NILPOTENT IDEALS IN POLYNOMIAL AND POWER SERIES RINGS

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ABSTRACT. Given a ring R and polynomials $f(x), g(x) \in R[x]$ satisfying $f(x)Rg(x) = 0$ we prove that the ideal generated by products of the coefficients of $f(x)$ and $g(x)$ is nilpotent. This result is generalized, and many well known facts, along with new ones, concerning nilpotent polynomials and power series are obtained. We also classify which of the standard nilpotence properties on ideals pass to polynomial rings, or from ideals in polynomial rings to ideals of coefficients in base rings. In particular, we prove that if $I \leq R[x]$ is a left T-nilpotent ideal, then the ideal formed by the coefficients of polynomials in I is also left T-nilpotent.

1. INTRODUCTION

Throughout this paper, all rings are associative rings with 1. It is well known that a polynomial $f(x)$ over a commutative ring is nilpotent if and only if each coefficient of $f(x)$ is nilpotent. But this result is not true in general for noncommutative rings. For example, let $R = \mathbb{M}_n(k)$, the $n \times n$ full matrix ring over some ring $k \neq 0$. Consider the polynomial $f(x) = e_{12} + (e_{11} - e_{22})x - e_{21}x^2$, where the e_{ij}'s are the matrix units. In this case $f(x)^2 = 0$, but $e_{11} - e_{22}$ is not nilpotent. (In fact, when $n = 2$ this is a unit.) The trouble seems to arise from that fact that if $f(x)$ is nilpotent, then so is the ideal $\langle f \rangle$ over a commutative ring, but not necessarily over a noncommutative ring. Indeed, for a general ring R, $f(x)R[x]$ is a nilpotent right ideal of $R[x]$ if and only if $A_f R$ is a nilpotent right ideal of R, where A_f is the set of coefficients of $f(x)$. We provide some extensions and generalizations of this result.

Fields [1, Corollary 1] proved that if R is a commutative Noetherian ring, then a power series $g(x)$ over a commutative ring R is nilpotent if and only if each coefficient of $g(x)$ is nilpotent. He also provided an example showing that the condition "R is Noetherian" is not redundant [1, Example 1]. On the other hand, when R is a general ring, X is a set of noncommuting indeterminates with $|X| \geq 2$, and I is a right ideal of $R[\![X]\!]$, it is known that if I is nil of bounded index then the ideal generated by the coefficients of the power series in I is nilpotent of the same index [9, Theorem 2]. The case for power series in one variable is much more difficult; and we refer the reader to [10] for more information. In this paper we prove a generalization of the fact that if we have a power series $f(x) \in R[\![x]\!]$, where $A_f R$ is finitely generated, then $f(x)R[\![x]\!]$ is nilpotent if and only if $A_f R$ is nilpotent. The methods we employ work for power series over any number of noncommuting indeterminates, without invoking hypotheses concerning ideals in such rings. Throughout the paper, we let A_f be the set in R consisting of all the "coefficients" of $f(x)$ whether $f(x)$ be a polynomial, power series, or something more exotic.

In the last few sections we collect much of the folklore concerning the behavior of nilpotence properties on (one-sided) ideals going to polynomial extensions, or from polynomials down to the base ring. It appears that both Example 15 and Theorem 18 are new. For more information about the behavior of radical properties under polynomial extensions, we refer the reader to the recent book [3].

2. Generating Nilpotent Ideals

As mentioned in the introduction, if R is a noncommutative ring and $f(x) \in R[x]$ is nilpotent, this does not imply that the coefficients of $f(x)$ are nilpotent. However, if we assume a small "commutativity-like" hypothesis, we obtain stronger conditions on the coefficients of $f(x)$.

Theorem 1. *Let R be a ring, and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be polynomials in $R[x]$. If $f(x)Rg(x)R = 0$, then $a_i R b_j R$ is a nilpotent right ideal, for each pair (i,j).*

Proof. We work by induction on $k = i + j$. If $k = 0$ then $i = j = 0$. We know $a_0 R b_0 R = 0$ by looking at the degree zero term in $f(x)Rg(x)R = 0$. Thus $a_0 R b_0 R$ is trivially nilpotent.

So, we may assume $k \geq 1$ and that the claim is true for all smaller values by inductive assumption. From the degree k coefficient in $f(x)Rg(x)R = 0$ we obtain

$$(1) \qquad \sum_{t=0}^{k} a_t r b_{k-t} s = 0$$

for arbitrary $r, s \in R$. In particular, specializing r in Equation (1) to $u b_0 v$ (with $u, v \in R$ arbitrary) we have $\sum_{t=0}^{k} a_t u b_0 v b_{k-t} s = 0$. But then

$$a_k u b_0 v b_0 s = -\sum_{t=0}^{k-1} a_t u b_0 v b_{k-t} s \in \sum_{t<k} a_t R b_0 R.$$

The right ideal on the right hand side is nilpotent (since it is a finite sum of such, by inductive hypothesis). Thus, $(a_k R b_0 R)^2 \subset a_k R b_0 R b_0 R$ is a nilpotent right ideal, hence so is $a_k R b_0 R$.

Repeat the argument in the previous paragraph by specializing r to $u b_j v$ instead of $u b_0 v$, and induct on j. Note that the previous paragraph consists of checking the base case. In this more general case, Equation (1) then yields

$$a_{k-j} u b_j v b_j s = -\left(\sum_{t=0}^{k-j-1} a_t u b_j v b_{k-t} s \right) - \left(\sum_{t=k-j+1}^{k} a_t u b_j v b_{k-t} s \right).$$

In particular, $a_{k-1} R b_j R b_j R \subseteq \left(\sum_{t<k-j} a_t R b_j R \right) + \left(\sum_{i'+j'=k,\, j'<j} a_{i'} R b_{j'} R \right)$, which is nilpotent by the inductive assumptions on k and j. By the same reasoning as in the previous paragraph, we obtain that $a_{k-j} R b_j R$ is nilpotent, finishing both inductions. □

This theorem can be proven much more quickly by using an idea brought to our notice by the referee. Notice that over either of the two rings $R' = R/\langle a_m \rangle$ or $R'' = R/\langle b_n \rangle$, we have reduced k. By the inductive hypothesis, the image of $T = a_i R b_j R$ is nilpotent in both of the quotient rings R' and R''. Thus, some power of T lives in the product $\langle a_m \rangle \cdot \langle b_n \rangle$, which is nilpotent in R.

We retain the original proof, as it allows some generalizations not available to the faster proof such as (we will see) to the case of power series rings. One should also note that the nilpotency index on $a_i R b_j R$ can easily be made effective in terms of the pair (i,j) by modifying the inductive hypothesis employed in the initial proof. There are many other ways to generalize this theorem. The first is to notice that we can make it into an "if and only if" statement as follows:

Proposition 2. *Let R be a ring, and let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. The set $f(x)Rg(x)R$ is nilpotent if and only if $a_i R b_j R$ is a nilpotent right ideal for each pair (i,j).*

Proof. The forward direction is a simple *mutatis mutandis* argument using the first proof. The backward direction is a consequence of the fact that there are only finitely many pairs (i,j). □

Note that since $x \in R[x]$ is central we have that the set $f(x)Rg(x)R$ is nilpotent if and only if the right ideal $f(x)R[x]g(x)R[x]$ is nilpotent. In other situations we will consider later, we don't have a similar "centrality" condition, which is why we stated the theorem as we did. However, using this information we do obtain the following classical results.

Corollary 3. *Let R be a ring. The right ideal $f(x)R[x]$ of $R[x]$ is nilpotent if and only if the right ideal $A_f R$ of R is nilpotent.*

Denote the sum of the nilpotent ideals of R, called the *Wedderburn radical*, by $W(R)$.

Corollary 4. *For any ring R, $W(R[x]) = W(R)[x]$.*

Another way to generalize Theorem 1 is to consider the product of more than two polynomials. We note that since our rings are possibly noncommutative, when we write a product $\prod_{i=1}^{n} a_i$ it signifies the left-to-right product: $a_1 a_2 \cdots a_n$.

Proposition 5. *Let $n \in \mathbb{Z}_{>0}$, let R be a ring, and for each $1 \leq \ell \leq n$ let $f_\ell(x) = \sum_{i=0}^{n_\ell} a_{i,\ell} x^i \in R[x]$. The set $\prod_{\ell=1}^{n}(f_\ell(x)R)$ is nilpotent if and only if $\prod_{\ell=1}^{n}(a_{i_\ell,\ell}R)$ is a nilpotent right ideal, for all choices $0 \leq i_\ell \leq n_\ell$ (for each ℓ).*

Proof. As before, the reverse implication is easy. For the forward implication, we simplify to the case $\prod_{\ell=1}^{n}(f_\ell(x)R) = 0$, as the more general case follows from this case but requires more extensive notations. Let $S = \{(i_1, i_2, \ldots, i_n) \mid 0 \leq i_\ell \leq n_\ell\}$ be the set of n-tuples whose entries consist of indices for the coefficients of the polynomials. We can order S by saying

$$(i_1, i_2, \ldots, i_n) < (j_1, j_2, \ldots, j_n)$$

if and only if there exists an integer m in the range $1 \leq m \leq n$ so that $\sum_{\ell=m'}^{n} i_\ell = \sum_{\ell=m'}^{n} j_\ell$ when $1 \leq m' < m$ but $\sum_{\ell=m}^{n} i_\ell < \sum_{\ell=m}^{n} j_\ell$. Given an n-tuple $s = (i_1, i_2, \ldots, i_n) \in S$, let $B_s = \prod_{\ell=1}^{n}(a_{i_\ell,\ell}R)$.

We work by induction on the well-ordered set S to prove that B_s is nilpotent for each $s \in S$. The claim is clearly true for the n-tuple $s_0 = (0, 0, \ldots, 0)$. Fix $s = (i_1, i_2, \ldots, i_n) \in S$, and suppose by induction that the claim is true for all n-tuples $t < s$. Set $k = \sum_{\ell=1}^{n} i_\ell$. Looking at the degree k coefficient in the equation $\prod_{\ell=1}^{n}(f_\ell(x)R) = 0$ yields

$$(2) \qquad \sum_{j_1+j_2+\cdots+j_n=k} \left(\prod_{\ell=1}^{n}(a_{j_\ell,\ell}r_\ell) \right) = 0$$

with $r_\ell \in R$ arbitrary. Specializing each r_ℓ to $s_{\ell,\ell} \prod_{\ell'=\ell+1}^{n}(a_{i_{\ell'},\ell'} s_{\ell',\ell})$ (for arbitrary $s_{\ell',\ell} \in R$, for all $\ell' > \ell$) then, just as in the proof of Theorem 1, Equation (2) implies that $C_s = \prod_{\ell=1}^{n} \left(\prod_{\ell'=\ell}^{n} a_{i_{\ell'},\ell'} R \right)$ belongs to $\sum_{t<s} B_t$, which is nilpotent. On the other hand $B_s^n \subseteq C_s$, and so B_s is similarly nilpotent. This completes the induction. $\square$

To generalize our theorem another way, note that the induction on k in Theorem 1 does not rely in any way on the fact that k is bounded by $m + n$. In other words, the same proof works for power series rings by merely replacing m and n by ∞.

Proposition 6. *Let $n \in \mathbb{Z}_{>0}$, let R be a ring, and for each $1 \leq \ell \leq n$ set $f_\ell = \sum_{i=0}^{\infty} a_{i,\ell} x^i \in R[\![x]\!]$. If $\prod_{\ell=1}^{n}(f_\ell(x)R)$ is nilpotent then $\prod_{\ell=1}^{n}(a_{i_\ell,\ell}R)$ is a nilpotent right ideal, for all choices $0 \leq i_\ell$ (for each ℓ).*

Corollary 7. *Let R be a ring and $f(x) \in R[\![x]\!]$. Suppose $A_f R$ is finitely generated. The right ideal $f(x)R[\![x]\!]$ is nilpotent if and only if $A_f R$ is nilpotent.*

The proof of Theorem 1 does not rely on the fact that we are working with polynomials, except when we claim the ability to order the products $a_i R b_j R$ in "compatible" ways. This point is emphasized in the proof of Propositions 5 and 6. More generally, we can work in the context of monoid rings.

Proposition 8. *Let $n \in \mathbb{Z}_{>0}$, let R be a ring, and let G be a monoid so that each finitely generated submonoid is an ordered monoid. Let $\{g_1, g_2, \ldots, g_u\} \subseteq G$. For each $1 \le \ell \le n$, set $f_\ell = \sum_{i=1}^{u} a_{i,\ell} g_i \in R[G]$. If $\prod_{\ell=1}^{n}(f_\ell R) = 0$ then $\prod_{\ell=1}^{n}(a_{i_\ell, \ell} R)$ is a nilpotent right ideal, for all choices $1 \le i_\ell \le u$ (for each ℓ).*

Proof. One follows the proof of Proposition 5, but since some of the needed changes are not obvious we clarify them here. First, fix an ordering on the submodule of G generated by $\{g_1, \ldots, g_u\}$. Second, define an ordering on n-tuples of indices by the rule:

$$(i_1, i_2, \ldots, i_n) < (j_1, j_2, \ldots, j_n)$$

if and only if there exists an integer m in the range $1 \le m \le n$ so that $\prod_{\ell=m'}^{n} g_{i_\ell} = \prod_{\ell=m'}^{n} g_{j_\ell}$ when $1 \le m' < m$ but $\prod_{\ell=m}^{n} g_{i_\ell} < \prod_{\ell=m}^{n} g_{j_\ell}$. One works by induction, as before, on the set S of these n-tuples under this well-ordering.

The third major change is that Equation (2) should be modified to include only those terms in the product $f_1 R f_2 R \cdots f_n R = 0$ with common support $g \in G$ (for some given element g). The rest of the proof involves only minor changes. $\square$

Corollary 9. *Let R be a ring, let X be a set of indeterminates, and let $f, g \in R\langle X \rangle$ be elements of the free ring in noncommuting indeterminates over X. We can write $f = \sum_{i \in I} a_i w_i$ and $g = \sum_{i \in I} b_i w_i$, where I is a finite set, $a_i, b_i \in R$ for all $i \in I$, and $\{w_i\}_{i \in I}$ is a subset (without repetitions) of the free monoid generated by X.*

If $fRgR = 0$ then $a_i R b_j R$ is a nilpotent right ideal for each pair (i, j).

In the context of the previous corollary, if $|X| \ge 2$ and we assume the stronger condition that $fR\langle X \rangle g = 0$ then one can easily prove $a_i R b_j = 0$.

Recall that if $\sigma : R \to R$ is a ring homomorphism we can define the skew-polynomial ring $R[x; \sigma]$ which, as an underlying set, is the same as the polynomial ring, but has the twisted multiplication

$$\left(\sum_{i=0}^{m} a_i x^i \right) \left(\sum_{j=0}^{n} b_j x^j \right) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i \sigma^i(b_j) \right) x^k.$$

Once again, the same proof as for Theorem 1 with only small changes yields:

Proposition 10. *Let R be a ring, and let σ be a ring epimorphism on R. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$. If $f(x) R g(x) R = 0$, then $a_i R \sigma^i(b_j) R$ is a nilpotent right ideal, for each pair (i, j).*

Note that these last few results can be appropriately extended to arbitrary finite products, or we may replace the condition $f(x) R g(x) R = 0$ with the hypothesis that $f(x) R g(x) R$ is nilpotent, exactly as when we extended Theorem 1 to Propositions 2 and 5 (with the usual *mutatis mutandis* caveat).

3. Nilpotence Properties

Let R be a ring. Following [6], we let $\mathrm{rad}(R)$, $\mathrm{Nil}^*(R)$, $\mathrm{L\text{-}rad}(R)$, and $\mathrm{Nil}_*(R)$ denote, respectively, the Jacobson radical (the intersection of all maximal left ideals), the upper nilradical (the sum of all nil ideals), the Levitsky radical (the sum of all locally nilpotent ideals), and the lower nilradical (the intersection of all prime ideals). One has the following containments $W(R) \subseteq \mathrm{Nil}_*(R) \subseteq \mathrm{L\text{-}rad}(R) \subseteq \mathrm{Nil}^*(R) \subseteq \mathrm{rad}(R)$, where each containment may be proper. More generally, let I be a right ideal in R. The following implications hold for some common nilpotence conditions on I:

$$(3) \quad \text{nilpotent} \Longrightarrow T\text{-nilpotent} \Longrightarrow \text{left } T\text{-nilpotent} \Longrightarrow \text{locally nilpotent} \Longrightarrow \text{nil}$$

nil of bounded index

We refer the reader to [6] for basic information about these properties, and proofs of all of these implications. When we say that I is T-*nilpotent* we mean that I is both left and right T-nilpotent. We note that there are no other implications (except those implied by composition) even if I is finitely generated as a right ideal, due to the following examples.

Example 11. (1) Let $R = \mathbb{Q}\langle x_1, x_2, \ldots \rangle$ with relations $x_1^2 = 0$, $x_i x_j = 0$ when $i, j > 1$, and $x_m x_1 x_n = 0$ whenever $n \geq m$. The right ideal $I = x_1 R$ is left T-nilpotent but not right T-nilpotent and not nil of bounded index since $y_n = x_1 x_{n+1} + x_1 x_n + \cdots + x_1 x_2$ has nilpotency index n. (Note: We can similarly choose the relations so that $I = x_1 R$ is right T-nilpotent but not left T-nilpotent and not nil of bounded index.)

(2) Let $R = \mathbb{Q}\langle x_1, x_2, \ldots \rangle$ with relations $x_1^2 = 0$, $x_i x_j = 0$ when $i, j > 1$, $x_i x_1 x_j = x_j x_1 x_i$ when $i, j > 1$, and $x_m \prod_{i=1}^{m}(x_1 x_{\ell_i}) = 0$ for any choice of indices ℓ_i. The right ideal $I = x_1 R$ is left and right T-nilpotent but not nil of bounded index.

(3) Let $R = \mathbb{F}_2\langle x_1, x_2, \ldots \rangle$ subject to the relations $x_1^2 = 0$, $x_i x_j = 0$ when $i, j > 1$, $x_i x_1 x_i = 0$ for all $i > 1$, and $x_i x_1 x_j = x_j x_1 x_i$ when $i, j > 1$. The right ideal $I = x_1 R$ has bounded index of nilpotency 3, but is neither left nor right T-nilpotent. We note that an ideal which has bounded index of nilpotency 2 is automatically nilpotent, so this example is optimal.

(4) Locally nilpotence is decidable on finite subsets. Any left or right T-nilpotent right ideal lies in $\mathrm{Nil}_*(R)$ by [6, Proposition 23.5], and similarly any right ideal which is nil of bounded index lies in $\mathrm{Nil}_*(R)$ by [5]. So the nonexistence of all other implications boils down to the possible strict inclusions $\mathrm{Nil}_*(R) \subset \mathrm{L\text{-}rad}(R) \subset \mathrm{Nil}^*(R)$. An example of the first inclusion being strict is constructed in [6, p. 167]. The first example of the other inclusion being strict arise from Golod's work in the 1960's. We refer the reader to [12] for a stronger example, which shows that the inclusion can be strict even for polynomial rings.

Remark 1. The previous examples must all be noncommutative due to the fact that: (1) in a commutative ring any nil right ideal is locally nilpotent, and (2) in a commutative ring, any finitely generated nil right ideal is nilpotent.

Remark 2. There are no further implications in Diagram (3) when I is a two-sided ideal by similar constructions to those above. We leave it to the reader to construct examples.

For a thorough treatment of the behavior of certain radical properties under polynomial extensions, we direct the reader to Chapter 4 of [3]. We compile some results which are scattered across the literature here. A finite sum of right ideals which are nil of bounded index is still of bounded index by [5, Theorem 3]. Following [5] we define

$$N(R) = \{a \in R \mid aR \text{ is nil of bounded index}\}$$

and we have $W(R) \subseteq N(R) \subseteq \mathrm{Nil}_*(R)$. One can define the higher Wedderburn radicals by setting $W_1(R) = W(R)$, $W_{\alpha+1}(R) = \{a \in R \mid aR + W_\alpha(R) \in W(R/W_\alpha)\}$, and if α is a limit ordinal then $W_\alpha(R) = \bigcup_{\beta < \alpha} W_\beta(R)$. It is an easy fact that these stabilize to the lower nilradical [6, Exercise 10.11]. Further, from our work in the previous section we know $W_\alpha(R[x]) = W_\alpha(R)[x]$, and hence we have another proof of Amitsur's theorem $\mathrm{Nil}_*(R[x]) = \mathrm{Nil}_*(R)[x]$, see [6, Theorem 10.19]. Note that Klein in [5] proved that $W_1(R) \subseteq N(R) \subseteq W_2(R)$, where either containment may be proper or an equality.

One may ask how the properties in Diagram (3) behave when passing to polynomial rings, or from polynomial rings down to the base ring. We will completely answer both of these questions in the remainder of this section. Most of the implications are classical, and our main contributions are Example 15 and Theorem 18.

The following "Going-up" theorem is a collection of results from the literature:

Theorem 12 (Going Up)**.** *Let $\mathcal{P}$ be any property in Diagram (3) except "nil." Let R be a ring, and let J be a right ideal in R. If J has property $\mathcal{P}$ then $J[x]$ has property $\mathcal{P}$. No other implications going*

from J to $J[x]$ exist among those properties listed in Diagram (3), even if J is an ideal, except those implied by transitivity.

Proof. The proofs of "$\mathcal{P}$ on J implies $\mathcal{P}$ on $J[x]$" when $\mathcal{P} \neq$ "nil" are either folklore or straightforward. Gardner proved that if R is a (nonunital) left T-nilpotent ring, then so is $R[x]$ in [2, Proposition 1.4]. The case $\mathcal{P} =$ "nil of bounded index" was dealt with by Klein [4]. Smoktunowicz and Puczyłowski [13] gave an example of a polynomial ring which is Jacobson radical but not upper nilradical, thus dealing with the case when $\mathcal{P} =$ "nil". All other non-implications are a trivial consequence of the non-implications in Diagram (3), by considering constant polynomials. □

Corollary 13. *There are no extra implications in Diagram (3), even when we restrict to ideals in polynomial rings.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be two properties of the diagram with $\mathcal{P} \nRightarrow \mathcal{Q}$. Fix some ideal $J \leq R$, for some ring R, so that J has property $\mathcal{P}$ but not $\mathcal{Q}$.

We first consider the case when $\mathcal{P} \neq$ "nil". By the previous theorem $J[x]$ has property $\mathcal{P}$. If $J[x]$ has property $\mathcal{Q}$, then so does J by considering the constant polynomials in $J[x]$. But this would contradiction our construction of J. Thus $J[x]$ does not have property $\mathcal{Q}$.

Now suppose $\mathcal{P} =$ "nil". By a result of Amitsur and McCoy, we have the equality $\mathrm{Nil}_*(R[x]) = \mathrm{Nil}_*(R)[x]$, as proven in [6, Theorem 10.19]. It is also easy to check the equality $\mathrm{L\text{-}rad}(R[x]) = \mathrm{L\text{-}rad}(R)[x]$ always holds. (To prove it, first reduce to the case when $\mathrm{L\text{-}rad}(R) = 0$, and then look at the nonzero coefficients in $\mathrm{L\text{-}rad}(R[x])$ in smallest possible degree.) Thus, the inclusion $\mathrm{Nil}_*(R[x]) \subset \mathrm{L\text{-}rad}(R[x])$ is proper when the corresponding inclusion is proper over R.

Finally, by [12], fix a ring R with $\mathrm{rad}(R[x,y])$ not nil. By another theorem of Amitsur we have $\mathrm{rad}(R[x,y]) \subseteq \mathrm{Nil}^*(R[x])[y]$, see [6, Theorem 5.10]. It cannot be the case that $\mathrm{Nil}^*(R[x])$ (which is a nil ideal) equals $\mathrm{L\text{-}rad}(R[x])$, since (as we just saw) local nilpotence passes up to the polynomial ring. □

The "Going-down" theorem is more complicated than the "Going-up" theorem.

Theorem 14 (Going Down). *Let R be a ring, let I be a right ideal in $R[x]$, and let A_I be the right ideal of coefficients of polynomials in I. Let*

$$\mathcal{P} \in \{\text{``$T$-nilpotent''}, \text{``left T-nilpotent''}, \text{``locally nilpotent''}, \text{``nil''}\}.$$

If I has property $\mathcal{P}$ then A_I has property $\mathcal{P}$. No other implications going from I to A_I exist among those properties listed in Diagram (3), even if I is an ideal, except those implied by transitivity.

We prove this theorem in a number of steps. We begin with an example of a commutative ring R with an ideal L in $R[x]$ which is nilpotent (in fact $L^2 = 0$), but A_L is not even nil of bounded index.

Example 15. We modify the example in [10, Theorem 11]. Let $F = K[s_1, s_2, \ldots]$ be the polynomial ring in a countably-infinite set of variables, over a field K of characteristic zero. Let I be the ideal of F generated by elements $h_k = \sum_{i=1}^{k-1} s_i s_{k-i}$, for each $k \geq 2$. Let H be the ideal of F consisting of polynomials with zero constant term.

Fix an integer $m \geq 3$, and fix another integer $n > m^{m^2}$ (which depends on m). Let

$$J_m = \langle s_{i_1} \cdots s_{i_t} \mid t > m \text{ or } i_1 + \cdots + i_t > m^{m^2} \rangle$$

which is an ideal of F, and put $I_n = \langle s_\alpha \mid \alpha > n \rangle$.

Let $R_n = F/(I + I_n)$. The ideal $H' = H/(I + I_n)$ of R_n is nilpotent, but we claim that the index of nilpotency is $> m$. The argument is as follows. Since $I_n \subseteq J_m$ it suffices to prove that $(H/(I + J_m))^m \neq 0$. Hence, it suffices to prove $(H^m + J_m)/J_m \nsubseteq (I + J_m)/J_m$, which follows by dimension arguments as worked out in [10].

We now make the dependence of n on m explicit, by writing $n = n_m$. Set $R = \prod_{m \geq 3} R_{n_m}$, and for each $m \geq 3$ set $f_m(x) = \sum_{j=1}^{n_m} \overline{s_j} x^j \in R_{n_m}$. Note that we have $(f_m(x))^2 = 0$ due to the relations in the ideal I. Finally form the ideal

$$L = \langle (f_3(x), 0, 0, 0, \ldots), (0, f_4(x), 0, 0, \ldots), (0, 0, f_5(x), 0, \ldots), \ldots \rangle$$

in $R[x]$. We have $L^2 = 0$, but A_L is not nil of bounded index, since the index of nilpotency for the mth coordinate is greater than m.

Corollary 3 demonstrates that if a single polynomial generates a nilpotent ideal, then so do the coefficients, and the same is true for any finitely generated ideal. On the other hand, the previous example shows that the result is not true, in general, for countably generated ideals. However, we can prove a slight extension of Corollary 3 by working with polynomials of bounded degree. For any right ideal I of $R[x]$, we denote by $A_{I,n}$ the subset of A_I generated by the coefficients of the polynomials in I of degree $\leq n$.

Proposition 16. *Let R be a ring and I a right ideal of $R[x]$. If $I^n = 0$ then $(A_{I,m})^{n^{m+1}} = 0$ for each $m \geq 0$. Similarly, if I is nil of bounded index then so is $A_{I,m}$ for each $m \geq 0$. In particular, if I is generated by polynomials of bounded degree, then A_I is nilpotent (respectively, nil of bounded index) when I is.*

Proof. Assume $I^n = 0$. We first claim that $A_{I,s}^n \subseteq A_{I,s-1}$. To prove this claim let $\{f_t\}_{t \leq n}$ be an arbitrary, but finite, set of polynomials in I. Write $f_t(x) = \sum_{i=1}^{n_t} a_{i,t} x^i$. It suffices to show that for $s \in \mathbb{Z}_{>0}$, an element $\alpha = a_{s,1} a_{s,2} \cdots a_{s,n}$ lies inside the right ideal of R generated by $a_{i,t}$ for $i < s$. To prove this subclaim, compute the degree sn coefficient of $f_1(x) f_2(x) \cdots f_n(x) = 0$. One term equals α while all of the other terms live inside the right ideal of R generated by $a_{i,t}$ for $i < s$. Since these terms all sum to zero, our claim is proven.

Note that $A_{I,0}^n = 0$. Since $A_{I,1}^n \subseteq A_{I,0}$ this means that $A_{I,1}^{n^2} = 0$. Repeating this argument, $A_{I,m}^{n^{m+1}} = 0$ for each $m \geq 0$.

Now assume instead that I is nil of bounded index n. First note that $A_{I,0}$ is clearly nil of bounded index, with index at most $k_0 = n$. Assume by induction that $A_{I,s-1}$ is nil of bounded index k_{s-1}. Letting $a \in A_{I,s}$, fix some polynomial $f(x) \in I$ with a the coefficient of x^s in $f(x)$. Looking at the degree sn term in $f(x)^n = 0$ we see that $a^n \in A_{I,s-1}$ (using the same argument as above). Thus, $A_{I,s}$ is nil of bounded index, with index at most k_{s-1}^n. The rest of the proof follows easily. $\square$

Corollary 17. *Letting R be a ring then $N(R[x]) = N(R)[x]$.*

Proof. We have $N(R)[x] \subseteq N(R[x])$ by Theorem 12. On the other hand $A_{N(R[x])} = \sum_{m \geq 0} A_{N(R[x]),m} \subseteq N(R)$ by the previous proposition, hence $N(R[x]) \subseteq A_{N(R[x])}[x] \subseteq N(R)[x]$. $\square$

We now continue the proof of Theorem 14. First, if I is nil then A_I is nil; and this follows from the fact we mentioned earlier, and first proved by Amitsur, that $\mathrm{rad}(R[x]) \subseteq \mathrm{Nil}^*(R)[x]$. Second, as also already mentioned, we have the equality $\mathrm{L\text{-}rad}(R[x]) = \mathrm{L\text{-}rad}(R)[x]$. Thus, if I is locally nilpotent then A_I is locally nilpotent.

The non-implications in Theorem 14 are all proved along similar lines. We give a single example of how this is done, and leave the other cases to the reader. Suppose, by way of contradiction, that I being nil implies A_I is locally nilpotent. By the "Going-up" theorem this would mean $A_I[x] \supset I$ is locally nilpotent. But there is an ideal $I \leq R[x]$ which is nil but not locally nilpotent, since (as we previously proved) there are no extra implications in Diagram (3) for polynomial rings. This is the desired contradiction.

All that remains for the "Going-down" theorem is:

Theorem 18. *Let R be a ring, let I be a right ideal in $R[x]$, and A_I the right ideal in R of coefficients of polynomials in I. If I is left (respectively, right) T-nilpotent then A_I is left (respectively, right) T-nilpotent.*

Proof. Assume I is left T-nilpotent, and let $\alpha = (a_1, a_2, a_3, \ldots)$ be a sequence of elements in A_I. Fix polynomials $f_i(x) \in I$ so that a_i is a coefficient of $f_i(x)$. Write $f_i(x) = \sum_{j=0}^{n_i} b_{i,j} x^j$ and let $B = \{b_{i,j} \mid \forall i, j\}$ be the set of all coefficients of the polynomials $\{f_1, f_2, \ldots\}$. We think of the polynomials $f_1, f_2, \ldots$ as distinct (even if they, technically, are the same element in the ring $R[x]$), and of each of their coefficients as distinct. More formally, we could let B be the set of ordered tuples (b, i, j), where $b \in R$ is the coefficient of $f_i(x)$ at degree j. However, it is easier to just think of the subscripts on these coefficients as place-holders, so we do not have to overly complicate notation. We leave it to the reader to verify that this slight equivocation causes no problems in the proof.

Due to cardinality reasoning, we can easily construct an injective set map $\varphi : B \to \mathbb{Z}_{>0}$ with the property that $\varphi(b_{i,j}) > i$. In effect, the map φ will tell us how to multiply polynomials to take advantage of the fact that I is left T-nilpotent, and in particular will give us a set of "reductions" that do not allow for looping. Let $B_i = \{b_{i,0}, \ldots, b_{i,n_i}\}$ be the set of coefficients of $f_i(x)$ (again, thought of as distinct from one another). Set $C_1 = \{1\}$ and inductively put $C_\ell = \{\varphi(b_{i,k}) \mid i \in C_{\ell-1}, k \text{ arbitrary}\} \cup C_{\ell-1}$. We also let

$$F_\ell = \left\{ f_i(x) r_i \mid i \in C_\ell, r_i \in \prod_{j=i+1}^{\varphi(b_{i,k})-1} B_j, \text{ where } k \text{ is arbitary} \right\}.$$

Note that C_ℓ is always a finite set, and hence so is F_ℓ, for each $\ell \geq 1$. By König's tree lemma, and the fact that I is left T-nilpotent, there exists some integer $n > 0$ so that

$$(4) \qquad\qquad F_1 F_2 \cdots F_n = 0.$$

Let N be the maximal element in C_{n+1}.

Let S be the set of all sequences $\delta = (d_1, d_2, \ldots)$ where $d_i \in B_i$, and note that $\alpha \in S$. We will prove that for any element $\delta \in S$ then $d_1 d_2 \cdots d_N = 0$. To do so, we need additional notations. Given $\delta = (d_1, d_2, \ldots) \in S$ we define the active subsequence $\delta' = (d_1', d_2', \ldots)$ by setting $d_1' = d_1$, and for $i > 1$ we put $d_i' = d_{\varphi(d_{i-1}')}$. The fact that $\varphi(b_{i,j}) > i$ implies that this really is a subsequence.

Given a sequence δ, with active subsequence δ' as written above, let $f_i'(x) = f_{\varphi(d_{i-1}')}(x)$ denote the polynomial with d_i' occurring as a coefficient. Write

$$d_1 d_2 \cdots d_N = d_1' r_1 d_2' r_2 \cdots d_n' r_n r$$

by combining terms as necessary. Note that $d_i' r_i$ is a coefficient in $f_i'(x) r_i \in F_i$. Hence $d_1' r_1 d_2' r_2 \cdots d_n' r_n r$ is a term in the product $\prod_{i=1}^{n} f_i'(x) r_i$, say at degree k. This product of polynomials is zero, by Equation (4). Hence

$$d_1' r_1 d_2' r_2 \cdots d_n' r_n r = - \sum_{(e_1', e_2', \ldots, e_n')} e_1' r_1 e_2' r_2 \cdots e_n' r_n r$$

where the sum is over all n-tuples where e_i' is a coefficient of $f_i'(x)$, the product $e_1' \cdots e_n'$ occurs at degree k, and $(e_1', e_2', \ldots, e_n') \neq (d_1', d_2', \ldots, d_n')$. We let $\mathcal{E}_\delta = \{\eta = (e_1, e_2, \ldots) \in S\}$ be the set of all sequences η that are obtained from δ by the following process: replace the first n terms of the active subsequence of δ with one of the sequences $(e_1', e_2', \ldots, e_n')$ satisfying the conditions above. In particular, we have the equality

$$(5) \qquad\qquad d_1 d_2 d_3 \cdots d_N = - \sum_{\eta \in \mathcal{E}_\delta} e_1 e_2 e_3 \cdots e_N.$$

Note that at least one of the e_i' (for each η) occurs in a higher degree than where d_i' occurs, unless the sum is over the empty set and hence equals 0.

Give two sequences $\delta, \eta \in S$ we say that one can make a "valid move" from δ to η, written $\delta \mapsto \eta$, if $\eta \in \mathcal{E}_\delta$. Equation (5) says that any sequence in S is a linear combination of sequences arrived at by valid moves. Valid moves can only change the first N entries of a sequence. In particular, either there is a cycle of valid moves $\delta_1 \mapsto \delta_2 \mapsto \cdots \mapsto \delta_p = \delta_1$ with $p > 1$, or after a fixed number of valid moves every sequence in S is sent to zero. If the second case holds then our theorem is established; for the product $a_1 a_2 \cdots a_N$ can be replaced by a linear combination of products arising from valid moves applied to α, and this process can be repeated, eventually terminating is a linear combination of zero products.

So, suppose by way of contradiction that there is a sequence of valid moves $\delta_1 \mapsto \delta_2 \mapsto \cdots \mapsto \delta_p = \delta_1$ with $p > 1$. We can define an ordering on active subsequences by lexicographical order. In other words $(d'_1, d'_2, \ldots) < (e'_1, e'_2, \ldots)$ if and only if there is an integer $m \geq 1$ so that $d'_i = e'_i$ when $i < m$, but d'_m occurs as the coefficient of a smaller power of x than e'_m does. After renumbering our loop, we may suppose that δ_1 has a maximal active subsequence. For each $\ell > 0$ write the active subsequence of δ_ℓ as $\delta'_\ell = (d'_{\ell,1}, d'_{\ell,2}, \ldots)$. When passing from δ_1 to δ_2, only the active subsequence changes. Further, one of the entries must change to a coefficient of higher degree, as mentioned immediately after Equation (5), say d'_{1,k_0} increases to d'_{2,k_0}, with k_0 minimal. But since we are in a cycle of valid moves, this coefficient must later change again. This can only happen, due to the injectivity of φ, if some later active subsequence begins $(d'_{1,1}, d'_{1,2}, \ldots, d'_{1,k_0-1}, d'_{2,k_0})$, contradicting the maximality of δ_1. Thus, there can be no cycles, and the theorem is proven. $\square$

Definition 19. Let I be an indexing set, and for each $i \in I$ let J_i be a subset of R. We say that the collection $\{J_i\}_{i \in I}$ is *right vanishing* if for each sequence $i_1, i_2, i_3, \ldots$ of *distinct* elements of I, and each sequence of elements $a_{i_j} \in J_{i_j}$, there exists an integers $n \geq 1$ so that $a_{i_1} a_{i_2} \cdots a_{i_n} = 0$.

This looks similar to the definition of left T-nilpotence, except that we are not allowed to repeat an index, and the elements are not restricted to a single set. However, we still have:

Proposition 20. *Let R be a ring, and let I be an indexing set. If $\{J_i\}_{i \in I}$ is a right vanishing set of left (or right) ideals, then $\{J_i[x]\}_{i \in I}$ is right vanishing. On the other hand, if $\{K_i\}_{i \in I}$ is a right vanishing set of left (or right) ideals in $R[x]$, then $\{A_{K_i}\}_{i \in I}$ is right vanishing.*

Proof. The first statement is a consequence of König's tree lemma. The second follows from the proof of the previous theorem. $\square$

We thank the referee for pointing out the following application of Theorem 18. Given a ring R, and an indexing set I, we let $RFM_I(R)$ denote the ring of row-finite matrices with rows and columns indexed by I. As proved by Patterson [8], one has the equality $\mathrm{rad}(RFM_I(R)) = RFM_I(\mathrm{rad}(R))$ if and only if $\mathrm{rad}(R)$ is left T-nilpotent. In fact, as proved by Sexauer and Warnock [11], and subsequently generalized by Zelmanowitz [14], we have $r \in \mathrm{rad}(RFM_I(R))$ if and only if $r \in RFM_I(\mathrm{rad}(R))$ and the column left ideals of r are right vanishing, where the column left ideals of $r = (r_{ij})_{i,j \in I}$ are of the form

$$B_j(r) = \left\{ \sum_{i \in I} k_i r_{ij} \ \middle|\ k_i \in R \text{ and } k_i = 0 \text{ for almost all } i \in I \right\}.$$

Putting this all together, we have:

Corollary 21. *The Jacobson radical of $RFM_I(R[x])$ is the set of matrices $r \in RFM_I(N[x])$, where $N = \mathrm{rad}(R[x]) \cap R$ is a nil ideal, and the coefficient ideals for each of the column left ideals of r are right vanishing.*

4. LOOSE ENDS

Proposition 10 leaves open the possibility that the condition "σ is an epimorphism of R" could be weakened to "σ is an endomorphism of R." But the following example shows this will not work, even for a monomorphism.

Example 22. We refer the reader to [7, Example 3.7]. Let R be the subset of the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ matrices over a field K defined as follows

$$R = \left\{ M \;\middle|\; M = \sum_{i,j=1}^{n} a_{ij}e_{ij} + a \sum_{i=n+1}^{\infty} e_{ii} \text{ for some } n \in \mathbb{Z}_{>0} \text{ and } a_{ij}, a \in K \right\},$$

where $\{e_{ij}\}_{i,j\in\mathbb{N}}$ denotes the set of matrix units. In other words, R is the set of infinite matrices with finite support, plus K-multiples of the identity matrix. The ring R is a prime ring. Define a map $\sigma : R \to R$ by

$$\sigma \left(\sum_{i,j=1}^{n} a_{ij}e_{ij} + a \sum_{i=n+1}^{\infty} e_{ii} \right) = ae_{11} + \sum_{i,j=1}^{n} a_{ij}e_{i+1,j+1} + a \sum_{i=n+2}^{\infty} e_{ii}.$$

In other words,

$$\begin{pmatrix} A & 0 & 0 & 0 & \cdots \\ 0 & a & 0 & 0 & \cdots \\ 0 & 0 & a & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & 0 & \cdots \\ 0 & A & 0 & 0 & \cdots \\ 0 & 0 & a & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where the 0 entries have the appropriate sizes. The map σ is an injective ring endomorphism of R. Notice that $e_{11}\sigma(R) = Ke_{11}$. Therefore, for any integer $k \geq 0$, we have $e_{11}xRx^ke_{11} = Ke_{11}e_{2+k,2+k}x^{k+1} = 0$, and so $e_{11}xR[x;\sigma]e_{11} = 0$. But $e_{11}R\sigma(e_{11})R$ is not nilpotent, as it contains $e_{11} = e_{11}e_{12}e_{22}e_{21}$.

Theorems 12 and 14 classify which properties pass up, or go down, in polynomial rings. One can ask the same question for finitely generated right ideals, and the result is much nicer.

Theorem 23. *Let R be a ring, and let $\mathcal{P}$ be a property in Diagram (3). Let J be a finitely generated right ideal of R, and let I be a finitely generated right ideal of $R[x]$. If J has property $\mathcal{P}$ then $J[x]$ necessarily has property $\mathcal{P}$, except in the case $\mathcal{P} = $ "nil." If I has property $\mathcal{P}$ then so does A_I. No other implications among these properties exist, going up or down, except those implied by transitivity.*

Proof. All claimed implications follow from Theorems 12 and 14 and Proposition 16. Nonexistence of additional implications follows from our earlier examples. □

5. Acknowledgements

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