# KEY

**Math 334 Midterm III**
**Fall 2007**
**section 004**
**Instructor: Scott Glasgow**

**Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering paper. Write your name, course, and section number on the blue book, or on your own pile of papers. Again, do not write this or any other type of information on this exam.**

1. Solve the following initial value problem in terms of the convolution integral:

$$y'' - 25y = g(t);\ y(0) = y'(0) = 0. \tag{1.1}$$

**15 points**

**Solution**

Laplace transformation of (1.1) gives

$$s^2 L[y] - 25L[y] = L[y''] - 25L[y] = L[g]$$
$$\Leftrightarrow$$
$$L[y] = \frac{1}{s^2 - 5^2} L[g] = L\left[\frac{1}{5}\sinh(5\cdot)\right] L[g] = L\left[\frac{1}{5}\sinh(5\cdot) * g\right] \tag{1.2}$$
$$\Leftrightarrow$$
$$y(t) = \left(\frac{1}{5}\sinh(5\cdot) * g\right)(t) = \int_0^t \frac{1}{5}\sinh\left(5(t-\tau)\right) g(\tau)d\tau.$$

**Check**

$$y(t) := \int_0^t \frac{1}{5}\sinh\left(5(t-\tau)\right) g(\tau)d\tau \ \Rightarrow$$
$$y'(t) = \frac{1}{5}\sinh\left(5(t-t)\right) g(t) + \int_0^t \frac{5}{5}\cosh\left(5(t-\tau)\right) g(\tau)d\tau$$
$$= \int_0^t \cosh\left(5(t-\tau)\right) g(\tau)d\tau,$$
$$y''(t) = \cosh\left(5(t-t)\right) g(t) + \int_0^t 5\sinh\left(5(t-\tau)\right) g(\tau)d\tau$$
$$= g(t) + 25\int_0^t \frac{1}{5}\sinh\left(5(t-\tau)\right) g(\tau)d\tau \tag{1.3}$$
$$= g(t) + 25y(t)$$
$$\Leftrightarrow$$
$$y''(t) - 25y(t) = g(t),\ y(0) = \int_0^0 \frac{1}{5}\sinh\left(5(t-\tau)\right) g(\tau)d\tau = 0,$$
$$y'(t) = \int_0^0 \cosh\left(5(t-\tau)\right) g(\tau)d\tau = 0.$$

2. Find the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \mathbf{x}\,. \tag{1.4}$$

**20 points**

**Solution**

The general solution of (1.4) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t} \,, \tag{1.5}$$

provided the $\boldsymbol{\xi}$'s and $\lambda$'s are independent eigenvectors and distinct eigenvalues of the matrix in (1.4):

$$\begin{bmatrix} 3-\lambda & -2 \\ 1 & 0-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} 3-\lambda & -2 \\ 1 & 0-\lambda \end{bmatrix} = (-\lambda)(3-\lambda) - (1)(-2) \tag{1.6}$$
$$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$
$$\Leftrightarrow$$
$$\lambda = 1, 2 =: \lambda_1, \lambda_2 .$$

So

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} 3-1 & -2 \\ 1 & 0-1 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_1 \Leftarrow \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \mathbf{0} &= \begin{bmatrix} 3-2 & -2 \\ 1 & 0-2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_2 \Leftarrow \boldsymbol{\xi}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned} \tag{1.7}$$

Thus, explicitly, (1.5) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}. \tag{1.8}$$

3. Solve the following initial value problem in terms of the convolution integral:

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \mathbf{g}(t), \;\; \mathbf{x}(0) = \mathbf{0} \,. \tag{1.9}$$

Here I expect you to write the solution $\mathbf{x} = \mathbf{x}(t)$ of (1.9) as

$$\mathbf{x}(t)=\int_0^t \begin{bmatrix} M_{11}(t-\tau) & M_{12}(t-\tau) \\ M_{21}(t-\tau) & M_{22}(t-\tau) \end{bmatrix} \mathbf{g}(\tau)d\tau =: \int_0^t M(t-\tau)\mathbf{g}(\tau)d\tau = \left(M*\mathbf{g}\right)(t), \quad (1.10)$$

so that the problem is effectively to determine $M_{11}(t-\tau), M_{12}(t-\tau), M_{21}(t-\tau),$ and $M_{22}(t-\tau)$ in (1.10). (Hint: Use the Laplace transform and the convolution theorem.)

**10 points**

**<u>Solution</u>**

By the Laplace Transform we have from (1.9) that

$$sL[\mathbf{x}]-\mathbf{x}(0)=sL[\mathbf{x}]=\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} L[\mathbf{x}]+L[\mathbf{g}]$$

$$\Leftrightarrow$$

$$\begin{bmatrix} s-3 & 2 \\ -1 & s \end{bmatrix} L[\mathbf{x}]=L[\mathbf{g}]$$

$$\Leftrightarrow$$

$$L[\mathbf{x}]=\begin{bmatrix} s-3 & 2 \\ -1 & s \end{bmatrix}^{-1} L[\mathbf{g}]=\frac{1}{(s-3)(s)-(-1)(2)}\begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}]$$

$$=\frac{1}{s^2-3s+2}\begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}]=\frac{1}{(s-1)(s-2)}\begin{bmatrix} s & -2 \\ 1 & s-3 \end{bmatrix} L[\mathbf{g}]$$

$$=\begin{bmatrix} \frac{s}{(s-1)(s-2)} & \frac{-2}{(s-1)(s-2)} \\ \frac{1}{(s-1)(s-2)} & \frac{s-3}{(s-1)(s-2)} \end{bmatrix} L[\mathbf{g}]$$

$$=\begin{bmatrix} \frac{\frac{1}{1-2}}{s-1}+\frac{\frac{2}{2-1}}{s-2} & \frac{\frac{-2}{1-2}}{s-1}+\frac{\frac{-2}{2-1}}{s-2} \\ \frac{\frac{1}{1-2}}{s-1}+\frac{\frac{1}{2-1}}{s-2} & \frac{\frac{1-3}{1-2}}{s-1}+\frac{\frac{2-3}{2-1}}{s-2} \end{bmatrix} L[\mathbf{g}]$$

$$=\begin{bmatrix} \frac{2}{s-2}-\frac{1}{s-1} & 2\left(\frac{1}{s-1}-\frac{1}{s-2}\right) \\ \frac{1}{s-2}-\frac{1}{s-1} & \frac{2}{s-1}-\frac{1}{s-2} \end{bmatrix} L[\mathbf{g}]=L[M*\mathbf{g}], \quad (1.11)$$

where

$$M = M(t) = \begin{bmatrix} 2e^{2t} - e^{t} & 2\left(e^{t} - e^{2t}\right) \\ e^{2t} - e^{t} & e^{t} - e^{2t} \end{bmatrix}. \tag{1.12}$$

Thus, from (1.11) and(1.12), we have

$$\begin{aligned} \mathbf{x} = \mathbf{x}(t) &= \left(M * \mathbf{g}\right)(t) = \int_0^t M(t-\tau)\mathbf{g}(\tau)d\tau \\ &= \int_0^t \begin{bmatrix} 2e^{2(t-\tau)} - e^{(t-\tau)} & 2\left(e^{(t-\tau)} - e^{2(t-\tau)}\right) \\ e^{2(t-\tau)} - e^{(t-\tau)} & e^{(t-\tau)} - e^{2(t-\tau)} \end{bmatrix} \mathbf{g}(\tau)d\tau. \end{aligned} \tag{1.13}$$

4. Find a real-valued representation of the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \mathbf{x} \tag{1.14}$$

**25 points**

**<u>Solution</u>**

The general solution of (1.14) can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_+ e^{\lambda_+ t} + c_2 \boldsymbol{\xi}_- e^{\lambda_- t}, \tag{1.15}$$

provided the $\boldsymbol{\xi}$'s and $\lambda$'s are independent eigenvectors and distinct eigenvalues of the matrix in (1.14):

$$\begin{bmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{bmatrix} = \left(\lambda - 3\right)\left(\lambda + 1\right) + 8 = \lambda^2 - 2\lambda + 5 = \left(\lambda - 1\right)^2 + 2^2 \tag{1.16}$$

$$\Leftrightarrow$$

$$\lambda = 1 \pm 2i = 1 + 2i, 1 - 2i =: \lambda_+, \lambda_-.$$

So

$$\mathbf{0} = \begin{bmatrix} -1-(1\pm 2i) & -4 \\ 2 & 3-(1\pm 2i) \end{bmatrix} \boldsymbol{\xi}_{\pm} = \begin{bmatrix} -2\mp 2i & -4 \\ 2 & 2\mp 2i \end{bmatrix} \boldsymbol{\xi}_{\pm} = \begin{bmatrix} -1\mp i & -2 \\ 1 & 1\mp i \end{bmatrix} \boldsymbol{\xi}_{\pm}$$

$$= \begin{bmatrix} -1\mp i & -2 \\ 1\cdot(-1\mp i) & (1\mp i)(-1\mp i) \end{bmatrix} \boldsymbol{\xi}_{\pm} = \begin{bmatrix} -1\mp i & -2 \\ -1\mp i & -1+i^2 \pm i \mp i \end{bmatrix} \boldsymbol{\xi}_{\pm} = \begin{bmatrix} -1\mp i & -2 \\ -1\mp i & -2 \end{bmatrix} \boldsymbol{\xi}_{\pm} \quad (1.17)$$

$$= \begin{bmatrix} -1\mp i & -2 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_{\pm} \Leftarrow \boldsymbol{\xi}_{\pm} = \begin{bmatrix} 2 \\ -1\mp i \end{bmatrix}.$$

Thus, explicitly, (1.15) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 2 \\ -1+i \end{bmatrix} e^{(1-2i)t}. \quad (1.18)$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) := \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{t}\left(\cos 2t + i\sin 2t\right) = \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix} e^{t} + i \begin{bmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{bmatrix} e^{t}, \quad (1.19)$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2\sin 2t \\ -\cos 2t - \sin 2t \end{bmatrix} e^{t}. \quad (1.20)$$

**Check**

$$\begin{aligned}
\mathbf{x}_+(t) &:= \begin{bmatrix} 2\cos 2t \\ -\cos 2t+\sin 2t \end{bmatrix} e^t \Rightarrow \mathbf{x}_+{}' - \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \mathbf{x}_+ \\
&= \begin{bmatrix} 2\cos 2t \\ -\cos 2t+\sin 2t \end{bmatrix} e^t + \begin{bmatrix} -4\sin 2t \\ 2\sin 2t+2\cos 2t \end{bmatrix} e^t - \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2\cos 2t \\ -\cos 2t+\sin 2t \end{bmatrix} e^t \\
&= \begin{bmatrix} 2\cos 2t-4\sin 2t \\ \cos 2t+3\sin 2t \end{bmatrix} e^t - \begin{bmatrix} -2\cos 2t-4\left(-\cos 2t+\sin 2t\right) \\ 2\cdot 2\cos 2t+3(-\cos 2t+\sin 2t) \end{bmatrix} e^t \\
&= \begin{bmatrix} 2\cos 2t-4\sin 2t \\ \cos 2t+3\sin 2t \end{bmatrix} e^t - \begin{bmatrix} 2\cos 2t-4\sin 2t \\ \cos 2t+3\sin 2t \end{bmatrix} e^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\mathbf{x}_-(t) &:= \begin{bmatrix} 2\sin 2t \\ -\cos 2t-\sin 2t \end{bmatrix} e^t \Rightarrow \mathbf{x}_-{}' - \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \mathbf{x}_- \\
&= \begin{bmatrix} 2\sin 2t \\ -\cos 2t-\sin 2t \end{bmatrix} e^t + \begin{bmatrix} 4\cos 2t \\ -2\cos 2t+2\sin 2t \end{bmatrix} e^t - \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2\sin 2t \\ -\cos 2t-\sin 2t \end{bmatrix} e^t \\
&= \begin{bmatrix} 4\cos 2t+2\sin 2t \\ -3\cos 2t+\sin 2t \end{bmatrix} e^t - \begin{bmatrix} -2\sin 2t-4\left(-\cos 2t-\sin 2t\right) \\ 2\cdot 2\sin 2t+3(-\cos 2t-\sin 2t) \end{bmatrix} e^t \\
&= \begin{bmatrix} 4\cos 2t+2\sin 2t \\ -3\cos 2t+\sin 2t \end{bmatrix} e^t - \begin{bmatrix} 4\cos 2t+2\sin 2t \\ -3\cos 2t+\sin 2t \end{bmatrix} e^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\det\left[\mathbf{x}_+(t)\mathbf{x}_-(t)\right] &= \det\left[\begin{bmatrix} 2\cos 2t \\ -\cos 2t+\sin 2t \end{bmatrix} e^t \begin{bmatrix} 2\sin 2t \\ -\cos 2t-\sin 2t \end{bmatrix} e^t\right] \\
&= e^{2t}\det\left[\begin{bmatrix} 2\cos 2t \\ -\cos 2t+\sin 2t \end{bmatrix} \begin{bmatrix} 2\sin 2t \\ -\cos 2t-\sin 2t \end{bmatrix}\right] \\
&= e^{2t}\left(\left(2\cos 2t\right)\left(-\cos 2t-\sin 2t\right)-\left(2\sin 2t\right)\left(-\cos 2t+\sin 2t\right)\right) \\
&= 2e^{2t}\left(-\cos^2 2t-\sin^2 2t\right) = -2e^{2t} \neq 0.
\end{aligned}$$

(1.21)

5. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ to the previous problem that has the property that $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**10 points**

**Solution**

A fundamental matrix of solutions $\Psi = \Psi(t)$, one not necessarily having the desired property, can be found from the above general solution (1.20) as in

$$\Psi(t) = e^t \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t+\sin 2t & -\cos 2t-\sin 2t \end{bmatrix}. \tag{1.22}$$

The desired fundamental matrix $\Phi = \Phi(t)$ can be obtained from $\Psi = \Psi(t)$ via

$$\Phi = \Phi(t) = \Psi(t)\Psi^{-1}(0) = e^t \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t + \sin 2t & -\cos 2t - \sin 2t \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}^{-1}$$
$$= e^t \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t + \sin 2t & -\cos 2t - \sin 2t \end{bmatrix} \frac{1}{2\cdot(-1)-(-1)\cdot 0} \begin{bmatrix} -1 & -0 \\ 1 & 2 \end{bmatrix}$$
$$= e^t \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t + \sin 2t & -\cos 2t - \sin 2t \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} = \frac{e^t}{2} \begin{bmatrix} 2\cos 2t - 2\sin 2t & -4\sin 2t \\ 2\sin 2t & 2\cos 2t + 2\sin 2t \end{bmatrix}$$
$$= e^t \begin{bmatrix} \cos 2t - \sin 2t & -2\sin 2t \\ \sin 2t & \cos 2t + \sin 2t \end{bmatrix}. \tag{1.23}$$

6. Solve the initial value problem given by the system of problem 4 and the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{1.24}$$

(Hint: rather than "reinventing the wheel", just use the fundamental matrix of problem 5.)

**7 points**

**<u>Solution</u>**

Using the fundamental matrix of problem 5 we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = e^t \begin{bmatrix} \cos 2t - \sin 2t & -2\sin 2t \\ \sin 2t & \cos 2t + \sin 2t \end{bmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = e^t \begin{bmatrix} 2(\cos 2t - \sin 2t) - 1(-2\sin 2t) \\ 2\sin 2t - 1(\cos 2t + \sin 2t) \end{bmatrix}$$
$$= e^t \begin{bmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}. \tag{1.25}$$

7. Calculate

$$e^{\begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}\pi}. \tag{1.26}$$

(Hint: use the result of problem 5).

**5 points**

**<u>Solution</u>**

We have, from problem 5 and the general theory,

$$e^{\begin{bmatrix}-1 & -4\\ 2 & 3\end{bmatrix}\pi}=\Phi(\pi)=e^{\pi}\begin{bmatrix}\cos 2\pi-\sin 2\pi & -2\sin 2\pi\\ \sin 2\pi & \cos 2\pi+\sin 2\pi\end{bmatrix}=e^{\pi}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}e^{\pi} & 0\\ 0 & e^{\pi}\end{bmatrix}. \quad (1.27)$$

8. Find a representation of the general solution of the system

$$\mathbf{x}'=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\mathbf{x}. \quad (1.28)$$

**20 points**

**<u>Solution</u>**

The matrix in (1.28) has a repeated eigenvalue with only one eigenvector. Hence the general solution is of the form

$$\mathbf{x}=\mathbf{x}(t)=c_1\boldsymbol{\xi}e^{\lambda t}+c_2\left(\boldsymbol{\xi}t+\boldsymbol{\eta}\right)e^{\lambda t} \quad (1.29)$$

where $\lambda$ is the sole eigenvalue, $\boldsymbol{\xi}$ is its eigenvector, and $\boldsymbol{\eta}$ is an associated pseudo eigenvector:

$$\begin{bmatrix}0-\lambda & 1\\ -1 & 2-\lambda\end{bmatrix}\boldsymbol{\xi}=\mathbf{0}\Leftrightarrow\boldsymbol{\xi}=\mathbf{0}$$

unless

$$0=\det\begin{bmatrix}-\lambda & 1\\ -1 & 2-\lambda\end{bmatrix}=\lambda^2-2\lambda+1=\left(\lambda-1\right)^2 \quad (1.30)$$

$$\Leftrightarrow$$

$$\lambda=1,1.$$

So

$$\mathbf{0}=\begin{bmatrix}0-1 & 1\\ -1 & 2-1\end{bmatrix}\xi=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\xi\Leftarrow\xi=\begin{bmatrix}1\\1\end{bmatrix},$$

and (1.31)

$$\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\boldsymbol{\eta}=\xi=\begin{bmatrix}1\\1\end{bmatrix}\Leftrightarrow\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{\eta}=\begin{bmatrix}1\\0\end{bmatrix}\Leftrightarrow\boldsymbol{\eta}=\begin{bmatrix}\eta_1\\ 1+\eta_1\end{bmatrix}.$$

Thus, explicitly, (1.29) is

$$\mathbf{x}=\mathbf{x}(t)=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{1t}+c_2\left(\begin{bmatrix}1\\1\end{bmatrix}t+\begin{bmatrix}\eta_1\\ 1+\eta_1\end{bmatrix}\right)e^{1t}. \tag{1.32}$$

9. Find the fundamental matrix of solutions $\Phi=\Phi(t)$ for the system of problem 8 that satisfies $\Phi(0)=I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

**15 points**

**<u>Solution</u>**

From (1.32) we have a fundamental matrix of solutions

$$\Psi(t)=e^t\begin{bmatrix}1 & t+\eta_1\\ 1 & t+1+\eta_1\end{bmatrix}, \tag{1.33}$$

whence the one desired is

$$\begin{aligned}\Phi(t)=\Psi(t)\Psi^{-1}(0)&=e^t\begin{bmatrix}1 & t+\eta_1\\ 1 & t+1+\eta_1\end{bmatrix}\begin{bmatrix}1 & \eta_1\\ 1 & 1+\eta_1\end{bmatrix}^{-1}\\ &=e^t\begin{bmatrix}1 & t+\eta_1\\ 1 & t+1+\eta_1\end{bmatrix}\frac{1}{1+\eta_1-\eta_1}\begin{bmatrix}1+\eta_1 & -\eta_1\\ -1 & 1\end{bmatrix}=e^t\begin{bmatrix}1 & t+\eta_1\\ 1 & t+1+\eta_1\end{bmatrix}\begin{bmatrix}1+\eta_1 & -\eta_1\\ -1 & 1\end{bmatrix} \\ &=e^t\begin{bmatrix}1+\eta_1-1(t+\eta_1) & -\eta_1+t+\eta_1\\ 1+\eta_1-1(t+1+\eta_1) & -\eta_1+t+1+\eta_1\end{bmatrix}=e^t\begin{bmatrix}1-t & t\\ -t & 1+t\end{bmatrix}.\end{aligned} \tag{1.34}$$

10. Solve the initial value problem obtained from combining the differential equation of problem 8 with the initial data

$$\mathbf{x}(0)=\begin{pmatrix}2\\4\end{pmatrix}. \tag{1.35}$$

(Hint: do not "reinvent the wheel", but rather use the result from problem 9.)

**6 points**

**Solution**

From problem 9 we have

$$\begin{aligned}\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) &= e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = e^t \begin{bmatrix} 2(1-t)+4t \\ 2(-t)+4(1+t) \end{bmatrix} \\ &= e^t \begin{bmatrix} 2+2t \\ 4+2t \end{bmatrix} = 2e^t \begin{bmatrix} 1+t \\ 2+t \end{bmatrix}.\end{aligned} \tag{1.36}$$