KEY

Math 334 Midterm II
Fall 2007
section 004
Instructor: Scott Glasgow

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1. Determine a lower bound for the radius of convergence of the power series representation of the general solution of the following differential equation about the point $x_0 = -2$:

$$x\left(\left(x-1\right)^2+4\right)y''+xy'+x^2y=0. \tag{1.1}$$

5 points

Solution

Equation (1.1) can be reduced to the equation

$$\left(\left(x-1\right)^2+4\right)y''+y'+xy=0 \tag{1.2}$$

which has singularities at the zeroes of the leading coefficient, which are $x = 1 \pm 2i$. In the complex plain the distance of the singularities to the expansion point $x_0 = -2 = -2+0i$ is $\sqrt{\left(-2-1\right)^2+\left(0\mp 2\right)^2} = \sqrt{13}$. Thus, even along the real axis, we cannot guarantee a radius of convergence beyond $\sqrt{13}$ without more information.

2. Find a (particular) solution of the following differential equation by the method of undetermined coefficients:

$$y''-2y'+y=3t+5. \tag{1.3}$$

7 points

Solution

The usual explanation of the ansatz for developing a particular solution to a linear constant coefficient differential equation (with a RHS that is in the null space of a linear constant coefficient differential operator) is to first find a basis for the span of the RHS together with all its derivatives. Then, barring the phenomena of *resonance* (which is that one or more elements of such a basis are in the null space of the specific differential operator in question), one then forms a general element of the space spanned by the basis, which general element constitutes the "method of undetermined coefficients ansatz" for a solution of the equation in question. For the problem at hand, and since the RHS of (1.3) is spanned by the set of functions $\{t,1\}$, whose first derivatives are both spanned by the set of functions $\{1\}\subset\{t,1\}$, so that all subsequent/higher derivatives are spanned by $\{t,1\}$, the relevant basis for a particular solution of (1.3) is, barring resonance, $\{t,1\}$. One soon finds that neither of these is a solution of the homogeneous version of(1.3), so that there is no resonance, and the ansatz for a solution of (1.3) is, together with relevant derivatives,

$$\begin{aligned} y &= At+B \\ y' &= 0t+A \\ y'' &= 0t+0. \end{aligned} \tag{1.4}$$

Weighted appropriate for the equation (1.3), the equations (1.4) are

$$\begin{aligned} y &= At+B \\ -2y' &= 0t-2A \\ +1y'' &= 0t+0 \end{aligned} \tag{1.5}$$

which sum to

$$y''-2y'+y=-2A+At+B=At+\left(B-2A\right)\cdot 1=3t+5\cdot 1\,. \tag{1.6}$$

The latter equation holds uniformly in t if and only if $A=3$ and $B-2A=5 \Rightarrow B=5+2A=5+2\cdot 3=11$. Thus the solution sought is

$$y=At+B=3t+11. \tag{1.7}$$

3. A 125 kilogram mass stretches a spring $1/5$ meter. If the mass is set in motion from the equilibrium position at 15 meters per second upward, and there is no damping, determine the displacement $u(t)$ of the mass above the equilibrium position at any subsequent time t. Use that the acceleration of gravity is 49/5 meters per second per second.

9 points

Solution

The relevant version of Newton's second law is

$$0 = mu'' + ku = 125\text{kg}u'' + ku \,. \tag{1.8}$$

Here we may determine the spring constant k from

$$k = F/s = ma/s = 125\text{kg}49/5meter/\sec^2/(1/5meter) = 125\cdot 7^2\,\text{kg}/\sec^2 \,, \tag{1.9}$$

so that (1.8) is

$$0 = 125\text{kg}u'' + 125\cdot 7^2\,\text{kg}/\sec^2 u \Leftrightarrow 0 = u'' + 7^2/\sec^2 u \,. \tag{1.10}$$

Rendering (1.10) unit-less, by measuring time in seconds, this is

$$0 = u'' + 7^2 u \,, \tag{1.11}$$

the general solution to which being

$$u = A\cos(7t) + B\sin(7t) \,. \tag{1.12}$$

The initial data specifies that

$$\begin{gathered} u(0) = 0 = A, u'(0) = 15 = 7B \\ \Leftrightarrow \\ A = 0, B = 15/7, \end{gathered} \tag{1.13}$$

so that the required solution to the initial value problem is

$$u = A\cos(7t) + B\sin(7t) = \left(15/7\right)\sin(7t). \tag{1.14}$$

4. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' - 3xy' + 13y = 0 \,. \tag{1.15}$$

11 points

Solution

The differential equation (1.15) defines a linear differential operator L_x, in terms of which (1.15) can be written $L_x[y]=0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \left(r(r-1)-3r+13\right)x^r = \left(r^2-4r+13\right)x^r = \left((r-2)^2+3^2\right)x^r, \tag{1.16}$$

so that complex solutions of (1.15) are clearly then
$y_{2+3i} = x^{2+3i} = x^2 e^{3i\ln x} = x^2\left(\cos(3\ln x)+i\sin(3\ln x)\right)$ and
$y_{2-3i} = x^{2-3i} = x^2 e^{-3i\ln x} = x^2\left(\cos(3\ln x)-i\sin(3\ln x)\right)$. Independent complex linear combinations of these linearly independent complex valued solutions gives the following real-representation of the general solution:

$$y = x^2\left(A\cos(3\ln x)+B\sin(3\ln x)\right). \tag{1.17}$$

5. Solve the following initial value problem:

$$y''-4y'+29y=0;\quad y(0)=1,\ y'(0)=3. \tag{1.18}$$

12 points

Solution

This linear homogeneous differential equation is associated with the following characteristic (polynomial) and characteristic exponents r:

$$\begin{aligned} &0 = r^2-4r+29 = r^2-4r+4+25 = (r-2)^2-(5i)^2 \\ &\Leftrightarrow \\ &r = 2\pm 5i. \end{aligned} \tag{1.19}$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$\begin{aligned} &y = e^{2t}\left(C_1\cos 5t + C_2\sin 5t\right) \\ &\quad\text{and} \\ &y' = e^{2t}\left((2C_1+5C_2)\cos 5t + (-5C_1+2C_2)\sin 5t\right). \end{aligned} \tag{1.20}$$

Inserting $t=0$ into (1.20), and using the initial data given in (1.18), one obtains

$$y(0) = C_1 = 1$$
$$\text{and} \tag{1.21}$$
$$y'(0) = 2C_1 + 5C_2 = 3,$$

the solution to which being $C_1 = 1$ and $C_2 = 1/5$. Thus the solution to the initial value problem is then

$$y = e^{2t}\left(\cos 5t + \frac{1}{5}\sin 5t\right). \tag{1.22}$$

6. Given that $y_1 = t^2$ is a solution of

$$t^2 y'' - 3ty' + 4y = 0 \tag{1.23}$$

for $t > 0$, find a second, linearly independent solution y_2 of (1.23) for $t > 0$ ***by making the D'Alembert ansatz*** $y_2 = vy_1 = vt^2$.

14 points

<u>Solution</u>

Using D'Alembert's ansatz in (1.23) gives

$$\begin{aligned} 0 &= t^2 y_2'' - 3ty_2' + 4y_2 = t^2\left(vt^2\right)'' - 3t\left(vt^2\right)' + 4vt^2 = t^2\left(v''t^2 + 4v't + 2v\right) - 3t\left(v't^2 + 2tv\right) + 4vt^2 \\ &= t^4 v'' + \left(4t^3 - 3t^3\right)v' + \left(2t^2 - 6t^2 + 4t^2\right)v \\ &= t^4 v'' + t^3 v' = t^4 u' + t^3 u = t^3\left(tu' + u\right), \end{aligned} \tag{1.24}$$

and where we defined $u = v'$. By any one of a number of standard techniques, one finds that the first order homogeneous equation (1.24) has a nontrivial solution $u = t^{-1} = v' \Leftarrow v = \ln t$. Thus a second, linearly independent solution is

$$y_2 = vt^2 = t^2 \ln t. \tag{1.25}$$

7. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' - 5xy' + 9y = 0\,. \tag{1.26}$$

16 points

Solution

The differential equation (1.26) defines a linear differential operator L_x, in terms of which (1.26) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \left(r(r-1) - 5r + 9\right)x^r = \left(r^2 - 6r + 9\right)x^r = \left(r-3\right)^2 x^r\,, \tag{1.27}$$

so that a solution of (1.26) is clearly then $y_3 = x^3$. To find the general solution to this second order differential equation we need to find a second, linearly independent solution. Since the ansatz $y_r = x^r$ only produces solutions dependent upon $y_3 = x^3$, we must use another ansatz. Fortunately the structure of (1.27), together with the fact that the differential operators $\frac{d}{dr}$ and L_x commute, suggest such an alternative ansatz: applying $\frac{d}{dr}$ to both sides of (1.27), and using the indicated commutivity, one obtains

$$L_x[\frac{d}{dr}y_r] = \left(r-3\right)^2 x^r \ln x + 2\left(r-3\right)^1 x^r\,, \tag{1.28}$$

so that $\frac{d}{dr}y_r\Big|_{r=3} = x^r \ln x\Big|_{r=3} = x^3 \ln x$ is clearly a second, linearly independent solution of (1.26). Thus the general solution to this linear homogeneous equation is

$$y = \left(A + B\ln x\right)x^3\,. \tag{1.29}$$

8. Find the first *two* ***nonzero*** terms (if there are that many) in the series representation of *one* of the 2 linearly independent solutions of the equation

$$\left(x^2 + 2x^3\right)y'' - \left(2x + 6x^2\right)y' + \left(2 + 6x\right)y = 0\,. \tag{1.30}$$

   about the point $x_0 = 0$.

18 points

Solution

The point $x_0 = 0$ is a singular point, so that the required series solution is not quite a Taylor series: insert $y = \sum_n a_n x^{n+r}$ (with the assumption that $a_n = 0$ for $n < 0$, and that the sum is over the integers, and that $a_0 \neq 0$)in (1.30) to obtain

$$
\begin{aligned}
0 &= \sum_n \left(x^2 + 2x^3\right)\left(n+r\right)(n+r-1)a_n x^{n+r-2} - \left(2x + 6x^2\right)\left(n+r\right)a_n x^{n+r-1} + \left(2+6x\right)a_n x^{n+r} \\
&= \sum_n \left\{ \begin{array}{l} \left(n+r\right)(n+r-1)a_n x^{n+r} + 2\left(n+r\right)(n+r-1)a_n x^{n+r+1} \\ -2\left(n+r\right)a_n x^{n+r} - 6\left(n+r\right)a_n x^{n+r+1} \\ +2a_n x^{n+r} + 6a_n x^{n+r+1} \end{array} \right\} \\
&= \sum_n \left\{ \begin{array}{l} \left(n+r\right)(n+r-1)a_n x^{n+r} + 2\left(n+r-1\right)(n+r-2)a_{n-1} x^{n+r} \\ -2\left(n+r\right)a_n x^{n+r} - 6\left(n+r-1\right)a_{n-1} x^{n+r} \\ +2a_n x^{n+r} + 6a_{n-1} x^{n+r} \end{array} \right\} \\
&= \sum_n \left\{ \left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1} \right\} x^{n+r} \\
&= \sum_{n=0}^{\infty} \left\{ \left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1} \right\} x^{n+r} \\
&= \left[r(r-1) - 2r + 2\right]a_0 x^r + \sum_{n=1}^{\infty} \left\{ \begin{array}{l} \left[\left(n+r\right)(n+r-1) - 2\left(n+r\right) + 2\right]a_n \\ + \left[2\left(n+r-1\right)(n+r-2) - 6\left(n+r-1\right) + 6\right]a_{n-1} \end{array} \right\} x^{n+r}.
\end{aligned}
\tag{1.31}
$$

Evidently we require

$$0 = r(r-1) - 2r + 2 = r^2 - 3r + 2 = \left(r-1\right)\left(r-2\right). \tag{1.32}$$

Since these roots differ by an integer, according to the general theory we should only use the larger of the two roots. However, here a miracle occurs and it turns out that both will work (so that you do not need to be aware of the general theory): even for the “wrong” choice $r = 1$ we get that (1.31) becomes

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty}\left\{\left[\left(n+1\right)n-2\left(n+1\right)+2\right]a_n+2\left[n(n-1)-3n+3\right]a_{n-1}\right\}x^{n+1} \\ &= \sum_{n=1}^{\infty}\left[n(n-1)a_n+2(n-1)(n-3)a_{n-1}\right]x^{n+1} \\ &= \sum_{n=1}^{\infty}(n-1)\left[na_n+2(n-3)a_{n-1}\right]x^{n+1} \\ &= 0+\sum_{n=2}^{\infty}(n-1)\left[na_n+2(n-3)a_{n-1}\right]x^{n+1} \end{aligned} \tag{1.33}$$

$$\Leftrightarrow$$

$$na_n+2(n-3)a_{n-1}=0,\ n=2,3,\ldots$$

$$\Leftrightarrow$$

$$a_n=-\frac{2(n-3)}{n}a_{n-1},\ n=2,3,\ldots.$$

Thus we have

$$\begin{aligned} a_2 &= -\frac{2(2-3)}{2}a_{2-1}=a_1, \\ a_3 &= -\frac{2(3-3)}{3}a_{3-1}=0, \\ a_4 &= -\frac{2(4-3)}{4}a_{4-1}=-\frac{1}{2}\cdot 0=0, \\ &\vdots \\ a_{n\ge 3} &= 0. \end{aligned} \tag{1.34}$$

Thus the "infinite" series terminates, and we get

$$y=\sum_n a_n x^{n+r}=a_0x^1+a_1x^2+a_2x^3=a_0x+a_1x^2+a_1x^3=a_0x+a_1\left(x^2+x^3\right), \tag{1.35}$$

which gives the general solution of (1.30).

9. Find the general solution of the following linear but non-homogeneous differential equation by the method of variation of parameters. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the "D'Alembert-like" ansatz that leads to that formula.

$$y''-3y'+2y=e^{3t} \tag{1.36}$$

22 points

Solution

The characteristic equation of the homogeneous version of the constant coefficient differential equation (1.36) is

$$0 = r^2 - 3r + 2 = (r-1)(r-2) \tag{1.37}$$

so that the general solution of the corresponding homogeneous equation is

$$y = Ae^t + Be^{2t}\,, \tag{1.38}$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (1.38), we have also an ansatz there for the solution of the non-homogeneous equation (1.36): with such an ansatz one immediately has

$$y' = Ae^t + 2Be^{2t} + \left(e^t A' + e^{2t} B'\right). \tag{1.39}$$

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$e^t A' + e^{2t} B' = 0\,, \tag{1.40}$$

so that then (1.39) becomes

$$y' = Ae^t + 2Be^{2t}\,. \tag{1.41}$$

Differentiating (1.41) gives

$$y'' = Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \tag{1.42}$$

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$\begin{aligned} 2y &= 2Ae^t + 2Be^{2t} \\ -3y' &= -3Ae^t - 6Be^{2t} \\ +1y'' &= Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \end{aligned} \tag{1.43}$$

and from which it is clear that the differential equation demands that

$$e^t A' + 2e^{2t} B' = e^{3t} . \tag{1.44}$$

Combining this with the "consistency ansatz" (1.40) we get

$$\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}, \tag{1.45}$$

which implies that

$$A' = \frac{\begin{vmatrix} 0 & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \frac{-e^{5t}}{e^{3t}} = -e^{2t}, \; B' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & e^{3t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \frac{e^{4t}}{e^{3t}} = e^t . \tag{1.46}$$

Solutions to (1.46) include the pair $A = -\frac{e^{2t}}{2}, \; B = e^t ,$ so that a solution to (1.36) is, according to (1.38),

$$y = Ae^t + Be^{2t} = -\frac{e^{2t}}{2} e^t + e^t e^{2t} = \frac{1}{2} e^{3t} , \tag{1.47}$$

and the general solution to (1.36) is

$$y = c_1 e^t + c_2 e^{2t} + \frac{1}{2} e^{3t} , \tag{1.48}$$

where c_1 and c_2 are (truly) constants now.

10. Solve the initial value problem obtained from combining the differential equation of problem 9 with the initial data $y(0) = 0, \; y'(0) = 0$. In order that errors don't "cascade", I will tell you that $y = Ae^t + Be^{2t} + \frac{1}{2} e^{3t}$ (A and B truly constant) is the general solution of the differential equation of problem 9. (So now if you just write down this solution to 9 without very convincing work, you will get 0 points on problem 9.) Thus, I am only testing if you understand the correct principles needed to construct the solution to the initial value problem given the general solution to the associated differential equation.

20 points

<u>Solution</u>

From the information given we have

$$\begin{aligned} y(0) &= 0 = A + B + \frac{1}{2} \\ y'(0) &= 0 = A + 2B + \frac{3}{2}, \end{aligned} \tag{1.49}$$

or, equivalently, the following augmented matrix for the column vector (A, B), which, together with row reduction, is

$$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 4 & -3 \end{bmatrix} \square \begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix} \square \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix} \square \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{1.50}$$

From (1.50) one has that

$$A = 1/2, \; B = -1. \tag{1.51}$$

and the solution sought is

$$y = Ae^{t} + Be^{2t} + \frac{1}{2}e^{3t} = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}. \tag{1.52}$$