm{U}\,(R)$, and first prove the following result which gives several explicit computations for the right ideal intersection $aR \cap bR$. The fact that this intersection is a principal right ideal does not seem to be well known as we have not been able to find an easy reference for it in the ring theory literature.

Proposition 2.1. *Given any equation* $a = b + u$ *where* $a, b \in R$ *and* $u \in \mathrm{U}\,(R)$*, we have*

$$aR \cap bR = au^{-1}bR = (b + bu^{-1}b)\,R = (a - au^{-1}a)\,R = bu^{-1}aR. \tag{2.2}$$

Proof. The second equality in (2.2) holds already on the generator level, since

$$au^{-1}b = (u+b)\,u^{-1}b = b + bu^{-1}b.$$

This implies, in particular, that $au^{-1}bR \subseteq aR \cap bR$. To prove the reverse inclusion, consider any element $ar \in bR$. From $ur = (a-b)\,r \in bR$, we get $ar = au^{-1}ur \in au^{-1}bR$. This proves the first equality in (2.2). Applying all this information to $b = a + (-u)$, we also get

$$bR \cap aR = bu^{-1}aR = (a - au^{-1}a)\,R,$$

with in fact $-bu^{-1}a = a - au^{-1}a$. This completes the proof of (2.2). □

Of course, the left ideal analogue of (2.2) for $Ra \cap R\,b$ also holds. Using this and (2.2), we shall now prove our first main result in this paper, a part of which (especially in the form of Corollary 2.4) was noted earlier in Călugăreanu's work [6]. In the conditions (3) and (4) below, we'll use for the first time the notation $\mathrm{I}\,(r)$ (introduced in [28]) for the set of inner inverses for an element r in a given ring R.

Theorem 2.3. *For any equation $a = b + u \in R$ where $u \in \mathrm{U}\,(R)$, the following six statements are equivalent:*

(1) $aR \cap bR = 0$ *("right zero-intersection condition").*
(2) $au^{-1}b = 0$ *("right zero-product condition").*
(3) $au^{-1}a = a$ *("inner inverse condition" $u^{-1} \in \mathrm{I}\,(a)$).*
(4) $b\,(-u^{-1})\,b = b$ *("inner inverse condition" $-u^{-1} \in \mathrm{I}\,(b)$).*
(5) $bu^{-1}a = 0$ *("left zero-product condition").*
(6) $Ra \cap Rb = 0$ *("left zero-intersection condition").*

Each of these statements implies that $R = aR \oplus bR = Ra \oplus Rb$. In the case where $ab = ba$ (or equivalently, $ub = bu$), the six statements above are equivalent to $ab = 0$.

Proof. The equivalence of the statements (1)–(5) follows directly from the crucial equations in (2.2). Since (3) and (4) are left-right symmetric, it follows that they are also equivalent to (6). If (1) holds, then $aR+bR$ is a *direct* sum which contains the unit u. Thus, $R = aR \oplus bR$, and similarly, $R = Ra \oplus Rb$. In the case where $ab = ba$ (or equivalently, $ub = bu$), the condition (2) can be rewritten as $abu^{-1} = 0$, which boils down to $ab = 0$. □

As an easy consequence of Theorem 2.3, we get the following *additive* characterization result for unit-regular elements in any ring R.

Corollary 2.4. *An element $a \in R$ is unit-regular iff $a \in b + \mathrm{U}\,(R)$ for some $b \in R$ such that $aR \cap bR = 0$.*

Proof. If $u := a - b \in \mathrm{U}\,(R)$ for some $b \in R$ such that $aR \cap bR = 0$, Theorem 2.3 implies that $a = au^{-1}a$ so a is unit-regular. Conversely, if a is unit-regular, writing $a = au^{-1}a$ for some $u \in \mathrm{U}\,(R)$ leads to an element $b := a - u$. Invoking Theorem 2.3 again yields the equation $aR \cap bR = 0$. □

A second consequence of Theorem 2.3 is the following special case of a result of Koliha and Rakočević [23, Theorem 3.2] for a difference of two idempotents a, b in a ring. The relationship between $(a-b)^{-1}$ and $(a+b)^{-1}$ is, however, new.

Corollary 2.5. *Let $a, b \in \mathrm{idem}\,(R)$ be such that $u := a - b \in \mathrm{U}\,(R)$. Then $u^{-1}(1-b)\,u = a$, and $R = aR \oplus bR = R\,a \oplus R\,b$. Also, we have $v := a + b \in \mathrm{U}\,(R)$, with $v^{-1} = u^{-1}vu^{-1}$; or equivalently, $(u^{-1}v)^2 = 1$.*

Proof. Keeping in mind that $(1-b)\,b = 0 = (1-a)\,a$, we have

$$u^{-1}(1-b)\,u = u^{-1}(1-b)\,a = u^{-1}(1-a+u)\,a = a.$$

Right multiplication by $u^{-1}b$ gives $au^{-1}b = 0$. Applying Theorem 2.3, we see that $R = aR \oplus bR = Ra \oplus Rb$. Also, the equations (2), (3), (4), (6) in Theorem 2.3 all hold, so upon writing $v = a+b$ we have

$$u = a - b = au^{-1}a + bu^{-1}b + au^{-1}b + bu^{-1}a = v\,u^{-1}v \in \mathrm{U}\,(R).$$

This implies that $v \in \mathrm{U}\,(R)$ too, with $v^{-1} = u^{-1}vu^{-1}$; or equivalently, $(u^{-1}v)^2 = 1$. ☐

In the rest of this paper, Theorem 2.3 will be applied to the case where b is an *idempotent* $e \in \mathrm{idem}\,(R)$. In this case, $a = e+u$ (with $u \in \mathrm{U}\,(R)$) would be a typical *clean decomposition* (and it would be a *strongly clean decomposition* if $eu = ue$). With Theorem 2.3 at our disposal, we say that a clean decomposition $a = e+u$ is **special** if it satisfies the equivalent conditions in Theorem 2.3 for $b = e$. If such a decomposition exists for $a \in R$, we say that the element a is **special clean** (in R). By Theorem 2.3, these are *left-right symmetric* notions. The set of special clean elements in R will henceforth be denoted by $\text{sp-cn}\,(R)$. In view of the inner inverse condition $a = au^{-1}a$ in Theorem 2.3, we have the fundamental inclusion relations

$$\text{sp-cn}\,(R) \subseteq \mathrm{cn}\,(R) \cap \mathrm{ureg}\,(R) \subseteq \mathrm{cn}\,(R) \cap \mathrm{reg}\,(R), \tag{2.6}$$

where (as in the introductory section) $\mathrm{cn}\,(R)$, $\mathrm{reg}\,(R)$ and $\mathrm{ureg}\,(R)$ denote, respectively, the sets of clean, regular and unit-regular elements in R. It also follows from Theorem 2.3 that, if $a = e+u$ is a special clean decomposition in R, then so is $\varphi\,(a) = \varphi\,(e) + \varphi\,(u)$ in S for any ring homomorphism $\varphi : R \to S$. This shows that $\varphi\big(\text{sp-cn}\,(R)\big) \subseteq \text{sp-cn}\,(S)$. In fact, this inclusion relation holds even for a *nonunital* ring homomorphism φ, if we use the semigroup characterization of the sets $\text{sp-cn}\,(R)$ and $\text{sp-cn}\,(S)$ in Theorem 2.13.

Historical Note. The significance of the extra condition $aR \cap eR = 0$ imposed on a clean decomposition $a = e+u$ first surfaced in the work of Camillo and Khurana in [7], where it was shown that a ring R is unit-regular iff every element $a \in R$ is special clean. The term "special clean" was coined by Abrams and Rangaswamy in their work [1] on regularity conditions for Leavitt path algebras, although they invoked this notion only when *all* elements of a ring satisfy the special clean condition (just as in [7]). This practice was followed by Akalan and Vaš in [2], while Chen [11, 12], Zhang-Camillo [43], Nielsen-Šter [34] and Altun-Özcan [3] later studied special cleanness as an *element-wise* notion. In all of these references, however, element-wise special cleanness was discussed only as a right-side condition. In May, 2016, we pointed out at the 33rd Ohio State-Denison Ring Theory Conference that the left-right symmetry of element-wise special cleanness can be deduced as a consequence of a general result in the paper [18] of Jain and Prasad.[1] To make §2 more self-contained, we have chosen to give a quicker and more direct proof of Theorem 2.3, which not only recaptures the left-right symmetry result (even in the more general case $a = b+u$ for any $a, b \in R$ and $u \in \mathrm{U}\,(R)$), but also offers several computations of $aR \cap bR$ in Proposition 2.1. In retrospect, in the case where $b = e \in \mathrm{idem}\,(R)$, it was these explicit computations that have provided us the new tools for the more extensive investigation of special clean elements in this paper.

[1] In fact, the same remark also applies to the element-wise notion of *special almost cleanness* studied by Akalan and Văs in [2], where "almost clean" means "idempotent + (non-0-divisor)".

For better motivation, some easy examples of special clean elements are given below.

Examples 2.7. **(A)** In any ring R, every $u \in \mathrm{U}(R)$ is special clean, with a unique special clean decomposition $0+u$. More precisely, *if the right annihilator of $a \in R$ is zero, then $a \in$ sp-cn(R) iff $a \in \mathrm{U}(R)$.* (Indeed, a unit-regular element is a unit as long as it is not a left 0-divisor.) To give an explicit example in the ring $R = \mathbb{M}_2(\mathbb{Z})$, the non-0-divisor $A = \mathrm{diag}(1,2) \in R$ has infinitely many clean decompositions (including $\begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$ for any $n \in \mathbb{Z}$). But $A \notin \mathrm{reg}(R)$, so $A \notin$ sp-cn(R).
(B) Every $f \in \mathrm{idem}(R)$ is special clean, with a special clean decomposition

$$f = (1-f) + (2f-1). \tag{†}$$

However, this special clean decomposition for f is in general not unique. In fact, for any $e \in \mathrm{idem}(R)$ such that $f - e \in \mathrm{U}(R)$, Corollary 2.5 implies that $R = fR \oplus eR$. This means that *any clean decomposition of f is automatically a special clean decomposition.* By a result of Wang-Chen [38, Theorem 2.9] and independently Lee-Zhou [30, Lemma 2.4], f is central iff (†) is its only clean decomposition. From what we said above, it follows that *f is central iff (†) is its only special clean decomposition.*
(C) For any $f \in \mathrm{idem}(R)$, we have $-f \in$ sp-cn(R) too, via the special clean decomposition $-f = (1-f) + (-1)$. However, although in general $f+1 \in \mathrm{cn}(R)$, we may have $f+1 \notin$ sp-cn(R), as is shown by the case $f=1$ over the ring $R = \mathbb{Z}$.
(D) One major way in which sp-cn(R) differs from $\mathrm{cn}(R)$ is that $\mathrm{cn}(R)$ is closed under the "Jacobson map" $a \mapsto 1-a$, while sp-cn(R) is (sometimes) not. For instance, $-1 \in$ sp-cn$(\mathbb{Z}_4)$, but $1-(-1) = 2 \notin$ sp-cn$(\mathbb{Z}_4)$. More generally, we observe that, while $\mathrm{cn}(R) \supseteq \mathrm{rad}(R)$ (the Jacobson radical of R), we have always

$$\text{sp-cn}(R) \cap \mathrm{rad}(R) \subseteq \mathrm{ureg}(R) \cap \mathrm{rad}(R) = 0.$$

(E) Let R be the ring of 2×2 upper triangular matrices over a ring S. For any $a \in \mathrm{U}(S)$ and any $b \in S$, the matrix $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$ has a clean decomposition $E + U$ where $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}$. We can check easily that $EU^{-1}E = -E$, so Theorem 2.3 implies that $A \in$ sp-cn(R).
(F) Let $R = \mathbb{M}_2(\mathbb{Z})$. According to [22, §1], the unit-regular matrix $A = \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix} \in R$ has *exactly* the following three clean decompositions:

$$A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}. \tag{$*$}$$

Again, an easy direct computation checking any of the conditions in Theorem 2.3 shows that *each* of the three clean decompositions above is special clean.
(G) If $a \in$ sp-cn(R), it need not follow that every clean decomposition of a is a special clean decomposition. For instance, in any division ring other than $\mathbb{F}_2$, any element $a \notin \{0,1\}$ has a special clean decomposition $a = 0 + a$, but also a nonspecial clean decomposition $a = 1 + (a-1)$. More nontrivially, taking $R = \mathbb{M}_2(\mathbb{Z})$ again, the matrix $B \in R$ below has (at least) two clean decompositions:

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \tag{$**$}$$

the first one of which is a special clean decomposition, but the second one is not.
(H) Let $a = e + u$ be a special clean decomposition in R. *For any* $w \in \mathrm{U}(R) \cap \mathrm{I}(e)$, *we have* $aw, wa \in \text{sp-cn}(R)$. Indeed, $aw = ew + uw$ is a special clean decomposition since $ew \in \text{idem}(R)$, $uw \in \mathrm{U}(R)$, and $awR \cap ewR = aR \cap eR = 0$. This shows that $aw \in \text{sp-cn}(R)$, and $wa \in \text{sp-cn}(R)$ can be seen similarly. Applying these remarks to $w = 2e - 1 \in \mathrm{U}(R) \cap \mathrm{I}(e)$ (for instance), we get $a(2e-1), (2e-1)a \in \text{sp-cn}(R)$. For an explicit example, if we take the second special clean decomposition in $(*)$ under (F), left multiplication by $2e - 1$ leads to a new one:

$$\begin{pmatrix} -5 & -3 \\ 20 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -5 & -3 \\ 18 & 11 \end{pmatrix} \in \text{sp-cn}\left(\mathbb{M}_2(\mathbb{Z})\right).$$

(I) If $(R, *)$ is a ring with involution, then every special clean decomposition $a = e + u \in R$ induces another special clean decomposition $a^* = e^* + u^*$, since $u^{-1} \in \mathrm{I}(a)$ implies that $(u^*)^{-1} \in \mathrm{I}(a^*)$. In particular, the set $\text{sp-cn}(R)$ is automatically $*$-invariant.

In general, the set $\text{sp-cn}(R)$ seems to have considerably richer properties than the set of clean elements $\text{cn}(R)$. For instance, while $\text{cn}(R)$ is usually not closed with respect to the "negation map" $a \mapsto -a$, the new set $\text{sp-cn}(R)$ (for any ring R) has the remarkable property that it is closed with respect to multiplication by central units, in a rather strong sense described in the following result.

Theorem 2.8. *Let* $a = e + u \in R$ *be a special clean decomposition, and let* $w \in \mathrm{U}(R)$.

(1) *If* $aw = wa$, *we have a special clean decomposition* $aw = e + (aw - e)$.
(2) *If* $ew = we$, *we have a special clean decomposition* $aw = e + (aw - e)$, *and also a special clean decomposition* $wa = e + (wa - e)$.

Thus, $aw, wa \in \text{sp-cn}(R)$ *if* $w \in \mathrm{U}(R)$ *commutes with* a *or with* e. *In particular,* $\text{sp-cn}(R)$ *is closed with respect to multiplication by any central unit of* R.

Proof. To begin with, we note that this result is rather different from that in Example 2.7(H) above. In both statement (1) and statement (2) here, the unit w is subject to different assumptions, and the conclusions are also different.
(1) Given that $aw = wa$, it suffices to check that $aw - e \in \mathrm{U}(R)$, since $awR \cap eR = aR \cap eR = 0$. Thinking of $aw - e$ as an endomorphism on R_R (by its left action), we need to check that $\ker(aw - e) = 0$ and $\operatorname{im}(aw - e) = R$. For the former, note that

$$(aw - e)(r) = 0 \Rightarrow aw(r) = er \in aR \cap eR = 0.$$

Since $aw = wa$ and $w \in \mathrm{U}(R)$, this gives $ar = 0$ too. Therefore, $r = 0$ (as $a - e = u \in \mathrm{U}(R)$). Next, we note that $\operatorname{im}(aw - e)$ contains $(wa - e)(u^{-1}e) = e$ (by (2) and (4) in Theorem 2.3), and hence aw. Thus, it contains $eR + awR = R$, as desired.
(2) Assume now that $ew = we$. Here we may not have $aw = wa$, so the proof method used in (1) no longer applies. Proceeding differently, we first check by a straightforward calculation (using $ew = we$) that

$$(\dagger) \qquad 1 - e(1 - w^{-1}) \in \mathrm{U}(R), \text{ with inverse } 1 - e(1 - w).$$

To show that $aw = e + (aw - e)$ is a special clean decomposition, it suffices (as in (1) above) to check that $aw - e = (e + u)w - e = uw + e(w - 1)$ is a unit. Right multiplying by $w^{-1}u^{-1}$ (and using now $w^{-1}e = ew^{-1}$), this amounts to checking that

$$1 + e(1 - w^{-1})u^{-1} = 1 + e(1 - w^{-1})eu^{-1} \in \mathrm{U}(R).$$

By Jacobson's Lemma, this further reduces to checking that $1 + eu^{-1}e\,(1 - w^{-1}) \in \mathrm{U}\,(R)$, which follows from (†) above since $eu^{-1}e = -e$. This completes the proof that $aw = e + (aw - e)$ is a special clean decomposition. In a similar way (or just using left-right symmetry), we can show that $wa = e + (wa - e)$ is also a special clean decomposition. □

Remark 2.9. By a relatively routine modification of the proof above, we see that the conclusion in Theorem 2.8(1) remains valid if we only assume that $w \in \mathrm{U}\,(R)$ is such that $aw = w'a$ for some $w' \in \mathrm{U}\,(R)$. For any such w, Theorem 2.8(1) would still yield the conclusion that $aw - e \in \mathrm{U}\,(R)$, which may not have been true if $a = e + u$ is merely a clean decomposition. For instance, taking $w = -1$, the conclusion $a + e \in \mathrm{U}\,(R)$ holds for the first clean decomposition of B in (∗∗) under Example 2.7(G), but it does not hold for the second.

Example 2.10. While (2.7)(C) provides a quick example of the closure of $\text{sp-cn}\,(R)$ with respect to multiplication by central units, it is of interest to double-check the conclusion of Theorem 2.8 in some less trivial cases. Take, for instance, the special clean decomposition for $A = \begin{pmatrix} -5 & -3 \\ 20 & 12 \end{pmatrix} = E + U$ in $R = \mathbb{M}_2(\mathbb{Z})$ obtained in Example 2.7(H). If we multiply A by $-I_2$ (and keep the same idempotent E), we do get a special clean decomposition $-A = \begin{pmatrix} 5 & 3 \\ -20 & -12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 3 \\ -22 & -13 \end{pmatrix}$, with the last matrix having determinant 1 (and hence in $\mathrm{U}\,(R)$). If we enlarge the ring R to $R' = \mathbb{M}_2\big(\mathbb{Z}\,[i]\big)$ where $i = \sqrt{-1}$, then multiplying A by $i\,I_2$ (and again keeping the same idempotent E) leads to the following special clean decomposition:

$$i\,A = \begin{pmatrix} -5\,i & -3\,i \\ 20\,i & 12\,i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -5\,i & -3\,i \\ 20\,i - 2 & 12\,i - 1 \end{pmatrix},$$

where the last matrix is in $\mathrm{U}\,(R')$ since it has determinant $-i$. On the other hand, if $A_1 \in \text{sp-cn}\,(R)$ but $W_1 \in \mathrm{U}\,(R)$ *fails* to commute with A_1, we may have $A_1W_1 \notin \text{sp-cn}\,(R)$. For instance, the idempotent matrix unit $A_1 := E_{11}$ is special clean by (2.7)(B). However, for $W_1 = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} \in \mathrm{U}\,(R)$, the product $A_1W_1 = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is not even a clean matrix according to [19, Example 4.5].

As another consequence of Theorem 2.3, we record below a necessary condition on special clean decompositions in terms of the unit group $\mathrm{U}\,(R)$. In this result, the notation $x \circ y$ denotes "Jacobson's circle product" $x + y - xy$, as defined in Jacobson's book [17, p. 8]. The result shows that there are certain units of R that arise naturally from any special clean decomposition $a = e + u$ in R.

Proposition 2.11. *If $a = e + u \in R$ is a special clean decomposition, then*

$$a \circ e = u \circ e = u + e - ue \in \mathrm{U}\,(R), \quad \textit{and} \quad e \circ a = e \circ u = u + e - eu \in \mathrm{U}\,(R),$$

but not conversely in general. In this case, we have

$$(u \circ e)^{-1} = u^{-1} \circ (-eu^{-1}), \quad \textit{and} \quad (e \circ u)^{-1} = (-u^{-1}e) \circ u^{-1}.$$

Proof. To begin with, $a \circ e = (e + u) + e - (e + u)\,e = u + e - ue = u \circ e$. Letting $v = u^{-1}$, we have $u + e - ue = u\,(1 + ve - e)$, so it suffices to show that $t := 1 - e + ve \in \mathrm{U}\,(R)$. This is the case since the equation $eve = eu^{-1}e = -e$ in (2.3)(4) implies that

$$t^2 = (1 - e + ve)\,(1 - e + ve) = (1 - e) + (1 - e)\,ve + ve\,(1 - e) + v(eve) = 1.$$

Using this, we see that $(u \circ e)^{-1} = (ut)^{-1} = tu^{-1} = v - ev + vev = v \circ (-ev)$. The conclusions about $e \circ u$ and $(e \circ u)^{-1}$ can be proved similarly. (Here, we use the parallel fact that $s := 1 - e + ev$ satisfies $s^2 = 1$.) To see that these conclusions have no converse (in general), we simply let $u = 1$. In this case, $u \circ e = e \circ u = 1 \in \mathrm{U}(R)$, but using (2.3)(4) again, $a = e + 1$ is *not* a special clean decomposition if $-e \neq e$. □

Our next characterization result for special clean elements is best expressed in terms of the notion of Bott-Duffin invertibility (first introduced in [5]).

Definition 2.12. For any $f \in \mathrm{idem}(R)$, an element $a \in R$ is said to be *Bott-Duffin invertible relative to* f if $faf \in \mathrm{U}(fRf)$. In this case, the inverse b of faf in the group $\mathrm{U}(fRf)$ is called the *Bott-Duffin inverse* of a relative to f. (The defining equations for b are thus $fb = b = bf$ and $fab = f = baf$.)

The following basic result gives three new characterizations for an element $a \in R$ to be special clean. Here, (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (4) may be called the BD-*criteria for special cleanness,* while (1) $\Leftrightarrow$ (3) may be called the *reflexive inverse criterion for special cleanness*, where reflexive inverses were defined in (1.3). Remarkably, all three criteria here are expressed solely in terms of the multiplicative structure of the ring R.

Theorem 2.13. *For any $a \in R$, the following statements are equivalent:*

(1) $a \in \text{sp-cn}(R)$.
(2) *a has a reflexive inverse b that is a Bott-Duffin inverse of a relative to some $f \in \mathrm{idem}(R)$.*
(3) *a has a reflexive inverse $b \in \mathrm{sreg}(R)$ (the set of strongly regular elements).*
(4) *There exists some $b \in \mathrm{I}(a)$ that is a Bott-Duffin inverse of a relative to some $f \in \mathrm{idem}(R)$.*

In statement (4), *we have a special clean decomposition $a = (1 - f) + (a - 1 + f)$.*

Proof. (1) $\Rightarrow$ (2). Assume that a has a special clean decomposition $a = e + u$, and let $f = 1 - e$, $v = u^{-1}$. Then $v \in \mathrm{I}(a)$ by Theorem 2.3. By a standard argument of von Neumann (see, e.g. [26, Theorem 2.15]), $b := vav$ is a reflexive inverse of a. Theorem 2.3 also gives $eb = evav = 0$, so we have $b \in fR$. It follows similarly that $b \in Rf$, and hence $b \in fRf$. Finally,

$$fab = favav = fav = f\,(e + u)\,v = f,$$

and similarly $baf = f$. Thus, a has Bott-Duffin inverse b relative to f.
(2) $\Rightarrow$ (3). For b, f as in (2), we have $b \in \mathrm{U}(fRf)$, say with inverse x in the corner ring. Then $b^2x = xb^2 = b$, so $b \in \mathrm{sreg}(R)$.
(3) $\Rightarrow$ (4). Given b as in (3), write it in the form $fu = uf \in fRf$ for some $f \in \mathrm{idem}(R)$ and some $u \in \mathrm{U}(R)$. (See [33, p. 3584], or [26, Theorem 3.12].) Then $fu = (fu)\,a\,(fu) \Rightarrow f = fuaf = b\,(faf)$, and we get similarly $f = (faf)\,b$. Thus, $faf \in \mathrm{U}(fRf)$, with inverse $b \in \mathrm{I}(a)$ in the ring fRf. This proves (4).
(4) $\Rightarrow$ (1). Given $b, f \in R$ as in (4), let $e := 1 - f$, and $v := b - (1 - ba)\,(1 - ab)$. Left multiplying the latter equation by a gives $av = ab$, so $ava = aba = a$. Also, since $b \in fRf$, we have $eb = 0$, so left multiplying $v = b - (1 - ba)\,(1 - ab)$ by e gives $ev = -e\,(1 - ab)$. Therefore,

$$av - ev = ab + e\,(1 - ab) = e + (1 - e)\,ab = e + fab = e + f = 1,$$

and we can similarly show that $va - ve = 1$. Thus, $v \in \mathrm{U}(R)$, with inverse $a - e$. Since $ava = a$, Theorem 2.3 shows that a has the special clean decomposition $a = e + v^{-1} = (1 - f) + (a - 1 + f)$. □

Remark 2.14. By the work of Azumaya [4], any $a \in \mathrm{sreg}(R)$ satisfies the statement (3) above by taking b to be the "group inverse" of a (which means the unique reflexive inverse of a satisfying $ab = ba$). Thus, Theorem 2.13 implies that

$$\mathrm{sreg}(R) \subseteq \text{sp-cn}(R) \tag{2.15}$$

for any ring R. Moreover, this inclusion will also follow from Theorem 5.1 in §5. However, if $a \in R$ has a reflexive inverse $b \in \text{sp-cn}(R)$, it need not follow that $a \in \text{sp-cn}(R)$. For instance, $A = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$ has reflexive inverse $B = \begin{pmatrix} -2 & 0 \\ 5 & 0 \end{pmatrix}$, which has a special clean decomposition $\begin{pmatrix} 3 & 1 \\ -6 & -2 \end{pmatrix} + \begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix}$. But the matrix A is not even clean in $\mathbb{M}_2(\mathbb{Z})$, again according to [19, Example 4.5].

Remark 2.16. Let $a = e + u \in R$ be a special clean decomposition. Since a is Bott-Duffin invertible relative to the idempotent $f = 1 - e$ (by the proof of $(1) \Rightarrow (2)$ in Theorem 2.13), we know from the general theory of Bott-Duffin invertibility that $1 - f + af \in \mathrm{U}(R)$; see [26, Theorem 10.21]. Since $1 - f + af = e + a(1 - e) = a \circ e$, we have retrieved the fact that $a \circ e \in \mathrm{U}(R)$ that was proved in Proposition 2.11. In fact, this was how we were led to the result in (2.11) in the first place.

We conclude this section by applying Theorem 2.13 to prove the following characterization result for unit-regular rings in terms of the notion of reflexive inverses.

Theorem 2.17. *For any ring R, the following four statements are equivalent:*

(1) *R is a unit-regular ring.*
(2) *Every element in R has an inner inverse in* $\mathrm{ureg}(R)$.
(3) *Every element in R has a reflexive inverse in* $\mathrm{ureg}(R)$.
(4) *Every element in R has a reflexive inverse in* $\mathrm{sreg}(R)$.

Proof. The implications $(4) \Rightarrow (3) \Rightarrow (2)$ are trivial. On the other hand, $(2) \Rightarrow (1)$ follows from the fact (proved in [26, Theorem 2.15(3)]) that, *in any ring*, an element is unit-regular as long as it has a unit-regular inner inverse. For the final implication $(1) \Rightarrow (4)$, let R be a unit-regular ring. By the main theorem of Camillo and Khurana in [7], we have $R = \text{sp-cn}(R)$. Thus, by $(1) \Rightarrow (3)$ in Theorem 2.13, every element in R has a reflexive inverse in $\mathrm{sreg}(R)$. □

3. Special Clean Through Three Other Specializations

In this section, we shall demonstrate the universal nature of the set $\text{sp-cn}(R)$ by showing that it can be reached not only (by definition) as a specialization of $\mathrm{cn}(R)$ (the set of clean elements), but also as a specialization of $\mathrm{suit}(R)$ (the set of suitable elements in the sense of Nicholson's papers [31, 32]), or $\mathrm{ureg}(R)$ (the set of unit-regular elements), or the set of elements satisfying an "exchange equation" in the sense of [20]. We will start with a discussion on the last specialization in this list.

In the recent paper [20], three of the present authors gave characterizations for suitable, clean, and strongly clean elements in rings in terms of the theory of *exchange equations*;

namely, linear equations of the form $xa - fx = 1$ where a is a given element in a ring R and f is an idempotent in R. It is natural to ask if special clean elements in R can be characterized as elements satisfying a special type of exchange equation. Recalling from [20] that the *exchange polynomial* associated with $a \in R$ is the quadratic polynomial $P_a(x) := (1 - xa)\,(1 + (1 - a)\,x)$, our result in this direction is as follows.

Theorem 3.1. *For any $a \in R$, the following statements are equivalent:*

(1) $a \in \text{sp-cn}\,(R)$.
(2) *There exist $f \in \text{idem}\,(R)$ and $x \in \mathrm{U}\,(R)$ such that $af = 0$ and $xa - fx = 1$.*
(3) *The equation $P_a(x) = 0$ has a root $x \in \mathrm{U}\,(R) \cap \mathrm{I}\,(a)$.*

Proof. (1) ⇒ (2). Assume that a has a special clean decomposition $a = e + u$. Letting $x = u^{-1}$ and $f = xex^{-1} \in \text{idem}\,(R)$, left multiplying $a = e+u$ by x gives $xa = xe+1 = fx+1$, so $xa - fx = 1$. By Theorem 2.3(2), we have $axe = 0$. Therefore, $af = a\,(xex^{-1}) = 0$.
(2) ⇒ (3). Let x and f be as in (2). Left multiplying $xa - fx = 1$ by a gives $axa = a$, so $x \in \mathrm{U}\,(R) \cap \mathrm{I}\,(a)$. By [20, Proposition 3.3(D)], we have $P_a(x) = 0$.
(3) ⇒ (1). For $x \in R$ as in (3), the final sentence of [20, Theorem 3.4] asserts that x is a *suitabilizer* for a, i.e. there is some $f \in \text{idem}\,(R)$ satisfying $xa - fx = 1$. Solving for a gives $a = x^{-1}fx + x^{-1}$. Since $a = axa$, the implication (3) ⇒ (1) in Theorem 2.3(3) shows that $a \in \text{sp-cn}\,(R)$. □

The next natural goal in this section is to try to characterize special clean elements in a ring R as a special type of the *suitable elements* in R. (According to Nicholson [32], an element $a \in R$ is *suitable* if there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. This is well known to be a left-right symmetric notion for elements.) This goal was first accomplished by Zhang and Camillo in [43, Lemma 8], where they proved the equivalence of (1) and (2) in the theorem below. Here we give an alternative simpler approach to the Zhang-Camillo result by adding a slightly varied third condition (3) to the equivalence list, while fully exploiting the method of exchange equations in formulating our different proof.

Theorem 3.2. *For any $a \in R$, the following statements are equivalent:*

(1) $a \in \text{sp-cn}\,(R)$.
(2) *There exist $e \in \text{idem}\,(R)$ and $v \in \mathrm{U}\,(R)$ such that*
$$ae = a, \quad e = eva, \quad \textit{and} \quad 1 - e = (1 - e)\,v\,(1 - a).$$
(3) *There exist $e \in \text{idem}\,(R)$ and $x \in \mathrm{U}\,(R)$ such that*
$$ae = a, \quad e = exa, \quad \textit{and} \quad 1 - e = -(1 - e)\,x\,(1 - a).$$

Proof. For any $e \in \text{idem}\,(R)$, one can quickly check that $(2e - 1)^2 = 1$. The equivalence of (2) and (3) is easily seen by noting that the "unit change" $v = (2e-1)\,x$ (with $v, x \in \mathrm{U}\,(R)$ and $e \in \text{idem}\,(R)$) entails $ev = ex$ and $(1 - e)\,v = -(1 - e)\,x$. Thus, we may complete the proof of the theorem by showing the equivalence of (1) and (3), independently of the proof of [43, Lemma 8].
(1) ⇒ (3). Assuming (1), we have by Theorem 3.1 an exchange equation $xa - fx = 1$ for some $x \in \mathrm{U}\,(R)$ and some $f \in \text{idem}\,(R)$ such that $af = 0$. Letting $e := 1 - f \in \text{idem}\,(R)$, we have $ae = a$, and left multiplying $xa - fx = 1$ by e gives $e = exa$. Also, rewriting $xa - fx = 1$ in the form $1 = ex - x\,(1 - a)$, left multiplying by f gives $f = -fx\,(1 - a)$. This proves (3).

(3) $\Rightarrow$ (1). Given e, x as in (3), the idempotent $f := 1 - e$ has the property that $af = 0$, and the last equation in (3) can be written in the form

$$1 - e = -fx + (1 - e)\, xa = -fx + xa - e.$$

Thus, $xa - fx = 1$ (with $x \in \mathrm{U}(R)$ and $af = 0$), so Theorem 3.1 gives (1). ☐

Remark 3.3. No understanding of the result above can be complete without citing some of its precedents in the earlier literature. The condition (3) in Theorem 3.2 *without the requirement* $ae = a$ has been investigated before by Chen-Cui in [13, Theorem 3.7] and by H. Zhang in [42, Proposition 2]. Indeed, such a modified version of (3) was shown to be equivalent to $a \in \mathrm{cn}(R)$, and if the requirement $x \in \mathrm{U}(R)$ is relaxed to just $x \in R$, then the further weakened version of (3) was shown to be equivalent to $a \in \mathrm{suit}(R)$.

Having proved Theorem 3.2, the remaining goal of this section is now the description of $\text{sp-cn}(R)$ as a "special subset" of the set $\mathrm{ureg}(R)$ of the unit-regular elements of R. Such a description turns out to be quite useful since it leads directly to a variety of new information on the structure of the set $\text{sp-cn}(R)$. As is well known [25, (4.14B)], we have $a \in \mathrm{ureg}(R)$ iff a has a (multiplicative) "ureg-factorization" $a = gu$ where $g \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$. Given this characterization, unit-regular elements may be thought of as multiplicative analogues of clean elements (which are elements of the form $e + u$ where $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$). Considering that $\mathrm{idem}(R) \subseteq \text{sp-cn}(R) \subseteq \mathrm{ureg}(R)$, we thus have

$$\mathrm{ureg}(R) = \mathrm{idem}(R) \cdot \mathrm{U}(R) = \text{sp-cn}(R) \cdot \mathrm{U}(R).$$

Since we have at our disposal the useful notion of special clean decompositions, it would be natural to ask if there is also a comparable notion of *special ureg-factorizations.* In the following proposition, we shall introduce such a notion, where we may think of the defining property (1) as largely inspired by the property (4) in Theorem 2.3.

Proposition 3.4. *For any ureg-factorization* $a = gu$ *where* $g \in \mathrm{idem}(R)$ *and* $u \in \mathrm{U}(R)$, *and for* $h := 1 - g$, *the following conditions are equivalent:*

(1) $-u \in \mathrm{I}(h)$ (*that is,* $-u$ *is an inner inverse of* h).
(2) $-hu \in \mathrm{idem}(R)$.
(3) $a - u \in \mathrm{idem}(R)$.

If these conditions hold, we'll say that $a = gu$ *is a special ureg-factorization. If such a factorization exists, we'll say that* a *is a special unit-regular element of* R.

Proof. Since (2) means $(hu)(hu) = -hu$ (where u is a unit), (2) is equivalent to (1). On the other hand, since $-hu = (g - 1)\, u = a - u$, (2) is also equivalent to (3). ☐

Strictly speaking, the two notions introduced in Proposition 3.4 should have been called *right* special ureg-factorization and *right* special unit-regular element (with the unit factor u sitting on the right). Similarly, a *left* ureg-factorization should mean ug' with $u \in \mathrm{U}(R)$ and $g' \in \mathrm{idem}(R)$, and we should use the term *left* special ureg-factorization if additionally $-u \in \mathrm{I}(1 - g')$. However, if $a = gu$ is a right ureg-factorization, then $a = u\,(u^{-1}gu)$ is a left ureg-factorization, and it is easy to show that the former one is special iff the latter one is special. Because of the existence of such a one-one correspondence, it will be sufficient for us to work only with the right ureg-factorizations and their "special" versions, conveniently omitting the adjective "right" in referring to them. Quite pleasantly, we have the following theorem relating special ureg-factorizations of $a \in R$ to the special clean decompositions of the same element a introduced in the last section. The last part of the theorem shows that, although it is not clear at all how to directly relate $\mathrm{cn}(R)$ to $\mathrm{ureg}(R)$ in general, the "special subsets" of these two sets turn out to be the same!

Theorem 3.5. *For any $a \in R$, there is a bijective map from the set of special clean decompositions of a to the set of special ureg-factorizations of a. In particular, a is special clean iff a is special unit-regular.*

Proof. From any special clean decomposition $a = e + u$, we may construct a special ureg-factorization of a as follows. Invoking the property (3) in Theorem 2.3, we have $a = au^{-1}a$, so $g := au^{-1} \in \mathrm{idem}(R)$, and $a = gu$ is a ureg-factorization. As $a - u = e \in \mathrm{idem}(R)$, Proposition 3.4 shows that we have arrived at a *special* ureg-factorization $a = gu$. Conversely, starting with a special ureg-factorization $a = gu$, let $h = 1 - g \in \mathrm{idem}(R)$. Proposition 3.4 shows that $e := -hu \in \mathrm{idem}(R)$. Subtraction gives $a - e = (g + h)\,u = u$, so $a = e + u$. As $aR \cap eR = gR \cap hR = 0$, we have arrived at a *special* clean decomposition $a = e + u$.

By easy inspection, we can check that the two "mappings" between special clean decompositions and special ureg-factorizations of the given element a defined in the last paragraph are inverse maps of each other, as desired. From this, it follows of course that an element $a \in R$ is special clean iff it is special unit-regular. □

In summary of what we have done so far, we state the following corollary in three parts, the first two of which follow, respectively, by using special clean decompositions and special ureg-factorizations. The last part follows from the last statement of (3.5) and the Camillo-Khurana theorem (from [7]) that unit-regular rings are special clean.

Corollary 3.6. (1) *The set* $\mathrm{sp\text{-}cn}(R)$ *consists of all sums* $e + u$ *where* $e \in \mathrm{idem}(R)$*,* $u \in \mathrm{U}(R)$*, and* $-u^{-1} \in \mathrm{I}(e)$.
(2) *The set* $\mathrm{sp\text{-}cn}(R)$ *also consists of all products* gu *where* $g \in \mathrm{idem}(R)$*,* $u \in \mathrm{U}(R)$ *and* $-u \in \mathrm{I}(1-g)$.
(3) *A ring R is unit-regular iff every element in R is special unit-regular.*

Another interesting application of Theorem 3.5 is that it induces a kind of involutive structure on the set of all special clean decompositions in any ring R.

Theorem 3.7. *If $a = e + u \in R$ is a special clean decomposition, so is its "dual":*

$$(1 - e)\,u^{-1} = (-eu^{-1}) + u^{-1}. \tag{$*$}$$

If we apply this dual construction procedure to the special clean decomposition ($*$), we get back the original special clean decomposition $a = e + u$.

Proof. First of all, the equation ($*$) trivially holds. By Theorem 2.3(4), we have $e\,(-u^{-1})\,e = e$, so $e_0 := -eu^{-1} \in \mathrm{idem}(R)$. This checks that ($*$) is a clean decomposition of $(1 - e)\,u^{-1}$. Finally, since

$$e_0 u e_0 = (-eu^{-1})\,u\,(-eu^{-1}) = e\,(eu^{-1}) = eu^{-1} = -e_0,$$

Theorem 2.3 implies that ($*$) is a special clean decomposition of $(1 - e)\,u^{-1}$.

If we apply the dual construction procedure to ($*$), we'll get a special clean decomposition whose "unit term" is u, and whose "idempotent term" is $-(-eu^{-1})\,u = e$. This means that we get back precisely the original decomposition $a = e + u$. □

With the involution defined above on the set of all special clean decompositions, it is of interest to describe the set of fixed points of this involution. This study will be undertaken a little bit later in Theorem 5.3.

Remark 3.8. The idea of defining an involutive structure on certain sets of two-term additive decompositions can also be utilized as follows. Let us say that $a = b + u$ is a **preclean decomposition** of $a \in R$ if $b \in \operatorname{ureg}(R)$ and $u \in \mathrm{U}(R)$. Then the passage from $a = b+u$ to $(1-b)\,u^{-1} = (-bu^{-1}) + u^{-1}$ can be easily checked to be an involution on the set of all preclean decompositions, upon noting that

$$b \in \operatorname{ureg}(R) \;\Rightarrow\; -bu^{-1} \in \operatorname{ureg}(R).$$

By restricting this involution to the set of all special clean decompositions, we get back the main conclusions of Theorem 3.7. However, this involution in general does not induce a self-map on the set of clean decompositions (since $b \in \operatorname{idem}(R)$ need not imply $-bu^{-1} \in \operatorname{idem}(R)$). Indeed, in view of the criterion in Theorem 2.3(4), the set of special clean decompositions is exactly the maximal subset of clean decompositions that is preserved by the involution on the set of all preclean decompositions.

Remark 3.9. As the reader might have noticed, there is in fact a *second* involutive structure on the set of special clean decompositions in R, which can be gotten by repeating the work above by using *left* ureg-factorizations. This second involution simply takes a special clean decomposition $e+u$ to $u^{-1}(1-e) = (-u^{-1}e) + u^{-1}$. If we compose the two involutions one way or the other, we'll get two mutually inverse permutations on the set of special clean decompositions, one taking $e+u$ to $ueu^{-1}+u$, and the other taking $e+u$ to $u^{-1}eu+u$. In particular, the n-th power of the first permutation (for any $n \in \mathbb{Z}$) would take a special clean decomposition $e+u$ to $u^n eu^{-n}+u$. In retrospect, the fact that $u^n eu^{-n}+u$ is a special clean decomposition (for all n) can be easily checked using the criterion (4) in Theorem 2.3.

Example 3.10. In illustration of Theorem 3.5, take again the matrix $A = \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix} \in R = \mathbb{M}_2(\mathbb{Z})$. Assuming what we said in Example 2.7(F), we may conclude from Theorem 3.5 that there are *precisely three special ureg-factorizations* for A in R. A routine computation (using the proof of (3.5)) shows that they are as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}. \tag{3.11}$$

On the other hand, while A has exactly three clean decompositions (as claimed in (2.7)(F) and [22]), it has *infinitely many* ureg-factorizations, including for instance:

$$A = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5-n\,(3+5\,m) & 3-n\,(2+3\,m) \\ 3+5\,m & 2+3\,m \end{pmatrix} \quad (\text{for any } n,\, m \in \mathbb{Z}\,), \tag{3.12}$$

where the second matrix on the right-hand side has visibly determinant 1. As for the matrix $B = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$, it certainly also has many ureg-factorizations; for instance $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}$. But *none of them* can be special, since otherwise B would be special clean by Theorem 3.5, while according to [19, (4.5)] B is not even clean in R.

4. Insufficient Conditions for Special Clean Elements

The first basic issue pertinent to the caption of this section is obviously the following. Since we know from (2.6) that special clean elements (in any ring) are always unit-regular and clean, it is natural to ask whether the *converse* of this statement also holds; that is, whether unit-regular clean elements are always special clean. A "yes" answer to this question would have given a relatively simple description of the set $\text{sp-cn}\,(R)$ in any ring R. However, in the following, we shall give an example to show that such a naive description of the set $\text{sp-cn}\,(R)$ is, in general, not possible.

Example 4.1. Let $R = \begin{pmatrix} \mathbb{Z} & 3\,\mathbb{Z} \\ 3\,\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, and let $A = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \in \text{ureg}\,(R)$ (with unit inner inverse $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$). Since $V := A - I_2 = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \in \mathrm{U}\,(R)$, we have $A \in \text{cn}\,(R)$. We claim that A *is uniquely clean in* R, which will show that $A \notin \text{sp-cn}\,(R)$ since $A = I_2 + V$ is obviously *not* a special clean decomposition. To prove our claim, assume instead that there is a clean decomposition $A = E + U$ in R with $E \neq I_2$. Clearly, $E \neq 0$ too, so we have $E = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x + w = \text{tr}\,(E) = 1$ and $xw - yz = \det\,(E) = 0$. Moreover, the invertibility of $A - E$ yields

$$\det\,(A - E) = -(2 - x)\,w + z\,(3 - y) \in \{\pm 1\}. \tag{4.2}$$

Reading these equations modulo 9 and recalling that $y, z \in 3\,\mathbb{Z}$, we have

$$\overline{x}\,\overline{w} = 0, \quad \overline{x} + \overline{w} = 1, \quad \text{and} \quad (2 - \overline{x})\,\overline{w} \in \{\pm 1\}. \tag{4.3}$$

The third equation implies that $\overline{w}$ is invertible, so the first equation gives $\overline{x} = 0$ and the second one gives $\overline{w} = 1$, which contradicts the third equation. This proves our claim that A is uniquely clean in R. In hindsight, we note that if we work in the larger ring $R' = \mathbb{M}_2(\mathbb{Z})$, A will no longer be uniquely clean as it has another clean decomposition $F + W$ where $F = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $W = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$. A quick calculation shows that $FW^{-1}F = -F$, so A turns out to be special clean in R'.

As for the above type of examples, there is also the finer problem of finding a *clean ring* R which would contain a unit-regular element that is not special clean. This can be done as follows, by mainly reiterating the idea of construction in Example 4.1 above.

Example 4.4. Let $S = \text{End}_k\,(M)$ where M is a right vector space over a field k with a countably infinite basis $\{e_1, e_2, \dots\}$. By a theorem of Ó Searcóid [35], S is a clean ring. Let J be the ideal of S consisting of all k-endomorphisms of finite rank. In the spirit of Example 4.1, we'll work with the ring $R = \begin{pmatrix} S & J \\ J & S \end{pmatrix} \subseteq \mathbb{M}_2(S)$. For the two idempotents given by the matrix units E_{11} and E_{22} (with sum I_2), the corner rings $E_{11}RE_{11}$ and $E_{11}RE_{11}$ are both isomorphic to the clean ring S, so by the Han-Nicholson theorem in [16], R is itself a clean ring. Now let $A = \text{diag}\,(a, 0) \in R$ where $a \in S$ is the "left shift operator", with $a\,(e_1) = 0$ and $a\,(e_i) = e_{i-1}$ for $i \geq 2$. If $b \in S$ is the "right shift operator" with $b\,(e_i) = e_{i+1}$ for all $i \geq 1$, we have $ab = 1$ and $\text{im}\,(1 - ba) = e_1 k$. In particular, $1 - ba \in J$, so $V := \begin{pmatrix} a & 0 \\ 1 - ba & b \end{pmatrix} \in R$. It is easy to check that $V \in \mathrm{U}\,(R)$, with inverse given by $\begin{pmatrix} b & 1 - ba \\ 0 & a \end{pmatrix} \in R$. Thus, A has a ureg-factorization $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V$, so $A \in \text{ureg}\,(R)$.

We claim that $A \notin \text{sp-cn}\,(R)$. To prove this, assume instead that A has a special clean decomposition $E+U$ in R; say $E = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. For any $s \in S$, we denote its image in $\overline{S} = S/J$ by $\overline{s}$. Since $\overline{E} = \text{diag}\,(\overline{x}, \overline{w}) = \overline{E}^2$, we have $\overline{w} \in \text{idem}\,(\overline{S})$. Moreover, the invertibility of $\overline{U} = \overline{A} - \overline{E} = \text{diag}\,(\overline{a-x}, -\overline{w})$ implies that $\overline{a} - \overline{x}, \overline{w} \in \mathrm{U}(\overline{S})$; in particular, $\overline{w} = 1$. Now, applying the special clean condition $\overline{AU^{-1}E} = 0$, we get $\overline{a}\,(\overline{a} - \overline{x})^{-1}\overline{x} = 0$, and hence $\overline{x} = 0$ (since $\overline{a} - \overline{x} \in \mathrm{U}(\overline{S})$, and $\overline{a} \in \mathrm{U}(\overline{S})$ too, with inverse $\overline{b}$). Therefore, $\overline{U} = \overline{A} - \overline{E} = \text{diag}\,(\overline{a}, -\overline{1})$. This is a contradiction, since U cannot be invertible even in the bigger ring $\mathbb{M}_2(S)$ since it is not injective as an endomorphism of M^2. This completes the proof for our claim that $A \notin \text{sp-cn}\,(R)$.

Remark 4.5. About Example 4.4, two more observations are in order. First, although the matrix $A = \text{diag}\,(a, 0)$ is not special clean in R, it does become special clean in the full matrix ring $\mathbb{M}_2(S)$. This fact can be deduced from our forthcoming result Corollary 6.6. Second, we note that the ring R is not only clean, but also *regular.* Indeed, R contains a regular ideal[2] $\hat{J} = \mathbb{M}_2(J)$ such that $R/\hat{J} \cong (S/J) \times (S/J)$ is regular, so the regularity of R follows from a well known classical result [15, Lemma 1.3].

With any element-wise property $\mathcal{P}$ in ring theory, it is always of interest to determine whether such a property would "satisfy Jacobson's Lemma"; that is, for any pair of elements a, b in any ring, whether $1 - ab$ having the property $\mathcal{P}$ would imply the same for $1 - ba$. If $\mathcal{P}_1$ is the property of being *unit-regular*, it is known from [10] and [28] that $\mathcal{P}_1$ satisfies Jacobson's Lemma. However, if $\mathcal{P}_2$ is the property of being *clean*, it was shown in [29, Proposition 4.2] that $\mathcal{P}_2$ does not satisfy Jacobson's Lemma. Since the property $\mathcal{P}$ of being *special clean* refines each of the properties $\mathcal{P}_1$ and $\mathcal{P}_2$, the two facts above do not give any quick way to tell if $\mathcal{P}$ itself would satisfy Jacobson's Lemma. Fortunately, a known result in [19] can be exploited to answer this question (negatively), as follows.

Example 4.6. For the two matrices $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix}$ in the ring $R = \mathbb{M}_2(\mathbb{Z})$, we have an obvious clean decomposition

$$I_2 - AB = \begin{pmatrix} 12 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 11 & 4 \\ 3 & 1 \end{pmatrix}, \tag{4.7}$$

which can be shown to be a special clean decomposition by a simple checking of the inner inverse condition (4) in Theorem 2.3. Thus, we have $I_2 - AB \in \text{sp-cn}\,(R)$. As for the matrix $X := I_2 - BA = \begin{pmatrix} 3 & 5 \\ 6 & 10 \end{pmatrix}$, letting $U = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \in \mathrm{U}(R)$ leads to the conjugated matrix $UXU^{-1} = \begin{pmatrix} 13 & 5 \\ 0 & 0 \end{pmatrix}$. This matrix is *not clean* in R, as was explicitly pointed out in Example 4.5 of [19]. From this, it follows in particular that $X = I_2 - BA \notin \text{sp-cn}\,(R)$. This means that, over a *general* ring R, the element-wise property of being special clean need not satisfy Jacobson's Lemma. Nevertheless, *over a clean ring* R', there "may be" a better chance for Jacobson's Lemma to hold. For instance, a brute-force computer check using *Mathematica* shows that, for the clean ring $R' = \mathbb{M}_2(\mathbb{Z}/4\mathbb{Z})$, $I_2 - AB \in \text{sp-cn}\,(R')$ does imply that $I_2 - BA \in \text{sp-cn}\,(R')$!

[2] An ideal in a ring is called *regular* if it is regular as a (possibly nonunital) ring.

In Example 2.7(D), we have pointed out that the set $\text{sp-cn}\,(R)$ is usually not closed with respect to the map $r \mapsto 1-r$. Therefore, Example 4.6 above is not sufficient to settle the question whether $xy \in \text{sp-cn}\,(R)$ would imply that $yx \in \text{sp-cn}\,(R)$. However, using a result from [29] also leads to a negative answer to this question, as follows.

Example 4.8. We work with the same ring $R = \mathbb{M}_2(\mathbb{Z})$ as in (4.6) above. Following the proof of [29, (4.2)], we consider the matrices $C = E_{11}$ and $D = n\,E_{11} + E_{12}$ in R, where $n \notin \{-1, 0, 1, 2\}$. As in [29, (4.4)], we have the following clean decomposition

$$CD = D = \begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ n-1 & 1 \end{pmatrix} + \begin{pmatrix} n & 1 \\ 1-n & -1 \end{pmatrix} \in R, \tag{4.9}$$

which can be shown to be again a special clean decomposition by an application of Theorem 2.3. Thus, $CD \in \text{sp-cn}\,(R)$. However, $DC = nE_{11} \notin \text{sp-cn}\,(R)$; in fact, it was shown in [29, p. 193] that $n\,E_{11}$ is not even a "suitable element" in R in the sense of Nicholson [31]. Here, R is not a clean ring. If one prefers a clean ring example, one can work in the ring $R' = \mathbb{M}_2\big(\mathbb{Z}/4\mathbb{Z}\big)$ mentioned in Example 4.6, and take instead $C = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$, $D = \text{diag}\,(1, 0)$. For these choices, $CD = C$ has a special clean decomposition $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$, but $DC = \text{diag}\,(2, 0)$ lies in the Jacobson radical of R', so it is not even a regular element in R'.

5. Sufficient Conditions for Special Clean Elements

The goal of this section is to demonstrate the ubiquity of special clean elements in rings by constructing various families of such elements. Roughly speaking, if a unit-regular element (or a clean element) $a \in R$ happens to have some extra properties beyond what is implied by unit-regularity (or cleanness), there seems to be generally a reasonable chance that a would be special clean. A number of examples of this nature will be given.

To state our first main result in this section, we first recall some basic terminology in the study of strongly π-regular (or Drazin invertible) elements in a ring R. Following Drazin [14], these are the elements $a \in R$ for which there exists an element a' called the Drazin inverse of a, satisfying

$$aa' = a'a,\ a'aa' = a',\ \text{and } a^{k+1}a' = a^k \text{ for some integer } k \geq 0.$$

The element $aa' = a'a$ is called the *associated idempotent* of a, while $e := 1 - a'a$ is called the *spectral idempotent* of a. It is well known (from work in [33, page 3589]) that $u := a - e \in \mathrm{U}\,(R)$, so that $a = e + u$ is a strongly clean decomposition of a, which is called the *spectral decomposition* of a.

If an element is strongly regular then it is trivially Drazin invertible, with the Drazin inverse exactly its "group inverse" as defined in Remark 2.14. The following theorem generalizes some results on abelian rings from [2], [11], and [3]. The "iff" statement in the first sentence of this theorem was also proved in [44].

Theorem 5.1. *For any ring R, $a \in \text{sreg}\,(R)$ iff a has a decomposition $e + u$ in R that is* both *special clean and strongly clean. In this case, $e + u$ is precisely the spectral decomposition of a. (In particular, $\text{sreg}\,(R) \subseteq \text{sp-cn}\,(R)$.) The special ureg-factorization corresponding to the spectral decomposition of a (in the sense of* Theorem 3.5*) is $a = fu$ where $f := 1 - e$ is the associated idempotent of a.*

Proof. If $a \in \operatorname{sreg}(R)$, then a is strongly π-regular, and it has a spectral decomposition $a = e + u$ with $e = 1 - a'a$. We have then $ea = (1 - a'a)\,a = a - aa'a = 0$ (since a' is the group inverse of a), so by the last sentence of Theorem 2.3, $a = e + u$ is a special clean decomposition (as well as a strongly clean one). In particular, we have $a \in \text{sp-cn}\,(R)$. Conversely, suppose a has a decomposition $a = e + u$ in R that is both special clean and strongly clean. Then $ea = 0$ by Theorem 2.3, so

$$a^2 = (e + u)\,a = ua = au.$$

Since $u \in \mathrm{U}(R)$, this shows that $a \in \operatorname{sreg}(R)$. To show that $a = e + u$ must be the spectral decomposition of a, we think of a and e as endomorphisms of R_R (acting by left multiplication). Letting $I = \ker(e)$ and $K = \operatorname{im}(e)$, it is clear that a acts as the automorphism u on I, and acts as zero on K (since $ae = 0$). Thus, $R = I \oplus K$ is exactly the Fitting decomposition of the action of $a \in \operatorname{sreg}(R)$ on R_R. In particular, e must be the spectral idempotent of a.

To prove the last statement in Theorem 5.1, we recall from the proof of Theorem 3.5 that the special ureg-factorization of a corresponding to the spectral decomposition $a = e + u$ is $a = (au^{-1})\,u$ where the idempotent factor in parentheses is

$$au^{-1} = (u + e)\,u^{-1} = 1 + eu^{-1} = 1 + eu^{-1}e = 1 - e,$$

in view of Theorem 2.3(4). This is exactly the associated idempotent of a. □

Remark 5.2. Note that the success of the proof of the "if" part of Theorem 5.1 depends crucially on working with a decomposition $a = e + u$ that is *simultaneously* strongly clean and special clean. In general, if an element $a \in R$ is strongly clean and special clean (with respect to two different decompositions), it need not follow that $a \in \operatorname{sreg}(R)$. For instance, in the ring $R = \mathbb{M}_2(\mathbb{Z})$, the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ has a trivial strongly clean decomposition $A = I_2 + (A - I_2)$ and a separate special clean decomposition $A = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. However, $A \notin \operatorname{sreg}(R)$ since otherwise $A \in A^2R \subseteq 2R$, which is not the case.

In Theorem 3.7, we have constructed an involution on the set of all special clean decompositions over a ring R, so that every special clean decomposition gives rise to a certain "dual" special clean decomposition. The following result closely related to Theorem 5.1 clarifies the formation of this dual on the spectral decompositions of the strongly regular elements of R, and also describes the set $\mathcal{F}$ of "fixed points" of this involution.

Theorem 5.3. (1) *If $a = e + u$ is the spectral decomposition of an element $a \in \operatorname{sreg}(R)$, then its dual in the sense of Theorem 3.7 is the spectral decomposition of the group inverse of a.*
(2) *Under the involution on the set of all special clean decompositions in R, the fixed point set $\mathcal{F}$ consists of* (*automatically special clean*) *decompositions of the form $e + u$ where $u^2 = 1$, $e^2 = e$, and $eu = -e$. Alternatively, $\mathcal{F}$ also consists of decompositions of the form $e + (a - e)$ where $e^2 = e$, $ea = 0$, and $(1 - a)\,e = 1 - a^2$.*
(3) *In the description of the set $\mathcal{F}$ in* (2)*, the element a is necessarily a tripotent; that is, $a^3 = a$. Any tripotent $a \in R$ always gives rise to a fixed point in $\mathcal{F}$ through its spectral decomposition $a = (1 - a^2) + (a^2 + a - 1)$.*

Proof. (1) By definition, the dual of the spectral decomposition $a = e + u \in \operatorname{sreg}(R)$ is given by $b := (1 - e)\,u^{-1} = (-eu^{-1}) + u^{-1} = e + u^{-1}$. From these equations (and the fact that $eu = ue = -e$), we check easily that $ab = ba$, $aba = a$, and $bab = b$, so b is the group inverse of a. Since the idempotent term e in the decomposition $b = e + u^{-1}$ is also the spectral idempotent of b, it follows that $b = e + u^{-1}$ is exactly the spectral decomposition of $b \in \operatorname{sreg}(R)$.

(2) Again from the definition of the construction of the dual, the condition for a special clean decomposition $e + u$ to be in the fixed point set $\mathcal{F}$ is $u^{-1} = u$ together with $-eu^{-1} = e$; or equivalently, $u^2 = 1$ together with $eu = -e$. (Note that these conditions imply that $eu^{-1}e = eue = -e^2 = -e$, so the decomposition $e + u$ would be automatically special clean by Theorem 2.3.) Next, we may transform the conditions above by introducing $a := e + u$ and eliminating u. Under this transformation, $eu = -e$ transforms into $ea = e\,(e + u) = e + eu = 0$, and $u^2 = 1$ transforms into

$$1 = (a - e)^2 = a^2 + e - ae\,; \text{ or equivalently, } 1 - a^2 = (1 - a)\,e.$$

(3) For any decomposition $a = e + u$ in $\mathcal{F}$, the relations shown in (2) imply that $a^2 = (e + u)\,a = ua$, and so $a^3 = aua = au^{-1}a = a$, as claimed. This means that a is its own group inverse, so the spectral idempotent of a is $1 - a^2$, which gives rise to its spectral decomposition $a = (1 - a^2) + (a^2 + a - 1)$ in the fixed point set $\mathcal{F}$. $\square$

Recall that an idempotent $e \in R$ is said to be *left semicentral* if $(1 - e)R\,e = 0$, and *right semicentral* if $eR\,(1 - e) = 0$. Given this terminology, a ring R is said to be *semiabelian* if every $e \in \operatorname{idem}(R)$ is either left semicentral or right semicentral. For such rings, as well as for rings of stable range one (see [37]), it turns out that special clean elements are none other than the regular elements.

Theorem 5.4. (1) *For any ring R of stable range one, the following holds:*

$$\text{sp-cn}\,(R) = \operatorname{ureg}(R) = \operatorname{reg}(R).$$

(2) *For any semiabelian ring R, the three sets above are all equal to* $\operatorname{sreg}(R)$.
(3) *A ring R is abelian iff every $a \in \operatorname{reg}(R)$ has a unique special clean decomposition, iff every $f \in \operatorname{idem}(R)$ has a unique special clean decomposition.*

Proof. (1) In view of (2.6), it is enough to check that $\operatorname{reg}(R) \subseteq \text{sp-cn}\,(R)$ for any ring R of stable range one. This was proved in [39, Theorem 3.3].

(2) For any ring R, (2.6) and Theorem 5.1 combine to give

$$\operatorname{sreg}(R) \subseteq \text{sp-cn}\,(R) \subseteq \operatorname{ureg}(R) \subseteq \operatorname{reg}(R).$$

If R is semiabelian, we have $\operatorname{reg}(R) = \operatorname{sreg}(R)$ by [27]. This proves (2).
(3) If R is abelian, every $a \in \operatorname{reg}(R) = \operatorname{sreg}(R)$ has a unique special clean decomposition by Theorem 5.1. Conversely, if every $f \in \operatorname{idem}(R)$ has a unique special clean decomposition, the last statement in Example 2.7(B) implies that f must be central, so R is an abelian ring. $\square$

Corollary 5.5. *Let $A \in R = \mathbb{M}_n(S)$ be a block matrix $\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}$ where $B \in \mathbb{M}_{r,n}(S)$ and S is a ring of stable range one. If B is "regular" in the sense that $BCB = B$ for some $C \in \mathbb{M}_{n,r}(S)$, then $A \in \text{sp-cn}\,(R)$.*

Proof. Since $A\,(C,\mathbf{0})A = \begin{pmatrix} BC & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} BCB \\ \mathbf{0} \end{pmatrix} = A$, we have $A \in \mathrm{reg}\,(R)$. According to a result of Vaserstein [37], the fact that S has stable range one implies that the matrix ring $R = \mathbb{M}_n(S)$ also has stable range one. Invoking part (1) of Theorem 5.4, we see that $A \in \text{sp-cn}\,(R)$. □

Remark 5.6. A special case of Corollary 5.5 is that, if $BC = I_r$ for some $C \in \mathbb{M}_{n,r}(S)$ where S has stable range one, then $A \in \text{sp-cn}\,(R)$. Even in this special case, the conclusion of Corollary 5.5 is not known in the literature. However, if S is a *commutative* ring of stable range one and $r = 1$, $n = 2$, the fact that $xS + yS = S$ implies that $A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ is a *clean* matrix in $\mathbb{M}_2(S)$ was first proved in [19, (3.8)]. Corollary 5.5 above may be regarded as a rather strong generalization of this fact.

In [34], a ring element $a \in R$ is defined to be *doubly unit-regular* if there exists $r \in R$ such that $a = ara$, and for $e = ra \in \mathrm{idem}\,(R)$, the element eae is unit-regular in the corner ring eRe. In [34, Theorem 3.14], it is proved that

(5.7) Every doubly unit-regular element is special clean.

One of the easiest examples of a doubly unit-regular element is a regular element a in any ring R with $a^2 = 0$. (If $a = ara$, letting $e = ra$ gives $eae = ra^2e = 0 \in eRe$.) In this case, (5.7) implies that $a \in \text{sp-cn}\,(R)$. An independent proof for this (using only the results of this paper) can be given as follows. Fixing a reflexive inverse r for a, let $s = (1-ar)\,r$. Then $s^2 = (1-ar)\,r\,(1-ar)\,r = 0$, and $asa = a\,(1-ar)\,ra = ara = a$. Similar short calculations show that $b := s\,(1+a)$ is an *idempotent* reflexive inverse of a. By (3) ⇒ (1) in Theorem 2.13, we have $a \in \text{sp-cn}\,(R)$. Indeed, it turns out that a is even a "superspecial clean element" in the sense of the forthcoming work [21].

In [34, Theorem 3.14], it was also shown that the double unit-regularity of an element $a \in R$ amounts to the existence of a special clean decomposition $a = e + u$ *together with* a second condition $a^2R \cap aeR = 0$. Our normal expectation from such a statement would be that the class of special clean elements is in general larger than that of doubly unit-regular elements. Indeed, it is not difficult to produce an example of a special clean element that fails to be doubly unit-regular. To do this, we exploit a construction idea of Patrício-Hartwig [36] and [41], as follows.

Example 5.8. Let $R = \mathbb{M}_2(S)$ where $S = \mathbb{Z}/4\,\mathbb{Z}$, and let $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \in R$, with $A^3 = 0$. We see easily that $\pm\,A \in \text{sp-cn}\,(R)$ for both signs, on account of the following (more or less obvious) special clean decompositions:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad -A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5.9}$$

On the other hand, $A^2 = 2A$ shows that $A^2 \notin \mathrm{reg}\,(R)$, since $(2A)\,X\,(2A) = 0 \neq 2A$ for any matrix $X \in R$. This implies that A is not doubly unit-regular, according to [34, Theorem 3.14]. (Just to "reinforce" this conclusion, if we take $A = E + U$ to be the first special clean decomposition in (5.9), an easy computation shows that the two right ideals A^2R and AER in R are both given by $\begin{pmatrix} 0 & 0 \\ 2S & 2S \end{pmatrix}$, so the "second condition" $A^2R \cap AER = 0$ alluded to in the last paragraph is indeed not satisfied here. A similar remark applies to the special clean decomposition for $-A$ in (5.9).)

6. Corner Ring Results

Since special clean rings are exactly the same as unit-regular rings, it goes without saying that Peirce corner rings of special clean rings are special clean; see [15, (4.7)]. However, this remark does not give any useful information about the nature of special clean elements in corner rings. One result we can prove in this direction is the following.

Theorem 6.1. *For any idempotent e_0 in a ring R, we have*

$$\text{sp-cn}\,(e_0Re_0) \subseteq e_0Re_0 \cap \text{sp-cn}\,(R). \tag{6.2}$$

Equality holds if e_0 is a left or right semicentral idempotent. In this case, we also have

$$\text{sp-cn}\,(e_0Re_0) = e_0\ \text{sp-cn}\,(R)\, e_0. \tag{6.3}$$

Proof. Let $S = e_0Re_0$. To prove (6.2), consider any $a \in \text{sp-cn}\,(S)$, with a special clean decomposition $e+u$ in S. Then $ava = a$ where v is the inverse of u in S. For $f_0 := 1-e_0$, $a = (e + f_0) + (u - f_0)$ is surely a clean decomposition of a in R. As $(u - f_0)^{-1} = v - f_0$ in R, it follows that

$$a\,(u - f_0)^{-1}a = a\,(v - f_0)\,a = ava - (ae_0)\,f_0a = a, \tag{6.4}$$

so Theorem 2.3 implies that $a \in \text{sp-cn}\,(R)$.

Next, consider any $a \in S \cap \text{sp-cn}\,(R)$, with a special clean decomposition $e + u$ in R. If e_0 is a *left or right semicentral* idempotent in R, a theorem of Chase [9, (2.1)] implies that the map $\varphi : r \mapsto e_0re_0$ is a ring homomorphism from R to S. By our earlier observation in the text following (2.6), the fact that $a \in S \cap \text{sp-cn}\,(R)$ would imply that $\varphi(a) = e_0ae_0 = a \in \text{sp-cn}\,(S)$. This proves that equality holds in (6.2).

Finally, continuing to assume that e_0 is a left or right semicentral idempotent, the fact that φ is a ring homomorphism implies that $e_0\ \text{sp-cn}\,(R)\, e_0 \subseteq \text{sp-cn}\,(S)$. Since we also have $\text{sp-cn}\,(S) \subseteq \text{sp-cn}\,(R)$ by (6.2), the equality in (6.3) follows. □

We conclude this paper by proving the following corner ring result which is another nice application of Theorem 2.13.

Theorem 6.5. *Let $a \in S = e_0Re_0$ where $e_0 \in \text{idem}\,(R)$, and let $\varphi : R \to S$ be the map defined by $r \mapsto e_0re_0$. If $\varphi\,(\text{idem}\,(R)) = S$, then the following statements are equivalent:*

(1) $a \in \text{reg}\,(S)$; (2) $a \in \text{reg}\,(R)$; (3) $a \in \text{ureg}\,(R)$; (4) $a \in \text{sp-cn}\,(R)$.

Proof. To begin with, the implications (4) ⇒ (3) ⇒ (2) hold for any ring R. Next, (2) ⇒ (1) holds upon writing $a = ara$ for some $r \in R$ and observing that $a = (ae_0)\,r\,(e_0a) = ar_0a$ where $r_0 := e_0re_0 \in S$. For (1) ⇒ (4), say $a = asa$ (for some $s \in S$), and write $s = \varphi(e)$ for some $e \in \text{idem}\,(R)$. Then $a = a\,(e_0ee_0)\,a = aea$, so $e \in \mathrm{I}\,(a)$ (in the ring R). This implies that $b := eae$ is a reflexive inverse of a, with $b^2 = (eae)\,(eae) = e\,(aea)\,e = b$, and so $b \in \text{sreg}\,(R)$. Applying Theorem 2.13 then shows that $a \in \text{sp-cn}\,(R)$. □

Corollary 6.6. *Let $R = \mathbb{M}_n(S)$, where S is any ring and $n \geq 2$. For any $a \in S$ and $A = \text{diag}\,(a, 0, \ldots, 0) \in R$, the following statements are equivalent:*

(1) $a \in \text{reg}\,(S)$; (2) $A \in \text{reg}\,(R)$; (3) $A \in \text{ureg}\,(R)$; (4) $A \in \text{sp-cn}\,(R)$.

Proof. To apply Theorem 6.5, we take e_0 to be the matrix unit E_{11} in R, and identify S with the corner ring e_0Re_0. The equivalence of (1)–(4) will follow if we can show that the map $\varphi : \text{idem}\,(R) \to S$ defined by $r \mapsto e_0re_0$ is surjective. This is clear, since for every

$s \in S$, the diagonal block sum $\begin{pmatrix} s & 1-s \\ s & 1-s \end{pmatrix} \oplus \mathbf{0}_{n-2}$ is an *idempotent* in R, and obviously φ maps this idempotent matrix to the given element $s \in S$. $\square$

Remark 6.7. Using the Corollary above, we can see retrospectively that, in the setting of Theorem 6.1, the inclusion relation in (6.2) may be strict if $e_0 \in R$ is an *arbitrary* idempotent in R. Indeed, if S is any ring with an element $a \in \mathrm{reg}\,(S) \setminus \mathrm{cn}\,(S)$, Corollary 6.6 shows that $A = \mathrm{diag}\,(a, 0)$ is special clean in the matrix ring $R = \mathbb{M}_2(S)$. However, a is not clean (let alone special clean) in its corner ring $e_0Re_0 \cong S$ if we take e_0 to be the matrix unit E_{11}.

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Department of Mathematics, Panjab University, Chandigarh 160 014, India
Email address: dkhurana@pu.ac.in

Department of Mathematics, University of California, Berkeley, CA 94720, USA
Email address: lam@math.berkeley.edu

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
Email address: pace@math.byu.edu

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia
Email address: janez.ster@fmf.uni-lj.si