KEY

Math 334 Midterm I
Fall 2007
section 004
Instructor: Scott Glasgow

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1. Solve the following initial value problem:

$$\frac{dy}{dt} = t^2 - t^2 y^2,\ y(0) = 0. \tag{1.1}$$

10 points

Solution

The equation separates to

$$\frac{dy}{1-y^2} = t^2 dt. \tag{1.2}$$

Thus, since for any constant C

$$\frac{dy}{1-y^2} = d \operatorname{arctanh} y,\ \text{and } t^2 dt = d\left(\frac{t^3}{3} + C\right), \tag{1.3}$$

one obtains from (1.2) the integrated equations

$$\operatorname{arctanh} y = \frac{t^3}{3} + C \Rightarrow y = \tanh\left(\frac{t^3}{3} + C\right). \tag{1.4}$$

Using the initial data from (1.1) in the first of equations (1.4) we have

$$0 = \operatorname{arctanh} 0 = \frac{0^3}{3} + C = C, \tag{1.5}$$

and then from (1.4) the solution

$$y = \tanh\left(\frac{t^3}{3}\right). \tag{1.6}$$

Check:

$$y = \tanh\left(\frac{t^3}{3}\right) \Rightarrow \frac{dy}{dt} = \operatorname{sech}^2\left(\frac{t^3}{3}\right)\cdot t^2 = \left(1 - \tanh^2\left(\frac{t^3}{3}\right)\right)\cdot t^2 = \left(1-y^2\right)t^2 = t^2 - y^2 t^2, \tag{1.7}$$

$$y(0) = \tanh\left(0\right) = 0.$$

2. Prove that the following differential equation is exact and then find an expression for its general solution.

$$e^{xy}\left(1+xy\right)dx+e^{xy}x^2dy=0\,. \tag{1.8}$$

14 points

Solution

The equation (1.8) is, by definition, *exact* if the left-hand side is the differential of a (continuously differentiable) function (of two variables x and y, in some simply-connected region of the x-y plane, etc., etc.), i.e. if there is a function $\psi(x,y)$ such that

$$d\psi(x,y)=e^{xy}\left(1+xy\right)dx+e^{xy}x^2dy\,. \tag{1.9}$$

But we have, by definition,

$$d\psi(x,y)=\psi_x(x,y)dx+\psi_y(x,y)dy, \tag{1.10}$$

so that the equation (1.9) is the (potentially) over-determined system of equations

$$\psi_x(x,y)=e^{xy}\left(1+xy\right),\ \text{and}\ \psi_y(x,y)=e^{xy}x^2\,. \tag{1.11}$$

This over-determined pair of equations is consistent (or *integrable*) iff $\left(\psi_x\right)_y=\left(\psi_y\right)_x$, i.e. iff

$$\left(e^{xy}\left(1+xy\right)\right)_y=\left(e^{xy}x^2\right)_x\,. \tag{1.12}$$

(1.12) holds true, so that the equation (1.8) is indeed exact, because either side of (1.12) is $e^{xy}x\left(2+xy\right)$.

As for developing the function $\psi(x,y)$, and then (an expression for) the general solution of(1.8), one notes that the equations (1.11) demand, respectively, that

$$\begin{aligned}\psi(x,y)&=\int e^{xy}\left(1+xy\right)dx=\int e^{yx}dx+\int yxe^{yx}dx=\frac{e^{yx}}{y}+\int xde^{yx}\\&=\frac{e^{yx}}{y}+xe^{yx}-\int e^{yx}dx=xe^{yx}+f(y),\end{aligned} \tag{1.13}$$

and $\psi(x,y)=\int x^2e^{xy}dy=x\int e^{xy}d\left(xy\right)\Big|_{x\text{ fixed}}=xe^{xy}+g(x),$

for some initially rather arbitrary functions $f(y)$ and $g(x)$. The two statements (1.13) are not contradictory iff $f(y)=g(x)$, which implies both are independent of both x and y. As far as finding the general solution of (1.8) is concerned, without loss of generality we can choose $f(y)=g(x)=0$ so that (1.13) becomes

$$\psi(x,y)=xe^{xy}\,. \tag{1.14}$$

(1.14) is NOT the general solution to the (exact) differential equation (1.8). It is not even a specific solution. Rather (1.14) defines a "potential (function) for the solution." Using it one notes that (1.8) can be written as

$$d\psi(x,y)=d\left(xe^{xy}\right)=0\,, \tag{1.15}$$

the general solution to which is clearly

$$xe^{xy}=C\,. \tag{1.16}$$

3. Form the Picard iterates $\phi_1(t)$ and $\phi_2(t)$ for the initial value problem

$$\frac{dy}{dt}=t^2-t^2y^2,\ y(0)=0\,, \tag{1.17}$$

by first defining $\phi_0(t)\equiv 0$, and then, for $n=0,1$, making the recursive definitions

$$\phi_{n+1}(t):=\int_0^t\left(s^2-s^2\phi_n^{\,2}(s)\right)ds\,. \tag{1.18}$$

(If you understand Taylor series, you can check your answer with problem 1. This check is not required, just a fact!)

10 points

Solution

(1.18), together with $\phi_0(t)\equiv 0$, gives

$$\begin{aligned}\phi_1(t)&:=\int_0^t\left(s^2-s^2\phi_0^{\,2}(s)\right)ds=\int_0^t s^2ds=\frac{1}{3}s^3\bigg|_0^t=\frac{1}{3}t^3,\\ \phi_2(t)&:=\int_0^t\left(s^2-s^2\phi_1^{\,2}(s)\right)ds=\int_0^t\left(s^2-\frac{1}{9}s^8\right)ds=\frac{1}{3}s^3-\frac{1}{81}s^9\bigg|_0^t=\frac{1}{3}t^3-\frac{1}{81}t^9.\end{aligned} \tag{1.19}$$

4. Use the *most relevant* of the two existence and uniqueness theorem to find x_0's for which the (linear) initial value problem

$$\frac{dy}{dx} = 2\frac{y}{x-1},\ y(x_0) = y_0 \tag{1.20}$$

certainly has a unique solution $y = y(x)$ persisting for an open interval (of x's) containing x_0. According to this most relevant theorem, for any such x_0 what is the guaranteed interval of existence? Now find the general solution of the differential equation. Use this general solution to investigate the possibility of none or more than one solution passing through the initial point (x_0, y_0). Your investigation should not contradict the theorem, but should render additional insights not addressable by the theorem.

15 points

Solution

The differential equation is linear, so by the existence and uniqueness theorem for linear first order ODE, we know there will be one and only one solution to the initial value problem persisting for the largest open interval containing x_0 over which the coefficient function

$$P(x) := \frac{2}{x-1} \tag{1.21}$$

is continuous. Thus, to answer the first question, IVP (1.20) is guaranteed by the existence and uniqueness theorem for linear first order ODE to have a unique solution $y = y(x)$ persisting for an open interval (of x's) containing x_0 provided $x_0 \neq 1$ (at which $P(x)$ is not defined and, hence, not continuous). And to answer the second question we note that for any such $x_0 \neq 1$ the solution is guaranteed to persist (and be unique) over one of the unbounded intervals

$$\begin{aligned} &(-\infty, 1) \text{ if } x_0 \in (-\infty, 1), \text{ or} \\ &(1, \infty) \text{ if } x_0 \in (1, \infty). \end{aligned} \tag{1.22}$$

On the other hand, separation of the differential equation of (1.20) gives the general solution

$$y = C(x-1)^2 \tag{1.23}$$

which certainly gives rise to a unique solution $y = y(x)$ of the initial value problem (1.20) provided $x_0 \neq 1$ for the intervals indicated in (1.22) (because in such case we can solve for a unique C). Indeed, in such case, such a solution persists for all real x's, not just for those avoiding $x = 1$ and containing x_0!! (But if the solution "crosses" $x = 1$, uniqueness no longer holds—consider the following discussion and, so, the possibility of smoothly gluing together piecewise versions of (1.23) for different choices of C.) On the other hand, if $x_0 = 1$, then (1.23) says that if there is a solution $y = y(x)$ to the initial value problem, then $y_0 = y(1) = 0$. But in such case there are many solutions—one for each choice of C! Finally if $x_0 = 1$ yet we choose in initial value $y_0 = y(1) \neq 0$, then (1.23) dictates that the initial value problem has no solutions.

5. After a time $t = 0$, a solution of constant concentration of 1 gram solute per liter solvent enters a (perfect) stirring tank at a constant rate of 3 liters per minute. The well-stirred mixture exits the tank at a constant rate of 2 liters per minute. Suppose the solute takes no volume in solution. If the tank contains 10 liters of fluid at a time $t = 0$, write down a (self-contained) differential equation for the time evolution of the grams of solute $Q(t)$ accumulated in the tank at time t, one that is valid for as long as the tank is not overflowing. Then, assuming there are 10 grams of solute in the tank at $t = 0$, give an expression for the grams $Q(t)$ of solute accumulated in the tank at time t (by solving the relevant IVP).

15 points

<u>Solution</u>

By stoichiometric / "unit-canceling" / "chain-rule" reasoning, one has

$$\begin{aligned}\frac{dQ}{dt} &= \left(\frac{dQ}{dt}\right)_{total} = \left(\frac{dQ}{dt}\right)_{in} - \left(\frac{dQ}{dt}\right)_{out} = \left(\frac{dQ}{dV}\right)_{in}\left(\frac{dV}{dt}\right)_{in} - \left(\frac{dQ}{dV}\right)_{out}\left(\frac{dV}{dt}\right)_{out} \\ &= C_{in}R_{in} - C_{out}R_{out} \qquad (1.24) \\ &= R_{in}C_{in} - R_{out}\frac{Q}{V} = 3\cdot 1 - 2\frac{Q}{V} = 3 - 2\frac{Q}{V},\end{aligned}$$

where the fluid tank volume $V = V(t)$ is specified by

$$\frac{dV}{dt} = R_{in} - R_{out} = 3 - 2 = 1,\ V(0) = V_0 = 10, \qquad (1.25)$$

the latter (trivial) initial value problem having the unique solution

$$V = V_0 + t(R_{in} - R_{out}) = 10 + t \cdot 1 = 10 + t\ . \tag{1.26}$$

Thus the required, "self-contained" differential equation is

$$\begin{aligned} \frac{dQ}{dt} &= R_{in}C_{in} - R_{out}\frac{Q}{V_0 + t(R_{in} - R_{out})} \\ &= 3 - \frac{2}{10+t}Q \\ &\Leftrightarrow \\ \frac{dQ}{dt} &+ \frac{2}{10+t}Q = 3. \end{aligned} \tag{1.27}$$

We solve the initial value problem which is ODE (1.27) together with initial data

$$Q(0) = 10\ . \tag{1.28}$$

An integrating factor for the ODE (1.27) is, according to the standard theory,

$$\begin{aligned} \mu &= \exp\int \frac{2}{10+t}dt = \exp\left(2\log\left(10+t\right)\right) \\ &= \left(10+t\right)^2. \end{aligned} \tag{1.29}$$

Use of the integrating factor (1.29) in (1.27) gives

$$\frac{d\left(\left(10+t\right)^2 Q\right)}{dt} = \left(10+t\right)^2\frac{dQ}{dt} + \left(10+t\right)^2\frac{2}{10+t}Q = 3\left(10+t\right)^2 = \frac{d\left(10+t\right)^3}{dt}\ . \tag{1.30}$$

Integration of (1.30) using relevant limits (and dummy variables) gives

$$\begin{aligned} &\left(10+t\right)^2 Q(t) - 10^3 = \left(10+t\right)^2 Q(t) - 10^2 \cdot 10 = \\ &\left(10+t\right)^2 Q(t) - \left(10+0\right)^2 Q(0) = \left(10+s\right)^2 Q(s)\Big|_0^t = \\ &\int_0^t d\left(\left(10+s\right)^2 Q(s)\right) = \int_0^t d\left(10+s\right)^3 \\ &= \left(10+s\right)^3\Big|_0^t = \left(10+t\right)^3 - \left(10+0\right)^3 = \left(10+t\right)^3 - 10^3, \end{aligned} \tag{1.31}$$

or, equivalently,

$$Q(t) = 10 + t\ . \tag{1.32}$$

6. Show that the following differential equation (1.33) is not exact, but can be rendered exact by multiplication by an integrating factor that is only a function of either x or of y. Find an expression of the general solution of the differential equation.

$$\left(3x^2y+y^2\right)dx+\left(2x^3+3xy\right)dy=0\,. \tag{1.33}$$

15 points

<u>Solution</u>

(1.33) is not exact since

$$\psi_{xy}=\left(\psi_x\right)_y:=\left(3x^2y+y^2\right)_y=3x^2+2y\neq 6x^2+3y=\left(2x^3+3xy\right)_x=:\left(\psi_y\right)_x=\psi_{yx}\,. \tag{1.34}$$

With an integrating factor μ, (1.33), which can be written as

$$\left(3x^2+y\right)ydx+x\left(2x^2+3y\right)dy\,, \tag{1.35}$$

can, by theorem, be made exact. We note from (1.34) that

$$\left(2x^3+3xy\right)_x-\left(3x^2y+y^2\right)_y=6x^2+3y-\left(3x^2+2y\right)=3x^2+y\,, \tag{1.36}$$

which is a factor in the first term in (1.35), the remaining factor being only a function of y. Thus we suspect the existence of an integrating factor only depending on y. At any rate, with the use of such a factor, ODE (1.33) becomes

$$\mu(y)\left(3x^2y+y^2\right)dx+\mu(y)\left(2x^3+3xy\right)dy=0\,, \tag{1.37}$$

and exactness demands that

$$\begin{aligned}0&=\left(\mu(y)\left(3x^2y+y^2\right)\right)_y-\left(\mu(y)\left(2x^3+3xy\right)\right)_x=\\&\left(3x^2y+y^2\right)\mu'(y)+\left(3x^2+2y\right)\mu(y)-\left(6x^2+3y\right)\mu(y)\\&=\left(3x^2+y\right)y\mu'(y)-\left(3x^2+y\right)\mu(y)\\&=\left(3x^2+y\right)\left(y\mu'(y)-\mu(y)\right)\\&\Leftarrow\end{aligned} \tag{1.38}$$

$$y\mu'(y)=\mu(y).$$

Thus, as suspected, there is an integrating factor depending only on y. A solution of (1.38) is evidently given by

$$\mu(y) = y\,, \tag{1.39}$$

in which case (1.37) becomes

$$\left(3x^2y^2 + y^3\right)dx + \left(2x^3y + 3xy^2\right)dy = 0\,. \tag{1.40}$$

Writing

$$\begin{aligned} \psi_x &= 3x^2y^2 + y^3, \\ \psi_y &= 2x^3y + 3xy^2, \end{aligned} \tag{1.41}$$

gives

$$\begin{aligned} \psi &= x^3y^2 + xy^3 + f(y), \\ \psi &= x^3y^2 + xy^3 + g(x), \end{aligned} \tag{1.42}$$

which can be reconciled by $f(y) = g(x) = 0$. Thus (1.40) can be written as

$$d\left(x^3y^2 + xy^3\right) = 0\,, \tag{1.43}$$

an expression of the general solution to which is clearly

$$x^3y^2 + xy^3 = C\,. \tag{1.44}$$

7. Find a linear, first order, ordinary differential equation with the property that *every* solution $y(t)$ of it approaches the function $f(t) = 1 + t^2$ arbitrarily closely as $t \to +\infty$. Note that the (too) simple equation

$$y'(t) = f'(t) = \left(1 + t^2\right)' = 2t \tag{1.45}$$

does not work since the general solution of (1.45) is

$$y(t) = \int 2t\,dt = C + t^2\,, \tag{1.46}$$

giving

$$\lim_{t\to+\infty}\left(y(t)-f(t)\right)=\lim_{t\to+\infty}\left(C+t^2-\left(1+t^2\right)\right)=C-1, \tag{1.47}$$

which is not zero for every choice of C.

10 points

<u>Solution</u>

Introduce a general solution of the form

$$y(t)=1+t^2+Ce^{-at} \tag{1.48}$$

with $a>0$ to get

$$\lim_{t\to+\infty}\left(y(t)-f(t)\right)=\lim_{t\to+\infty}\left(1+t^2+Ce^{-at}-\left(1+t^2\right)\right)=\lim_{t\to+\infty}Ce^{-at}=0 \tag{1.49}$$

for every choice of C, as demanded by the problem. Thus, to get a first order ODE with the required property we differentiate (1.48) with respect to t and eliminate C between (1.48) and this new result. Differentiating (1.48) gives

$$y'(t)=2t-aCe^{-at}, \tag{1.50}$$

and elimination of C between (1.48) and (1.50) gives the required first order ODE, namely

$$\begin{aligned} y'(t)&=2t-aCe^{-at}=2t-a\left(y(t)-(1+t^2)\right)\\ &=-ay(t)+a+2t+at^2. \end{aligned} \tag{1.51}$$

8. Solve the following initial value problem. State the properties of the solution as $t\to\infty$ for all choices of the initial value y_0.

$$y'(t)=-y(t)+\left(1+t\right)^2,\ y(0)=y_0. \tag{1.52}$$

15 points

<u>Solution</u>

The ODE in (1.52) can be written as

$$y'(t)+1y(t)=\left(1+t\right)^2. \tag{1.53}$$

(1.53) suggests the integrating factor

$$\mu(t) = \exp\int 1dt = e^t \,, \tag{1.54}$$

which renders the ODE (1.53) as

$$\begin{aligned}
&\frac{d}{dt}e^t y(t) = \\
&e^t y'(t) + e^t y(t) = (1+t)^2 e^t = \frac{d}{dt}(1+t)^2 e^t - e^t \frac{d}{dt}(1+t)^2 \\
&= \frac{d}{dt}(1+t)^2 e^t - 2(1+t)e^t = \frac{d}{dt}\left((1+t)^2 e^t - 2(1+t)e^t\right) + e^t \frac{d}{dt}2(1+t) \\
&= \frac{d}{dt}\left((1+t)^2 e^t - 2(1+t)e^t\right) + 2e^t \\
&= \frac{d}{dt}\left((1+t)^2 e^t - 2(1+t)e^t + 2e^t\right) = \frac{d}{dt}\left((1+t)^2 - 2t\right)e^t \\
&= \frac{d}{dt}\left(1+t^2\right)e^t,
\end{aligned} \tag{1.55}$$

which, with the initial data specified in (1.52), integrates to

$$\begin{aligned}
&e^t y(t) - y_0 = e^t y(t) - 1\cdot y_0 = e^t y(t) - e^0 y(0) = e^s y(s)\Big|_0^t = \\
&\int_0^t de^s y(s) = \int_0^t d\left(1+s^2\right)e^s \\
&= \left(1+s^2\right)e^s\Big|_0^t = \left(1+t^2\right)e^t - \left(1+0^2\right)e^0 = \left(1+t^2\right)e^t - 1,
\end{aligned} \tag{1.56}$$

or, equivalently,

$$y(t) = 1 + t^2 + \left(y_0 - 1\right)e^{-t}. \tag{1.57}$$

Note then the differential equation in (1.52) gives a solution to problem 7, which indicates the desired properties.

9. Carefully state the (nonlinear) existence and uniqueness theorem for a single first order ODE.

15 points

<u>Solution</u>

Consider the initial value problem

$$y'(t) = f(t, y), y(t_0) = y_0. \qquad (1.58)$$

Suppose $f(t, y)$ and $f_y(t, y)$ are both continuous in an open rectangle $(t_{-1}, t_{+1}) \times (y_{-1}, y_{+1})$ containing the point (t_0, y_0). Then there exists an $h > 0$ such that (1.58) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $(t_0 - h, t_0 + h)$ (potentially much smaller than the interval (t_{-1}, t_{+1})).

10. Carefully state the linear existence and uniqueness theorem for a single first order ODE. Explain in general terms how it is proven.

15 points

Solution

Consider the initial value problem

$$y'(t) = p(t)y + q(t), y(t_0) = y_0. \qquad (1.59)$$

Suppose $p(t)$ and $q(t)$ are both continuous in an open interval (t_{-1}, t_{+1}) containing the point t_0. Then (1.59) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval (t_{-1}, t_{+1}).

The theorem is proven by explicitly integrating (1.59), using the various theorems of calculus, including that the integral of a continuous function exists, etc.