KEY

Math 334 Final
Fall 2007
section 004
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering paper. Write your name, course, and section number on the blue book, or on your own pile of papers. Again, do not write this or any other type of information on this exam.

Warning: check your solutions to each problem via a method independent of the one used to obtain your initial solution.

1. Consider the system of equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.1}$$

Write down the general solution of(1.1), and then sketch a phase portrait of this system, and also separately sketch a plot of both $x_1(t)$ versus t as well as $x_2(t)$ versus t, for $t \in [-1,1]$, given that

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.2}$$

(Note: These are three plots and one formula for the general solution.)

25 points

Solution

The general solution of (1.1) is

$$\mathbf{x}(t) = C_- e^{\lambda_- t}\mathbf{z}_- + C_+ e^{\lambda_+ t}\mathbf{z}_+ \tag{1.3}$$

provided $\mathbf{z}_-$ and $\mathbf{z}_+$ are linearly independent eigenvectors of matrix A corresponding to respective eigenvalues λ_- and λ_+. Now λ is an eigenvalue of A iff

$$\begin{aligned} 0 &= \det(\lambda I - A) = \det\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}\right) = \det\begin{bmatrix} \lambda - 2 & 2 \\ 2 & \lambda + 1 \end{bmatrix} = (\lambda - 2)(\lambda + 1) - 4 \\ &= \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3) \Leftrightarrow \lambda = \lambda_- := -2, \text{ or } \lambda = \lambda_+ := 3. \end{aligned} \tag{1.4}$$

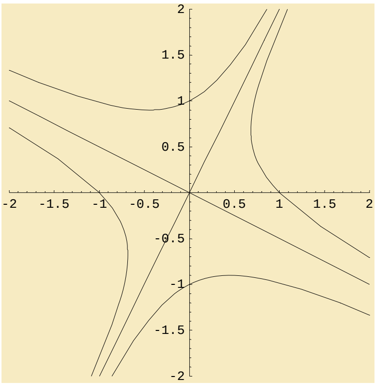
Since these eigenvalues are distinct, their corresponding eigenvectors are linearly independent and (1.3) holds, with $\mathbf{z}_-$ and $\mathbf{z}_+$ determined as follows:

$$\begin{aligned} \mathbf{0} \neq \mathbf{z}_- &\in \text{Nul}(\lambda_- I - A) = \text{Nul}\begin{bmatrix} \lambda_- - 2 & 2 \\ 2 & \lambda_- + 1 \end{bmatrix} = \text{Nul}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \text{Nul}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}, \\ \mathbf{0} \neq \mathbf{z}_+ &\in \text{Nul}(\lambda_+ I - A) = \text{Nul}\begin{bmatrix} \lambda_+ - 2 & 2 \\ 2 & \lambda_+ + 1 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}. \end{aligned} \tag{1.5}$$

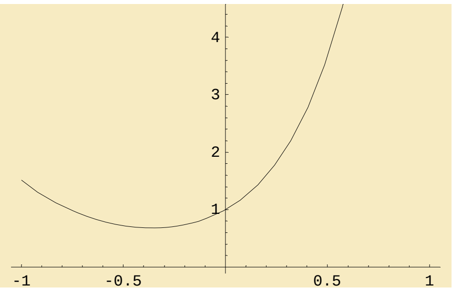
Thus, according to (1.4) and (1.5), we can take (1.3) as the general solution in the form of

$$\mathbf{x}(t) = C_- e^{\lambda_- t}\mathbf{z}_- + C_+ e^{\lambda_+ t}\mathbf{z}_+ = C_- e^{-2t}\begin{bmatrix}1\\2\end{bmatrix} + C_+ e^{3t}\begin{bmatrix}2\\-1\end{bmatrix}. \tag{1.6}$$

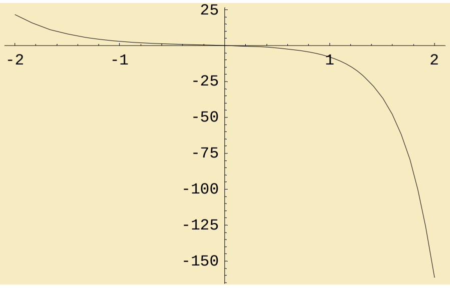
The phase portrait is given then by using these eigenvectors to form asymptotes of generic trajectories (as well as to indicate non-generic/“straight” trajectories), given that the $\begin{bmatrix}1\\2\end{bmatrix}$ direction corresponds to inward motion to the origin while the orthogonal $\begin{bmatrix}2\\-1\end{bmatrix}$ direction corresponds to outward motion from the origin. A Mathematica generated phase portrait (lacking arrows to indicate the forward passage of time t) is as follows:



Note that we have used here a trajectory that, at some time or another—one could choose it to be at $t = 0$, went through the point (1.2). Staring at that phase plane trajectory a moment suggests the final two graphs required here, which, generated by Mathematica, are



and



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2. Solve the following initial value problem in terms of the convolution integral:

$$\mathbf{x}' = \begin{bmatrix}3 & -2\\1 & 0\end{bmatrix}\mathbf{x} + \mathbf{g}(t), \ \mathbf{x}(0) = \mathbf{0}. \tag{1.7}$$

Here I expect you to write the solution $\mathbf{x} = \mathbf{x}(t)$ of (1.7) as

$$\mathbf{x}(t)=\int_0^t \begin{bmatrix} M_{11}(t-\tau) & M_{12}(t-\tau) \\ M_{21}(t-\tau) & M_{22}(t-\tau) \end{bmatrix} \mathbf{g}(\tau)d\tau =: \int_0^t M(t-\tau)\mathbf{g}(\tau)d\tau = \left(M*\mathbf{g}\right)(t), \qquad (1.8)$$

so that the problem is effectively to determine $M_{11}(t-\tau), M_{12}(t-\tau), M_{21}(t-\tau),$ and $M_{22}(t-\tau)$ in (1.8). (Hint: Use the Laplace transform and the convolution theorem.)

25 points

Solution

By the Laplace Transform we have from (1.7) that

$$sL[\mathbf{x}]-\mathbf{x}(0)=sL[\mathbf{x}]=\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} L[\mathbf{x}]+L[\mathbf{g}]$$

$$\Leftrightarrow$$

$$\begin{bmatrix} s-3 & 2 \\ -1 & s \end{bmatrix} L[\mathbf{x}]=L[\mathbf{g}]$$

$$\Leftrightarrow$$

$$\begin{aligned}
L[\mathbf{x}] &= \begin{bmatrix} s-3 & 2 \\ -1 & s \end{bmatrix}^{-1} L[\mathbf{g}] = \frac{1}{\left(s-3\right)\left(s\right)-\left(-1\right)\left(2\right)} \begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}] \\
&= \frac{1}{s^2-3s+2} \begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}] = \frac{1}{\left(s-1\right)\left(s-2\right)} \begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}] \\
&= \begin{bmatrix} \dfrac{s}{\left(s-1\right)\left(s-2\right)} & \dfrac{-2}{\left(s-1\right)\left(s-2\right)} \\ \dfrac{1}{\left(s-1\right)\left(s-2\right)} & \dfrac{s-3}{\left(s-1\right)\left(s-2\right)} \end{bmatrix} L[\mathbf{g}] \\
&= \begin{bmatrix} \dfrac{\frac{1}{1-2}}{s-1}+\dfrac{\frac{2}{2-1}}{s-2} & \dfrac{\frac{-2}{1-2}}{s-1}+\dfrac{\frac{-2}{2-1}}{s-2} \\ \dfrac{\frac{1}{1-2}}{s-1}+\dfrac{\frac{1}{2-1}}{s-2} & \dfrac{\frac{1-3}{1-2}}{s-1}+\dfrac{\frac{2-3}{2-1}}{s-2} \end{bmatrix} L[\mathbf{g}] \\
&= \begin{bmatrix} \dfrac{2}{s-2}-\dfrac{1}{s-1} & 2\left(\dfrac{1}{s-1}-\dfrac{1}{s-2}\right) \\ \dfrac{1}{s-2}-\dfrac{1}{s-1} & \dfrac{2}{s-1}-\dfrac{1}{s-2} \end{bmatrix} L[\mathbf{g}] = L[M*\mathbf{g}], \qquad (1.9)
\end{aligned}$$

so that

$$M = M(t) = \begin{bmatrix} 2e^{2t} - e^{t} & 2\left(e^{t} - e^{2t}\right) \\ e^{2t} - e^{t} & 2e^{t} - e^{2t} \end{bmatrix}. \tag{1.10}$$

Thus, from (1.8) and(1.10), we have

$$\begin{aligned} \mathbf{x} = \mathbf{x}(t) &= \left(M * \mathbf{g}\right)(t) = \int_0^t M(t-\tau)\mathbf{g}(\tau)d\tau \\ &= \int_0^t \begin{bmatrix} 2e^{2(t-\tau)} - e^{(t-\tau)} & 2\left(e^{(t-\tau)} - e^{2(t-\tau)}\right) \\ e^{2(t-\tau)} - e^{(t-\tau)} & 2e^{(t-\tau)} - e^{2(t-\tau)} \end{bmatrix} \mathbf{g}(\tau)d\tau. \end{aligned} \tag{1.11}$$

3. Find a real-valued representation of the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \mathbf{x} \tag{1.12}$$

25 points

Solution

The general solution of (1.12) can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_+ e^{\lambda_+ t} + c_2 \boldsymbol{\xi}_- e^{\lambda_- t}, \tag{1.13}$$

provided the $\boldsymbol{\xi}$'s and λ's are independent eigenvectors and distinct eigenvalues of the matrix in (1.12):

$$\begin{bmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} = \left(\lambda - 3\right)\left(\lambda + 1\right) + 8 = \lambda^2 - 2\lambda + 5 = \left(\lambda - 1\right)^2 + 2^2 \tag{1.14}$$

$$\Leftrightarrow$$

$$\lambda = 1 \pm 2i = 1 + 2i, 1 - 2i =: \lambda_+, \lambda_-.$$

So

$$\begin{aligned}\mathbf{0} &= \begin{bmatrix} -1-(1\pm 2i) & -4 \\ 2 & 3-(1\pm 2i) \end{bmatrix}\boldsymbol{\xi}_\pm = \begin{bmatrix} -2\mp 2i & -4 \\ 2 & 2\mp 2i \end{bmatrix}\boldsymbol{\xi}_\pm = \begin{bmatrix} -1\mp i & -2 \\ 1 & 1\mp i \end{bmatrix}\boldsymbol{\xi}_\pm \\ &= \begin{bmatrix} -1\mp i & -2 \\ 1\cdot(-1\mp i) & (1\mp i)(-1\mp i) \end{bmatrix}\boldsymbol{\xi}_\pm = \begin{bmatrix} -1\mp i & -2 \\ -1\mp i & -1+i^2 \pm i \mp i \end{bmatrix}\boldsymbol{\xi}_\pm = \begin{bmatrix} -1\mp i & -2 \\ -1\mp i & -2 \end{bmatrix}\boldsymbol{\xi}_\pm \quad (1.15) \\ &= \begin{bmatrix} -1\mp i & -2 \\ 0 & 0 \end{bmatrix}\boldsymbol{\xi}_\pm \Leftarrow \boldsymbol{\xi}_\pm = \begin{bmatrix} 2 \\ -1\mp i \end{bmatrix}.\end{aligned}$$

Thus, explicitly, (1.13) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 2 \\ -1+i \end{bmatrix} e^{(1-2i)t}. \qquad (1.16)$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) := \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^t \left(\cos 2t + i \sin 2t\right) = \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix} e^t + i \begin{bmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{bmatrix} e^t, \qquad (1.17)$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix} e^t + c_2 \begin{bmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{bmatrix} e^t. \qquad (1.18)$$

4. Find a representation of the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{x}. \qquad (1.19)$$

25 points

<u>Solution</u>

The matrix in (1.19) has a repeated eigenvalue with only one eigenvector. Hence the general solution is of the form

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi} e^{\lambda t} + c_2 \left(\boldsymbol{\xi} t + \boldsymbol{\eta}\right) e^{\lambda t} \qquad (1.20)$$

where λ is the sole eigenvalue, $\boldsymbol{\xi}$ is its eigenvector, and $\boldsymbol{\eta}$ is an associated pseudo eigenvector:

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det\begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \tag{1.21}$$

$$\Leftrightarrow$$

$$\lambda = 1,1.$$

So

$$\mathbf{0} = \begin{bmatrix} 0-1 & 1 \\ -1 & 2-1 \end{bmatrix}\boldsymbol{\xi} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\boldsymbol{\xi} \Leftarrow \boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and (1.22)

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\boldsymbol{\eta} = \boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ 1+\eta_1 \end{bmatrix}.$$

Thus, explicitly, **(1.33)** is

$$\mathbf{x} = \mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{1t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}t + \begin{bmatrix} \eta_1 \\ 1+\eta_1 \end{bmatrix}\right)e^{1t}. \tag{1.23}$$

5. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2y'' - 3xy' + 13y = 0. \tag{1.24}$$

25 points

Solution

The differential equation (1.24) defines a linear differential operator L_x, in terms of which (1.24) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \big(r(r-1) - 3r + 13\big)x^r = \big(r^2 - 4r + 13\big)x^r = \Big((r-2)^2 + 3^2\Big)x^r, \tag{1.25}$$

so that complex solutions of (1.24) are clearly then
$y_{2+3i} = x^{2+3i} = x^2 e^{3i\ln x} = x^2\left(\cos(3\ln x) + i\sin(3\ln x)\right)$ and
$y_{2-3i} = x^{2-3i} = x^2 e^{-3i\ln x} = x^2\left(\cos(3\ln x) - i\sin(3\ln x)\right)$. Independent complex linear combinations of these linearly independent complex valued solutions gives the following real-representation of the general solution:

$$y = x^2\left(A\cos(3\ln x) + B\sin(3\ln x)\right). \tag{1.26}$$

6. Solve the following initial value problem:

$$y'' - 4y' + 29y = 0; \;\; y(0) = 1, \; y'(0) = 3. \tag{1.27}$$

25 points

Solution

This linear homogeneous differential equation is associated with the following characteristic equation and characteristic exponents r:

$$\begin{aligned} &0 = r^2 - 4r + 29 = r^2 - 4r + 4 + 25 = (r-2)^2 - (5i)^2 \\ &\Leftrightarrow \\ &r = 2 \pm 5i. \end{aligned} \tag{1.28}$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$\begin{aligned} &y = e^{2t}\left(C_1\cos 5t + C_2\sin 5t\right) \\ &\quad\text{and} \\ &y' = e^{2t}\left(\left(2C_1 + 5C_2\right)\cos 5t + \left(-5C_1 + 2C_2\right)\sin 5t\right). \end{aligned} \tag{1.29}$$

Inserting $t = 0$ into(1.29), and using the initial data given in(1.27), one obtains

$$\begin{aligned} &y(0) = C_1 = 1 \\ &\quad\text{and} \\ &y'(0) = 2C_1 + 5C_2 = 3, \end{aligned} \tag{1.30}$$

the solution to which being $C_1 = 1$ and $C_2 = 1/5$. Thus the solution to the initial value problem is then

$$y = e^{2t}\left(\cos 5t + \frac{1}{5}\sin 5t\right). \qquad (1.31)$$

7. Find the first *two* ***nonzero*** terms (if there are that many) in the series representation of *one* of the 2 linearly independent solutions of the equation

$$\left(x^2+2x^3\right)y'' - \left(2x+6x^2\right)y' + \left(2+6x\right)y = 0. \qquad (1.32)$$

about the point $x_0 = 0$.

25 points

Solution

The point $x_0 = 0$ is a (regular) singular point, so that the required series solution is not quite a Taylor series: rather insert $y = \sum_n a_n x^{n+r}$ (with the assumption that $a_n = 0$ for $n < 0$, and that the sum is over the integers, and that $a_0 \neq 0$) in (1.32) to obtain

$$\begin{aligned}
0 &= \sum_n \left(x^2+2x^3\right)\left(n+r\right)(n+r-1)a_n x^{n+r-2} - \left(2x+6x^2\right)\left(n+r\right)a_n x^{n+r-1} + \left(2+6x\right)a_n x^{n+r} \\
&= \sum_n \left\{\begin{array}{l} \left(n+r\right)(n+r-1)a_n x^{n+r} + 2\left(n+r\right)(n+r-1)a_n x^{n+r+1} \\ -2\left(n+r\right)a_n x^{n+r} - 6\left(n+r\right)a_n x^{n+r+1} \\ +2a_n x^{n+r} + 6a_n x^{n+r+1} \end{array}\right\} \\
&= \sum_n \left\{\begin{array}{l} \left(n+r\right)(n+r-1)a_n x^{n+r} + 2\left(n+r-1\right)(n+r-2)a_{n-1} x^{n+r} \\ -2\left(n+r\right)a_n x^{n+r} - 6\left(n+r-1\right)a_{n-1} x^{n+r} \\ +2a_n x^{n+r} + 6a_{n-1} x^{n+r} \end{array}\right\} \\
&= \sum_n \left\{\left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1}\right\} x^{n+r} \\
&= \sum_{n=0}^{\infty} \left\{\left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1}\right\} x^{n+r} \\
&= \left[r(r-1) - 2r + 2\right]a_0 x^r + \sum_{n=1}^{\infty} \left\{\begin{array}{l} \left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n \\ + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1} \end{array}\right\} x^{n+r}.
\end{aligned}$$

(1.33)

Evidently we require

$$0 = r(r-1) - 2r + 2 = r^2 - 3r + 2 = \left(r-1\right)\left(r-2\right). \tag{1.34}$$

Since these roots differ by an integer, according to the general theory we should only use the larger of the two roots. However, here a miracle occurs and it turns out that both will work (so that you do not need to be aware of the general theory): even for the "wrong" choice $r = 1$ we get that (1.33) becomes

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} \left\{ \left[\left(n+1\right)n - 2\left(n+1\right) + 2 \right] a_n + 2\left[n(n-1) - 3n + 3 \right] a_{n-1} \right\} x^{n+1} \\
&= \sum_{n=1}^{\infty} \left[n(n-1)a_n + 2(n-1)(n-3)a_{n-1} \right] x^{n+1} \\
&= \sum_{n=1}^{\infty} (n-1)\left[na_n + 2(n-3)a_{n-1} \right] x^{n+1} \\
&= 0 + \sum_{n=2}^{\infty} (n-1)\left[na_n + 2(n-3)a_{n-1} \right] x^{n+1}
\end{aligned} \tag{1.35}$$

$$\Leftrightarrow$$

$$na_n + 2(n-3)a_{n-1} = 0,\ n = 2, 3, \ldots$$

$$\Leftrightarrow$$

$$a_n = -\frac{2(n-3)}{n} a_{n-1},\ n = 2, 3, \ldots.$$

Thus we have

$$\begin{aligned}
a_2 &= -\frac{2(2-3)}{2} a_{2-1} = a_1, \\
a_3 &= -\frac{2(3-3)}{3} a_{3-1} = 0, \\
a_4 &= -\frac{2(4-3)}{4} a_{4-1} = -\frac{1}{2} \cdot 0 = 0, \\
&\vdots \\
a_{n\geq 3} &= 0.
\end{aligned} \tag{1.36}$$

Thus the "infinite" series terminates, and we get

$$y = \sum_n a_n x^{n+r} = a_0 x^1 + a_1 x^2 + a_2 x^3 = a_0 x + a_1 x^2 + a_1 x^3 = a_0 x + a_1 \left(x^2 + x^3\right), \tag{1.37}$$

which gives the general solution of (1.32).

8. Solve the following initial value problem:

$$\frac{dy}{dt} = t^2 - t^2 y^2,\ y(0) = 0\,. \tag{1.38}$$

25 points

Solution

The equation separates to

$$\frac{dy}{1 - y^2} = t^2 dt. \tag{1.39}$$

Thus, since for any constant C

$$\frac{dy}{1-y^2} = d \operatorname{arctanh} y,\ \text{and } t^2 dt = d\left(\frac{t^3}{3} + C\right), \tag{1.40}$$

one obtains from (1.39) the integrated equations

$$\operatorname{arctanh} y = \frac{t^3}{3} + C \Rightarrow y = \tanh\left(\frac{t^3}{3} + C\right). \tag{1.41}$$

Using the initial data from (1.38) in the first of equations (1.41) we have

$$0 = \operatorname{arctanh} 0 = \frac{0^3}{3} + C = C, \tag{1.42}$$

and then from second of equations (1.41) the solution

$$y = \tanh\left(\frac{t^3}{3}\right). \tag{1.43}$$

9. Show that the following differential equation (1.44) is not exact, but can be rendered exact by multiplication by an integrating factor that is only a function of either x or of y —please generate such a factor “systematically”, i.e. without guessing. Find an expression of the general solution of the differential equation.

$$\left(3x^2 y + y^2\right) dx + \left(2x^3 + 3xy\right) dy = 0\,. \tag{1.44}$$

25 points

Solution

(1.44) is not exact since

$$\psi_{xy} = \left(\psi_x\right)_y := \left(3x^2y + y^2\right)_y = 3x^2 + 2y \neq 6x^2 + 3y = \left(2x^3 + 3xy\right)_x =: \left(\psi_y\right)_x = \psi_{yx} . \tag{1.45}$$

With an integrating factor μ , (1.44), which can be written as

$$\left(3x^2 + y\right)y\,dx + x\left(2x^2 + 3y\right)dy\,, \tag{1.46}$$

can, by theorem, be made exact. We note from (1.45) that

$$\left(2x^3 + 3xy\right)_x - \left(3x^2y + y^2\right)_y = 6x^2 + 3y - \left(3x^2 + 2y\right) = 3x^2 + y\,, \tag{1.47}$$

which is a factor in the first term in (1.46), the remaining factor being only a function of y . Thus we suspect the existence of an integrating factor only depending on y . At any rate, with the use of such a factor, ODE (1.44) becomes

$$\mu(y)\left(3x^2y + y^2\right)dx + \mu(y)\left(2x^3 + 3xy\right)dy = 0\,, \tag{1.48}$$

and exactness demands that

$$\begin{aligned} 0 &= \left(\mu(y)\left(3x^2y + y^2\right)\right)_y - \left(\mu(y)\left(2x^3 + 3xy\right)\right)_x = \\ &\left(3x^2y + y^2\right)\mu'(y) + \left(3x^2 + 2y\right)\mu(y) - \left(6x^2 + 3y\right)\mu(y) \\ &= \left(3x^2 + y\right)y\mu'(y) - \left(3x^2 + y\right)\mu(y) \\ &= \left(3x^2 + y\right)\left(y\mu'(y) - \mu(y)\right) \\ &\Leftarrow \\ y\mu'(y) &= \mu(y). \end{aligned} \tag{1.49}$$

Thus, as suspected, there is an integrating factor depending only on y . A solution of (1.49) is evidently given by

$$\mu(y) = y\,, \tag{1.50}$$

in which case (1.48) becomes

$$\left(3x^2y^2 + y^3\right)dx + \left(2x^3y + 3xy^2\right)dy = 0\,. \tag{1.51}$$

Writing

$$\begin{aligned}\psi_x &= 3x^2y^2 + y^3, \\ \psi_y &= 2x^3y + 3xy^2,\end{aligned} \tag{1.52}$$

gives

$$\begin{aligned}\psi &= x^3y^2 + xy^3 + f(y), \\ \psi &= x^3y^2 + xy^3 + g(x),\end{aligned} \tag{1.53}$$

which can be reconciled by $f(y) = g(x) = 0$. Thus (1.51) can be written as

$$d\left(x^3y^2 + xy^3\right) = 0\,, \tag{1.54}$$

an expression of the general solution to which clearly being

$$x^3y^2 + xy^3 = C\,. \tag{1.55}$$

10. Solve the following initial value problem. State the properties of the solution as $t \to \infty$ for all choices of the initial value y_0.

$$y'(t) = -y(t) + \left(1+t\right)^2,\ y(0) = y_0\,. \tag{1.56}$$

25 points

Solution

The ODE in (1.56) can be written as

$$y'(t) + 1y(t) = \left(1+t\right)^2. \tag{1.57}$$

(1.57) suggests the integrating factor

$$\mu(t) = \exp\int 1dt = e^t\,, \tag{1.58}$$

which renders the ODE (1.57) as

$$
\begin{aligned}
&\frac{d}{dt}e^t y(t) = \\
e^t y'(t) + e^t y(t) &= (1+t)^2 e^t = \frac{d}{dt}(1+t)^2 e^t - e^t \frac{d}{dt}(1+t)^2 \\
&= \frac{d}{dt}(1+t)^2 e^t - 2(1+t)e^t = \frac{d}{dt}\Big((1+t)^2 e^t - 2(1+t)e^t\Big) + e^t \frac{d}{dt}2(1+t) \\
&= \frac{d}{dt}\Big((1+t)^2 e^t - 2(1+t)e^t\Big) + 2e^t \\
&= \frac{d}{dt}\Big((1+t)^2 e^t - 2(1+t)e^t + 2e^t\Big) = \frac{d}{dt}\Big((1+t)^2 - 2t\Big)e^t \\
&= \frac{d}{dt}\left(1+t^2\right)e^t,
\end{aligned}
\tag{1.59}
$$

which, with the initial data specified in (1.56), integrates to

$$
\begin{aligned}
&e^t y(t) - y_0 = e^t y(t) - 1\cdot y_0 = e^t y(t) - e^0 y(0) = e^s y(s)\Big|_0^t = \\
&\int_0^t de^s y(s) = \int_0^t d\left(1+s^2\right)e^s \\
&\qquad = \left(1+s^2\right)e^s\Big|_0^t = \left(1+t^2\right)e^t - \left(1+0^2\right)e^0 = \left(1+t^2\right)e^t - 1,
\end{aligned}
\tag{1.60}
$$

or, equivalently,

$$y(t) = 1 + t^2 + \left(y_0 - 1\right)e^{-t}. \tag{1.61}$$

Note that irrespective of the initial data y_0

$$\lim_{t\to\infty}\Big(y(t) - \left(1+t^2\right)\Big) = 0. \tag{1.62}$$