KEY

Math 334 Final
Fall 2008
sections 001 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering paper. Write your name, course, and section number on the blue book, or on your own pile of papers. Again, do not write this or any other type of information on this exam.

Warning: check any given candidate solution to a problem via a method independent of the one used to first obtain it. In particular, a differential equation has the property that one can check whether a given function satisfies it. So check your solutions by plugging them back into the equation! If a solution doesn't work, try again, or at least note that your proposed solution doesn't work.

Good practice: Look the test over from beginning to end before beginning. Look for easy problems first, then proceed to harder ones.

1. Consider the initial value problem (IVP)

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{x}_0. \tag{1.1}$$

Here $P(t)$ is an $N \times N$ matrix of known functions of the variable t, the functions being continuous in some interval (t_{-1}, t_1) of t's containing t_0 (i.e. $t_0 \in (t_{-1}, t_1)$), and $\mathbf{x} = \mathbf{x}(t)$ is a column vector of , initially, N unknown functions to be solved for, $\mathbf{x}_0$ being another column vector of N known, fixed values. ($\mathbf{x}_0$ is the "initial data".) Thus the differential equation in (1.1) constitutes a first, order, linear, homogeneous first order system of ordinary differential equations for the initially unknown entries of $\mathbf{x} = \mathbf{x}(t)$. It turns out that there is a unique solution $\mathbf{x} = \mathbf{x}(t)$ to (1.1) over the interval (t_{-1}, t_1), and that it can be written as

$$\mathbf{x}(t) = \Phi_{t_0}(t)\mathbf{x}_0, \tag{1.2}$$

where $\Phi_{t_0}(t)$ is the one and only $N \times N$ fundamental matrix of solutions (of the system of ODE's in (1.1)) sporting the initial data $\Phi_{t_0}(t_0) = I$. (I denotes the $N \times N$ identity matrix; it has 1's on the diagonal and 0's elsewhere.) One can show that, according to the definition suggested, $\Phi_{t_0}(t)$ satisfies its own initial value problem, one that is rather analogous to(1.1), namely

$$\frac{d\Phi_{t_0}(t)}{dt} = P(t)\Phi_{t_0}(t),\ \Phi_{t_0}(t_0) = I. \tag{1.3}$$

So the only difference between $\Phi_{t_0} = \Phi_{t_0}(t)$ satisfying (1.3) and $\mathbf{x} = \mathbf{x}(t)$ satisfying (1.1) is that a) $\Phi_{t_0}(t)$ is an $N \times N$ matrix of functions (whereas $\mathbf{x} = \mathbf{x}(t)$ is a column vector of functions), and b) $\Phi_{t_0}(t)$ has some very specific initial data—a bunch of 1's and 0's in appropriate places.

Given (1.3), it is easy to show that, via the associative property of matrix multiplication, etc., (1.2) does in fact give a solution to the IVP (1.1): with (1.2) (together with (1.3)) as the definition of $\mathbf{x} = \mathbf{x}(t)$, one finds that

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} - P(t)\mathbf{x} &= \frac{d\Phi_{t_0}(t)\mathbf{x}_0}{dt} - P(t)\left(\Phi_{t_0}(t)\mathbf{x}_0\right) = \frac{d\Phi_{t_0}(t)}{dt}\mathbf{x}_0 - P(t)\left(\Phi_{t_0}(t)\mathbf{x}_0\right) \\
&= \left(P(t)\Phi_{t_0}(t)\right)\mathbf{x}_0 - P(t)\left(\Phi_{t_0}(t)\mathbf{x}_0\right) = P(t)\Phi_{t_0}(t)\mathbf{x}_0 - P(t)\Phi_{t_0}(t)\mathbf{x}_0 \quad (1.4) \\
&= \mathbf{0}, \\
\mathbf{x}(t_0) - \mathbf{x}_0 &= \Phi_{t_0}(t_0)\mathbf{x}_0 - \mathbf{x}_0 = I\mathbf{x}_0 - \mathbf{x}_0 = \mathbf{x}_0 - \mathbf{x}_0 = \mathbf{0},
\end{aligned}$$

which shows that both of the equations defining the IVP in (1.1) are satisfied.

So now I would like you to show that for the non-homogeneous IVP

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x} + \mathbf{g}(t), \ \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.5}$$

$\mathbf{g}(t)$ here being a nonzero column vector of N known functions, a solution $\mathbf{x} = \mathbf{x}(t)$ over the interval (t_{-1}, t_1) is given by a certain generalization of (1.2), namely

$$\mathbf{x}(t) = \Phi_{t_0}(t)\mathbf{u}(t) = \Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}^{-1}(\tau)\mathbf{g}(\tau)\,d\tau\right). \tag{1.6}$$

(Recall that a fundamental matrix of solutions such as $\Phi_{t_0}(t)$ will remain nonsingular and, so, invertible as long as $P(t)$ is continuous—there's no problem with the inverse $\Phi_{t_0}^{-1}(\tau)$ not being defined over τ 's in (t_{-1}, t_1).) Note that what I am asking you to do here is a bit easier than what I said I would ask for on the final: you do not have to derive (1.6) (via variation of parameters, say), but rather, analogous to what was done in (1.4), you just have to show that formula (1.6) "works" in (*both equations in*) (1.5). You will of course use (1.3) as the defining property of $\Phi_{t_0}(t)$.

This problem is very easy as long as you recall simple things like the fact that in matrix multiplication the order of factors matters, and that the product rule works as usual (with that last proviso).

20 points

<u>Solution</u>

From definition (1.6) and the data in (1.3) we immediately have

$$\mathbf{x}(t_0) = \Phi_{t_0}(t_0)\left(\mathbf{x}_0 + \int_{t_0}^{t_0} \Phi_{t_0}^{-1}(\tau)\mathbf{g}(\tau)\,d\tau\right) = I\left(\mathbf{x}_0 + \mathbf{0}\right) = \mathbf{x}_0, \tag{1.7}$$

so that the last, "data" equation in (1.5) holds. By using Liebniz rule (the product rule), and the associative property of matrix multiplication, etc., as well as the differential equation in(1.3), one also computes that the definition (1.6) gives

$$
\begin{aligned}
\frac{d\mathbf{x}}{dt} - P(t)\mathbf{x} - \mathbf{g}(t) &= \frac{d}{dt}\left\{\Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right)\right\} - P(t)\Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right) - \mathbf{g}(t) \\
&= \frac{d\Phi_{t_0}(t)}{dt}\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right) + \Phi_{t_0}(t)\frac{d}{dt}\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right) \\
&\quad - P(t)\Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right) - \mathbf{g}(t) \\
&= \cancel{P(t)\Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right)} + \Phi_{t_0}(t)\,\Phi_{t_0}{}^{-1}(t)\,\mathbf{g}(t) - \\
&\quad \cancel{P(t)\Phi_{t_0}(t)\left(\mathbf{x}_0 + \int_{t_0}^{t} \Phi_{t_0}{}^{-1}(\tau)\,\mathbf{g}(\tau)\,d\tau\right)} - \mathbf{g}(t) \\
&= I\mathbf{g}(t) - \mathbf{g}(t) = \mathbf{0},
\end{aligned}
\tag{1.8}
$$

so that the first, differential equation in (1.5) holds.

2. Note that for a constant coefficient problem such as

$$\frac{d\Phi_0(t)}{dt} = A\Phi_0(t),\ \Phi_0(0) = I, \tag{1.9}$$

we can compute the fundamental matrix $\Phi_0(t)$ via the Laplace transform: transforming the first, differential equation in (1.9), keeping in mind the second, we get

$$\mathcal{L}\left[\Phi_0{}'\right](s) = s\mathcal{L}\left[\Phi_0\right](s) - \Phi_0(0) = s\mathcal{L}\left[\Phi_0\right](s) - I = \mathcal{L}\left[A\Phi_0\right](s) = A\mathcal{L}\left[\Phi_0\right](s) \tag{1.10}$$

which is equivalent to

$$(sI - A)\mathcal{L}\left[\Phi_0\right](s) = I\ , \tag{1.11}$$

and then to

$$\mathcal{L}\left[\Phi_0\right](s) = (sI - A)^{-1} I = (sI - A)^{-1}, \tag{1.12}$$

so that finally one has the formula

$$\Phi_0(t) = \mathsf{L}^{-1}\left[\left(sI - A\right)^{-1}\right](t). \tag{1.13}$$

Thus, for example, if one wanted to compute the fundamental matrix $\Phi_0(t)$ associated with a 2×2 matrix A given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{1.14}$$

one would get

$$\begin{aligned}
\Phi_0(t) &= \mathsf{L}^{-1}\left[\left(sI - A\right)^{-1}\right](t) = \mathsf{L}^{-1}\left[\begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}^{-1}\right](t) = \mathsf{L}^{-1}\left[\frac{1}{\left(s-1\right)^2 - \left(-1\right)^2}\begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}\right](t) \\
&= \mathsf{L}^{-1}\left[\frac{1}{\left(s-1+1\right)\left(s-1-1\right)}\begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}\right](t) = \mathsf{L}^{-1}\left[\frac{1}{s\left(s-2\right)}\begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}\right](t) \\
&= \mathsf{L}^{-1}\left[\frac{1}{s\left(0-2\right)}\begin{bmatrix} 0-1 & 1 \\ 1 & 0-1 \end{bmatrix} + \frac{1}{2\left(s-2\right)}\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\right](t) \\
&= \frac{e^{0t}}{-2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix}.
\end{aligned} \tag{1.15}$$

Here we did some partial fractions, and used the supplied table to come back to the time domain. Note also that this really does work: we get (1.9) because, with $\Phi_0(t)$ defined by (1.15), we have

$$\begin{aligned}
\Phi_0(0) &= \frac{1}{2}\begin{bmatrix} e^{2\cdot 0}+1 & e^{2\cdot 0}-1 \\ e^{2\cdot 0}-1 & e^{2\cdot 0}+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = I_{2\times 2}, \\
\frac{d}{dt}\Phi_0\left(t\right) - A\Phi_0\left(t\right) &= \frac{d}{dt}\frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2e^{2t} & 2e^{2t} \\ 2e^{2t} & 2e^{2t} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} e^{2t}+1+e^{2t}-1 & e^{2t}-1+e^{2t}+1 \\ e^{2t}+1+e^{2t}-1 & e^{2t}-1+e^{2t}+1 \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - e^{2t}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}.
\end{aligned} \tag{1.16}$$

So here's what I want you to do: compute the fundamental matrix $\Phi_0(t)$ for a constant coefficient system such as (1.9) in which

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{1.17}$$

Again, it might be lightening fast to compute it via formula/algorithm (1.13). Check your answer as in my computations in(1.16).

20 points

<u>Solution</u>

From formula (1.13) and (1.17) we have

$$\begin{aligned} \Phi_0(t) &= \mathcal{L}^{-1}\left[(sI-A)^{-1}\right](t) = \mathcal{L}^{-1}\left[\begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}^{-1}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2-0}\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}\right](t) \\ &= \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}\right](t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{1.18}$$

Similar to (1.16) we check that with $\Phi_0(t)$ defined by (1.18) we have

$$\begin{aligned} \Phi_0(0) &= e^0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = I_{2\times 2}, \\ \frac{d}{dt}\Phi_0(t) - A\Phi_0(t) &= \frac{d}{dt}e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ &= e^t\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) - e^t\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ &= e^t\begin{bmatrix} 1 & 1+t \\ 0 & 1 \end{bmatrix} - e^t\begin{bmatrix} 1 & t+1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2} \end{aligned} \tag{1.19}$$

as required.

3. Using the formulas (1.6) and (1.13) developed in problems 1 and 2, solve the IVP

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t),\ \ \mathbf{x}(0) = \mathbf{x}_0, \tag{1.20}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ \mathbf{g}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \ \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{1.21}$$

To expedite matters, you may want to know that

$$\frac{d}{d\tau}\tau e^{-\tau} = (-\tau + 1)e^{-\tau}. \tag{1.22}$$

40 points

Solution

From (1.6) we have

$$\mathbf{x}(t) = \Phi_0(t)\left(\mathbf{x}_0 + \int_0^t \Phi_0^{-1}(\tau)\mathbf{g}(\tau)\,d\tau\right), \tag{1.23}$$

where, according to (1.13) and then, specifically, (1.18), we have

$$\Phi_0(t) = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Rightarrow \Phi_0^{-1}(t) = \Phi_0(-t) = e^{-t}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}. \tag{1.24}$$

Consequently, for our specific problem, and using the hint (1.22), we have

$$\begin{aligned} \int_0^t \Phi_0^{-1}(\tau)\mathbf{g}(\tau)\,d\tau &= \int_0^t e^{-\tau}\begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau = \int_0^t e^{-\tau}\begin{bmatrix} 1-\tau \\ 1 \end{bmatrix} d\tau = \left.\begin{bmatrix} \tau e^{-\tau} \\ -e^{-\tau} \end{bmatrix}\right|_{\tau=0}^{\tau=t} \\ &= \begin{bmatrix} te^{-t} \\ -e^{-t} \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} te^{-t} \\ 1-e^{-t} \end{bmatrix}, \end{aligned} \tag{1.25}$$

and then

$$\mathbf{x}(t) = \Phi_0(t)\left(\mathbf{x}_0 + \int_0^t \Phi_0^{-1}(\tau)\mathbf{g}(\tau)\,d\tau\right) = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} te^{-t} \\ 1-e^{-t} \end{bmatrix}\right)$$

$$= e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1+te^{-t} \\ 2-e^{-t} \end{bmatrix} = e^t \begin{bmatrix} 1+te^{-t}+t\left(2-e^{-t}\right) \\ 2-e^{-t} \end{bmatrix} = e^t \begin{bmatrix} 1+2t \\ 2-e^{-t} \end{bmatrix} \qquad (1.26)$$

$$= \begin{bmatrix} (1+2t)e^t \\ 2e^t - 1 \end{bmatrix}.$$

We note that this works: with (1.26) as definition we have

$$\mathbf{x}(0) = \begin{bmatrix} (1+2\cdot 0)e^0 \\ 2e^0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}_0,$$

$$\frac{d\mathbf{x}}{dt} - A\mathbf{x} - \mathbf{g}(t) = \frac{d}{dt}\begin{bmatrix} (1+2t)e^t \\ 2e^t - 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+2t)e^t \\ 2e^t - 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2t+2)e^t \\ 2e^t \end{bmatrix} - \begin{bmatrix} (1+2t)e^t + 2e^t - 1 \\ 2e^t - 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (1.27)$$

$$= \begin{bmatrix} (3+2t)e^t \\ 2e^t \end{bmatrix} - \begin{bmatrix} (3+2t)e^t - 1 \\ 2e^t - 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

as required.

4. Find an expression of the general solution of the following linear, constant coefficient ODE

$$(D+2)(D-3)y = \left(D^2 - D - 6\right)y = y'' - y' - 6y = 0. \qquad (1.28)$$

In (1.28) we have $Dy := y'$, i.e. D is the linear operator that differentiates functions. In expressing (1.28) the way I have, I have effectively endeavored to write the problem in a way that makes discovering the general solution easy. Nevertheless if you do not think that the notation used in the left most expressions in (1.28) is as helpful as I am suggesting, by all means please ignore those expressions and just concentrate on the version of the equation written

$$y'' - y' - 6y = 0. \qquad (1.29)$$

My guess is that even if you find the notation used in the left most expressions in (1.28) unhelpful, after finding the solution in the "usual way", i.e. using only

(1.29), you will then have a Eureka moment and discover why the notation in (1.28) was in fact helpful.

20 points

Solution

Because $Dy = ry$ when $y = y(t) = e^{rt}$ (and when D indicates differentiation with respect to t), the ansatz $y = e^{rt}$ in (1.28) gives

$$0 = (D+2)(D-3)y = (r+2)(r-3)y\,, \tag{1.30}$$

which holds (uniformly on an interval of t's) iff $r = -2$ or $r = 3$, indicating then some special solutions to, say, (1.29). Since (1.29) is linear and second order, and since the functions $t \mapsto e^{-2t}$ and $t \mapsto e^{3t}$ are linearly independent, the general solution of (1.29) can be expressed as

$$y = c_1 e^{-2t} + c_2 e^{3t}\,, \tag{1.31}$$

where c_1, c_2 are independent of t. This means that every specific solution of (1.29) can be expressed as in (1.31) via some particular (and unique) choice of the c's.

5. Find a (particular) solution of the following differential equation by the method of undetermined coefficients:

$$y'' + 2y' + y = t^2\,. \tag{1.32}$$

20 points

Solution

The usual explanation of the ansatz for developing a particular solution to a linear constant coefficient differential equation *with a right-hand-side (RHS) that is itself in the null space of a linear, constant coefficient differential operator* is to first find a basis for the span of the RHS of the equation together with all of its derivatives. Then, barring the phenomena of *resonance*, which is that one or more elements of such a basis are in the null space of the linear differential operator defining the equation, one then forms a general element of the space spanned by the basis, which general element constitutes the "method of undetermined coefficients ansatz" for a particular solution of the equation in question. For equation **Error! Reference source not found.** the span of the RHS together with all its derivatives is the same as the span of $\{t^2, t, 1\}$. Thus, if there is no resonance, the relevant basis for a particular solution of **Error! Reference source not found.** is $\{t^2, t, 1\}$. One finds that none of these proposed basis elements is a solution of the homogeneous version of**Error! Reference source not found.**, so that there is no resonance, and the ansatz for a solution of **Error! Reference source not found.** is, together with relevant derivatives of this candidate solution, of the form

$$\begin{aligned} y &= At^2 + Bt + C\cdot 1, \\ y' &= 0t^2 + 2At + B, \\ y'' &= 0t^2 + 0t + 2A. \end{aligned} \tag{1.33}$$

Weighted appropriate for the equation**Error! Reference source not found.**, the equations (1.33) are

$$\begin{aligned} y &= At^2 + Bt + C\cdot 1, \\ 2y' &= 0t^2 + 4At + 2B, \\ 1y'' &= 0t^2 + 0t + 2A, \end{aligned} \tag{1.34}$$

which sum to

$$\begin{aligned} y'' + 2y' + y &= (0+0+A)t^2 + (0+4A+B)t + (2A+2B+C)\cdot 1 \\ &= At^2 + (4A+B)t + (2A+2B+C)\cdot 1. \end{aligned} \tag{1.35}$$

To solve **Error! Reference source not found.** we thus demand that

$$\begin{aligned} At^2 + (4A+B)t + (2A+2B+C)\cdot 1 = t^2 + 0t + 0\cdot 1 \\ \Leftrightarrow \\ (A-1)t^2 + (4A+B)t + (2A+2B+C)\cdot 1 = 0. \end{aligned} \tag{1.36}$$

Since the set of functions $\{t^2, t, 1\}$ is a linearly independent set, the latter equation holds uniformly in t over any interval if and only if

$$\begin{aligned} &A - 1 = 4A + B = 2A + 2B + C = 0 \\ &\Leftrightarrow \\ &A = 1, B = -4A = -4, C = -2A - 2B = -2 + 8 = 6. \end{aligned} \tag{1.37}$$

Thus the solution sought is

$$y = At^2 + Bt + C = t^2 - 4t + 6. \tag{1.38}$$

6. Prove that the following differential equation is exact and then find an expression for its general solution.

$$\left(2x + 3x^2y^2\right)dx + \left(2x^3y + 3y^2\right)dy = 0. \tag{1.39}$$

40 points

<u>Solution</u>

The equation (1.39) is, by definition, *exact* if the left-hand side is the differential of a (continuously differentiable) function (of two variables x and y, in some simply-connected region of the x-y plane, etc., etc.), i.e. if there is a function $\psi(x, y)$ such that

$$d\psi(x, y) = \left(2x + 3x^2y^2\right)dx + \left(2x^3y + 3y^2\right)dy. \tag{1.40}$$

But we have, by definition,

$$d\psi(x, y) = \psi_x(x, y)dx + \psi_y(x, y)dy, \tag{1.41}$$

so that the equation (1.40) is the (potentially) over-determined system of equations

$$\psi_x(x, y) = 2x + 3x^2y^2, \text{ and } \psi_y(x, y) = 2x^3y + 3y^2. \tag{1.42}$$

This over-determined pair of equations is consistent (or *integrable*) iff $\left(\psi_x\right)_y = \left(\psi_y\right)_x$, i.e. iff

$$\left(2x + 3x^2y^2\right)_y = \left(2x^3y + 3y^2\right)_x. \tag{1.43}$$

(1.43) holds true, so that the equation (1.39) is indeed exact, because either side of (1.43) is $6x^2y$.

As for developing the function $\psi(x, y)$, and then (an expression for) the general solution of(1.39), one notes that the equations (1.42) demand, respectively, that

$$\begin{aligned}\psi(x,y) &= \int\left(2x+3x^2y^2\right)dx = x^2+x^3y^2+f(y),\\ \text{and } \psi(x,y) &= \int\left(2x^3y+3y^2\right)dy = x^3y^2+y^3+g(x),\end{aligned} \tag{1.44}$$

for some initially rather arbitrary functions $f(y)$ and $g(x)$. The two statements (1.44) are not contradictory iff $x^2+x^3y^2+f(y)=x^3y^2+y^3+g(x) \Leftrightarrow f(y)-y^3=g(x)-x^2$, which implies both sides of the equation are independent of both x and y. As far as finding the general solution of (1.39) is concerned, without loss of generality we can choose $f(y)-y^3=g(x)-x^2=0$ so that (1.44) becomes ("in either case")

$$\psi(x,y)=x^2+x^3y^2+y^3\,. \tag{1.45}$$

(1.45) is NOT the general solution to the (exact) differential equation (1.39). It is not even a specific solution. Rather (1.45) defines a "potential (function) for the solution." Using it one notes that (1.39) can be written as

$$d\psi(x,y)=d\left(x^2+x^3y^2+y^3\right)=0\,, \tag{1.46}$$

the general solution to which is clearly

$$x^2+x^3y^2+y^3=C\,. \tag{1.47}$$

7. Find the *general solution* of the following linear but non-homogeneous differential equation *by the method of variation of parameters*. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the "D'Alembert-like" ansatz that leads to that formula. (Also, do not use the method of undetermined coefficients.)

$$y''-3y'+2y=e^{2t} \tag{1.48}$$

40 points

Solution

The characteristic equation of the homogeneous version of the constant coefficient differential equation (1.48) is

$$0=r^2-3r+2=(r-1)(r-2) \tag{1.49}$$

so that the general solution of the corresponding homogeneous equation is

$$y=Ae^{t}+Be^{2t}\,, \tag{1.50}$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (1.50), we have also an ansatz there for the solution of the non-homogeneous equation (1.48): with such an ansatz one immediately has

$$y' = Ae^t + 2Be^{2t} + \left(e^t A' + e^{2t} B'\right). \tag{1.51}$$

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$e^t A' + e^{2t} B' = 0, \tag{1.52}$$

so that then (1.51) becomes simply

$$y' = Ae^t + 2Be^{2t}. \tag{1.53}$$

Differentiating (1.53) gives

$$y'' = Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \tag{1.54}$$

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$\begin{aligned} 2y &= 2Ae^t + 2Be^{2t} \\ -3y' &= -3Ae^t - 6Be^{2t} \\ +1y'' &= Ae^t + 4Be^{2t} + \left(e^t A' + 2e^{2t} B'\right). \end{aligned} \tag{1.55}$$

and from which it is clear that the differential equation demands that

$$e^t A' + 2e^{2t} B' = e^{2t}. \tag{1.56}$$

Combining this with the "consistency ansatz" (1.52) we get

$$\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}, \tag{1.57}$$

which implies that

$$A' = \frac{\begin{vmatrix} 0 & e^{2t} \\ e^{2t} & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix}} = \frac{-e^{4t}}{e^{3t}} = -e^{t}, \; B' = \frac{\begin{vmatrix} e^{t} & 0 \\ e^{t} & e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix}} = \frac{e^{3t}}{e^{3t}} = 1. \tag{1.58}$$

Solutions to (1.58) include the pair $A = -e^{t}, \; B = t,$ so that a solution to (1.48) is, according to(1.50),

$$y = Ae^{t} + Be^{2t} = -e^{t}e^{t} + te^{2t} = te^{2t} - e^{2t} = (t-1)e^{2t}, \tag{1.59}$$

and the general solution to (1.48) is

$$\begin{aligned} y &= c_1e^{t} + c_2e^{2t} + (t-1)e^{2t} \\ &= d_1e^{t} + d_2e^{2t} + te^{2t}, \end{aligned} \tag{1.60}$$

where $d_1 = c_1$ and $d_2 = c_2 - 1$ are (truly) constants now.

8. Solve the initial value problem obtained from combining the differential equation of problem 11 with the initial data $y(0) = 0, \; y'(0) = 0$. In order that errors don’t “cascade”, I will tell you that $y = Ae^{t} + Be^{2t} + te^{2t}$ is the general solution of the differential equation of problem 11. (So now if you just write down this solution to 11 without very convincing work, you will get 0 points on problem 11.) Thus, I am only testing if you understand the correct principles needed to construct the solution to the initial value problem given the general solution to the associated differential equation.

20 points

Solution

From the information given we have

$$\begin{aligned} y(0) = 0 &= Ae^{t} + Be^{2t} + te^{2t}\Big|_{t=0} = A + B \\ y'(0) = 0 &= Ae^{t} + 2Be^{2t} + (2t+1)e^{2t}\Big|_{t=0} = A + 2B + 1, \end{aligned} \tag{1.61}$$

or, equivalently, the following augmented matrix for the column vector (A, B), which, together with row reduction, is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{1.62}$$

From (1.62) one has that

$$A = 1,\ B = -1. \tag{1.63}$$

and the solution sought is

$$\begin{aligned} y &= Ae^t + Be^{2t} + te^{2t} = e^t - e^{2t} + te^{2t} \\ &= e^t + (t-1)e^{2t}. \end{aligned} \tag{1.64}$$

9. Find (a real-valued representation of) the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' - 5xy' + 13y = 0. \tag{1.65}$$

40 points

Solution

The differential equation (1.65) defines a linear differential operator L_x, in terms of which (1.65) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$\begin{aligned} L_x[y_r] &= \left(r(r-1) - 5r + 13\right)x^r = \left(r^2 - 6r + 13\right)x^r = \left((r-3)^2 + 2^2\right)x^r \\ &= (r - 3 + 2i)(r - 3 - 2i)\,x^r = \left[r - (3-2i)\right]\left[r - (3+2i)\right]x^r, \end{aligned} \tag{1.66}$$

so that complex solutions of (1.65) are clearly then
$y_{3+2i} = x^{3+2i} = x^3 e^{2i\ln x} = x^3\left(\cos(2\ln x) + i\sin(2\ln x)\right)$ and
$y_{3-2i} = x^{3-2i} = x^3 e^{-2i\ln x} = x^3\left(\cos(2\ln x) - i\sin(2\ln x)\right)$. Independent complex linear combinations of these (linearly independent, complex-valued) solutions give, as required, the following real-representation of the general solution:

$$y = x^3\left(A\cos(2\ln x) + B\sin(2\ln x)\right). \tag{1.67}$$

10. Solve the following initial value problem:

$$y'' - 10y' + 29y = 0;\quad y(0) = 1,\ y'(0) = 3. \tag{1.68}$$

40 points

Solution

This linear homogeneous differential equation is associated with the following characteristic (polynomial) and characteristic exponents r :

$$\begin{aligned}&0 = r^2 - 10r + 29 = r^2 - 10r + 25 + 4 = (r-5)^2 - (2i)^2 \\ &\Leftrightarrow \\ &r = 5 \pm 2i.\end{aligned} \tag{1.69}$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$\begin{aligned}&y = e^{5t}\left(C_1 \cos 2t + C_2 \sin 2t\right) \\ &\quad \text{and} \\ &y' = e^{5t}\left(\left(5C_1 + 2C_2\right)\cos 2t + \left(-2C_1 + 5C_2\right)\sin 2t\right).\end{aligned} \tag{1.70}$$

Inserting $t = 0$ into(1.70), and using the initial data given in(1.68), one obtains

$$\begin{aligned}&y(0) = C_1 = 1 \\ &\quad \text{and} \\ &y'(0) = 5C_1 + 2C_2 = 3,\end{aligned} \tag{1.71}$$

the solution to which being $C_1 = 1$ and $C_2 = -1$. Thus the solution to the initial value problem is then

$$y = e^{5t}\left(\cos 2t - \sin 2t\right). \tag{1.72}$$