Math 334 Midterm III KEY
Fall 2006
sections 001 and 004
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering.

Warning: check your solutions to each problem via a method independent of the one used to obtain your initial solution.

1. Solve the following initial value problem in terms of the convolution integral:

$$y'' + 9y = g(t);\ y(0) = y'(0) = 0. \tag{1.1}$$

15 points

Solution

Laplace transformation of (1.1) gives

$$\begin{gathered} s^2 L[y] + 9L[y] = L[y''] + 9L[y] = L[g] \\ \Leftrightarrow \\ L[y] = \frac{1}{s^2+3^2} L[g] = L\left[\frac{1}{3}\sin(3\cdot) * g\right] \\ \Leftrightarrow \\ y(t) = \left(\frac{1}{3}\sin(3\cdot) * g\right)(t) = \int_0^t \frac{1}{3}\sin\left(3(t-\tau)\right) g(\tau)d\tau. \end{gathered} \tag{1.2}$$

2. Find the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} \tag{1.3}$$

20 points

Solution

The general solution of (1.3) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t}, \tag{1.4}$$

where the $\boldsymbol{\xi}$'s and λ's are independent eigenvectors and distinct eigenvalues of the matrix in (1.3):

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (\lambda - 1)^2 - 2^2 \tag{1.5}$$

$$\Leftrightarrow$$

$$\lambda = 1 \pm 2 = 3, -1 =: \lambda_1, \lambda_2.$$

So

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_1 \Leftarrow \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ \mathbf{0} &= \begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_2 \Leftarrow \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{aligned} \tag{1.6}$$

Thus, explicitly, (1.4) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}. \tag{1.7}$$

3. Show that

$$L[f](s) = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}} \tag{1.8}$$

for $s, T > 0$ provided $f(t+T) = f(t)$ for all $t \in [0, +\infty]$ (and provided $\int_0^T e^{-st} f(t)dt$ exists).

10 points

Solution

We have

$$L[f](s) := \int_0^{+\infty} e^{-st} f(t)dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st} f(t)dt = \sum_{n=0}^{\infty}\int_0^{T} e^{-s(t+nT)} f(t+nT)dt$$
$$= \sum_{n=0}^{\infty} e^{-snT}\int_0^T e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt \sum_{n=0}^{\infty}\left(e^{-sT}\right)^n = \left(\int_0^T e^{-st} f(t)dt\right)\cdot \frac{1}{1-e^{-sT}} \,. \quad (1.9)$$
$$= \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}.$$

4. Find a real-valued representation of the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}\mathbf{x} \quad (1.10)$$

25 points

Solution

The general solution of (1.10) can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = c_1\boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2\boldsymbol{\xi}_2 e^{\lambda_2 t}\,, \quad (1.11)$$

where the $\boldsymbol{\xi}$'s and λ 's are independent eigenvectors and distinct eigenvalues of the matrix in (1.10):

$$\begin{bmatrix} 1-\lambda & 5 \\ -1 & -3-\lambda \end{bmatrix}\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det\begin{bmatrix} 1-\lambda & 5 \\ -1 & -3-\lambda \end{bmatrix} = (\lambda-1)(\lambda+3)+5 = \lambda^2+2\lambda+2 = (\lambda+1)^2 - i^2 \quad (1.12)$$

$$\Leftrightarrow$$

$$\lambda = -1 \pm i = -1+i, -1-i =: \lambda_1, \lambda_2.$$

So

$$\mathbf{0} = \begin{bmatrix} 1-(-1+i) & 5 \\ -1 & -3-(-1+i) \end{bmatrix}\boldsymbol{\xi}_1 = \begin{bmatrix} 2-i & 5 \\ -1 & -2-i \end{bmatrix}\boldsymbol{\xi}_1 = \begin{bmatrix} 2-i & 5 \\ -2+i & -5 \end{bmatrix}\boldsymbol{\xi}_1 = \begin{bmatrix} 2-i & 5 \\ 0 & 0 \end{bmatrix}\boldsymbol{\xi}_1 \Leftarrow \boldsymbol{\xi}_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix},$$
$$\mathbf{0} = \begin{bmatrix} 1-(-1-i) & 5 \\ -1 & -3-(-1-i) \end{bmatrix}\boldsymbol{\xi}_2 = \begin{bmatrix} 2+i & 5 \\ -1 & -2+i \end{bmatrix}\boldsymbol{\xi}_2 = \begin{bmatrix} 2+i & 5 \\ -2-i & -5 \end{bmatrix}\boldsymbol{\xi}_2 = \begin{bmatrix} 2+i & 5 \\ 0 & 0 \end{bmatrix}\boldsymbol{\xi}_2 \Leftarrow \boldsymbol{\xi}_2 = \overline{\boldsymbol{\xi}_1} = \begin{bmatrix} 2-i \\ -1 \end{bmatrix}.$$
$$(1.13)$$

Thus, explicitly, (1.11) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 2-i \\ -1 \end{bmatrix} e^{(-1-i)t}. \tag{1.14}$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{-t}\left(\cos t + i \sin t\right) = \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^{-t} + i \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix} e^{-t}, \tag{1.15}$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix} e^{-t}. \tag{1.16}$$

5. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ to the above problem that has the property that $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 points

<u>Solution</u>

A fundamental matrix of solutions $\Psi = \Psi(t)$, one not necessarily having the desired property, can be found from the above general solution (1.16):

$$\Psi(t) = e^{-t} \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix}. \tag{1.17}$$

The desired fundamental matrix $\Phi = \Phi(t)$ can be obtained from $\Psi = \Psi(t)$ via

$$\begin{aligned} \Phi = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{-t} \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \\ &= e^{-t} \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{bmatrix}. \end{aligned} \tag{1.18}$$

6. Solve the initial value problem given by the system of problem 4 and the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{1.19}$$

(Hint: rather than "reinventing the wheel", just use the fundamental matrix of problem 5.)

7 points

Solution

Using the fundamental matrix of problem 5 we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = e^{-t}\begin{bmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{bmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = e^{-t}\begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix}. \tag{1.20}$$

7. Calculate

$$e^{\begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}\pi}. \tag{1.21}$$

(Hint: use the result of problem 5).

5 points

Solution

We have, from problem 5,

$$e^{\begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}\pi} = \Phi(\pi) = e^{-\pi}\begin{bmatrix} \cos\pi + 2\sin\pi & 5\sin\pi \\ -\sin\pi & \cos\pi - 2\sin\pi \end{bmatrix} = \begin{bmatrix} -e^{-\pi} & 0 \\ 0 & -e^{-\pi} \end{bmatrix}. \tag{1.22}$$

8. Find a representation of the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\mathbf{x}. \tag{1.23}$$

20 points

Solution

The matrix in (1.23) has a repeated eigenvalue with only one eigenvector. Hence the general solution is of the form

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi} e^{\lambda t} + c_2 \left(\boldsymbol{\xi} t + \boldsymbol{\eta} \right) e^{\lambda t} \tag{1.24}$$

where $\boldsymbol{\xi}$ is an eigenvector and $\boldsymbol{\eta}$ is an associated pseudo eigenvector:

$$\begin{bmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + 3\lambda - 3\lambda - 9 + 9 = \lambda^2 \tag{1.25}$$

$$\Leftrightarrow$$

$$\lambda = 0, 0,$$

So

$$\mathbf{0} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \boldsymbol{\xi} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi} \Leftarrow \boldsymbol{\xi} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

and (1.26)

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \boldsymbol{\eta} = \boldsymbol{\xi} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{\eta} = \begin{bmatrix} 1-3\eta_2 \\ \eta_2 \end{bmatrix}.$$

Thus, explicitly, (1.24) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1-3\eta_2 \\ \eta_2 \end{bmatrix} \right). \tag{1.27}$$

9. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ for the system of problem 8 that satisfies $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

15 points

<u>Solution</u>

From (1.27) we have a fundamental matrix of solutions

$$\Psi(t) = \begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix}, \tag{1.28}$$

whence the one desired is

$$\begin{aligned} \Phi(t) = \Psi(t)\Psi^{-1}(0) &= \begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix} \begin{bmatrix} 3 & 1-3\eta_2 \\ -1 & \eta_2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix} \begin{bmatrix} \eta_2 & -1+3\eta_2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix}. \end{aligned} \tag{1.29}$$

10. Solve the initial value problem obtained from combining the differential equation of problem 8 with the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{1.30}$$

(Hint: do not "reinvent the wheel", but rather use the result from problem 9.)

6 points

<u>Solution</u>

From problem 9 we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = \begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{bmatrix} 2+42t \\ 4-14t \end{bmatrix}. \tag{1.31}$$