KEY

Math 334 Midterm II
Fall 2008
sections 001 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering paper. Write your name, course, and section number on the blue book, or on your own pile of papers. Again, do not write this or any other type of information on this exam.

Good Practice: A differential equation has the property that one can check whether a given function satisfies it. So check your solutions! If one doesn't work, try again, or at least note that your proposed solution doesn't work. Showing me that you are checking each problem will result in high amounts of partial credit—if not correct answers!

1. Determine a lower bound for the radius of convergence of the power series representation of the general solution of the following differential equation about the point $x_0 = -2$:

$$x^2\left((x-2)^2+9\right)y''+x^2y'+(1-\cos x)\,y=0. \tag{1.1}$$

WARNING: Handle the $(1-\cos x)$ term with extreme care. Specifically, consider whether or not the "$p(x)$" and the "$q(x)$" for an equation such as (1.1) are analytic at various places despite "removable/cosmetic" singularities.

5 points

Solution

Equation (1.1) can be reduced to the standard form

$$y''+p(x)y'+q(x)y = y''+\frac{1}{(x-2)^2+9}y'+\frac{1-\cos x}{x^2}\frac{1}{(x-2)^2+9}y=0. \tag{1.2}$$

which has coefficients $p(x)$ and $q(x)$ with singularities at the zeroes of $(x-2)^2+9$, which zeroes are $x = 2\pm 3i$. Note that there are no singularities anywhere else. In particular there is no singularity of our "$q(x)$" at $x=0$ other than a cosmetic/removable one: the function

$$f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ \lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}, & x=0\end{cases} = \begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ \dfrac{1}{2}, & x=0\end{cases} \tag{1.3}$$

is analytic everywhere, including $x=0$. To establish analyticity at $x=0$ one simply notes that f 's power series representation about $x=0$ is that of $1-\cos x$ divided (term wise) by x^2, which is also a proper power series with nonzero (in fact infinite) radius of convergence: we have for $f(x)$ defined by (1.3) that

$$f(x)=\frac{1-\sum\limits_{m=0}^{\infty}\dfrac{(-1)^m}{(2m)!}x^{2m}}{x^2}=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m)!}x^{2(m-1)}=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+2)!}x^{2m} \tag{1.4}$$

the latter (proper) power series representation holding for all x including $x = 0$.

In the complex plain the distance of the singularities $x = 2 \pm 3i$ to the expansion point $x_0 = -2 = -2 + 0i$ is $\sqrt{\left(2-(-2)\right)^2 + \left(\pm 3 - 0\right)^2} = \sqrt{16+9} = \sqrt{25} = 5$. Thus, even along the real axis, we cannot guarantee a radius of convergence beyond 5 without more information. (But by theorem we can guarantee a radius of convergence of at least that much with the information given, namely that of the location of the non-cosmetic singularities of the coefficients of the equation in standard form(1.2).)

2. Find a (particular) solution of the following differential equation by the method of undetermined coefficients:

$$y'' + 2y' + y = t^2 e^{-t}. \tag{1.5}$$

7 points

Solution

The usual explanation of the ansatz for developing a particular solution to a linear constant coefficient differential equation *with a right-hand-side (RHS) that is itself in the null space of a linear, constant coefficient differential operator* is to first find a basis for the span of the RHS of the equation together with *all* of its derivatives (which will be finite dimensional since the RHS is in the null space of a linear, constant coefficient differential operator). Then, barring the phenomena of *resonance*, which is that one or more elements of such a basis are in the null space of the linear differential operator defining the equation, one then forms a general element of the space spanned by the basis, which general element constitutes the "method of undetermined coefficients ansatz" for a particular solution of the equation in question. For equation (1.5) the span of the RHS together with all its derivatives is the same as the span of $\left\{t^2e^{-t}, te^{-t}, e^{-t}\right\}$. Thus, if there is no resonance, the relevant basis for a particular solution of (1.5) is $\left\{t^2e^{-t}, te^{-t}, e^{-t}\right\}$. One finds that the last 2 of these proposed basis elements are solutions of the homogeneous version of (1.5), so that there is a "double resonance", and the relevant ansatz for a solution of (1.5) is, together with relevant derivatives of this candidate solution, of the form

$$\begin{aligned}
y &= \left(At^4 + Bt^3 + Ct^2 + 0t + 0\right)e^{-t} \\
y' &= \left(-At^4 + (4A-B)t^3 + (3B-C)t^2 + 2Ct + 0\right)e^{-t}, \\
y'' &= \left(At^4 + (-8A+B)t^3 + (12A-6B+C)t^2 + (6B-4C)t + 2C\right)e^{-t}.
\end{aligned} \tag{1.6}$$

Weighted appropriate for the equation(1.5), the equations (1.6) are

$$\begin{aligned} y &= \left(\quad At^4 \quad + \quad Bt^3 \quad + \quad Ct^2 \quad + \ 0t + \ 0\right)e^{-t} \\ 2y' &= \left(-2At^4 + (8A-2B)t^3 + \quad (6B-2C)t^2 + \quad 4Ct + \ 0\right)e^{-t}, \\ y'' &= \left(\quad At^4 + (-8A+B)t^3 + (12A - \ 6B + C)t^2 + (6B - \ 4C)t + 2C\right)e^{-t}. \end{aligned} \tag{1.7}$$

which sum to

$$y'' + 2y' + y = \left(0t^4 + 0t^3 + 12At^2 + 6Bt + 2C\right)e^{-t} = \left(12At^2 + 6Bt + 2C\right)e^{-t}. \tag{1.8}$$

To solve (1.5) we thus demand that

$$\begin{gathered} \left(12At^2 + 6Bt + 2C\right)e^{-t} = t^2e^{-t} \\ \Leftrightarrow \\ (12A-1)t^2 + 6Bt + 2C \cdot 1 = 0 \end{gathered} \tag{1.9}$$

over some interval of t 's, with, evidently, A, B, C independent of t. Since the set of functions $\left\{t^2, t, 1\right\}$ is a linearly independent set, the latter equation holds uniformly in t over any interval if and only if

$$\begin{gathered} 12A - 1 = 6B = 2C = 0 \\ \Leftrightarrow \\ A = \frac{1}{12}, B = C = 0. \end{gathered} \tag{1.10}$$

Thus, by(1.6), the solution sought is

$$y = \left(At^4 + Bt^3 + Ct^2\right)e^{-t} = \frac{1}{12}t^4e^{-t}. \tag{1.11}$$

3. A 5 kilogram mass stretches a (linear, Hooke's law) spring 1/5 meter. If the mass is set in motion from the equilibrium position at 6 meters per second *upward* at time $t = 0$, and there is no damping, determine the displacement $u(t)$ of the mass *above* the equilibrium position at any subsequent time t. Use that the acceleration of gravity is 49/5 meters per second per second.

9 points

Solution

The relevant version of Newton's second law is

$$0 = mu'' + ku = 5\text{kg}u'' + ku\,. \tag{1.12}$$

Here we may determine the spring constant k from

$$k = F/\Delta x = mg/\Delta x = \frac{5\text{kg}\left(49/5\right)\frac{\text{m}}{s^2}}{1/5\text{m}} = 5\cdot 7^2\,\text{kg}/s^2, \tag{1.13}$$

so that (1.12) is

$$0 = 5\text{kg}u'' + 5\cdot 7^2\,\text{kg}/s^2 u \Leftrightarrow 0 = u'' + 7^2/s^2 u\,. \tag{1.14}$$

Rendering (1.14) unit-less (by measuring time in seconds), we have

$$0 = u'' + 7^2 u\,, \tag{1.15}$$

the general solution to which being

$$u = A\cos(7t) + B\sin(7t)\,. \tag{1.16}$$

The initial data specifies that

$$\begin{gathered} u(0) = 0 = A, u'(0) = 6 = 7B \\ \Leftrightarrow \\ A = 0, B = 6/7, \end{gathered} \tag{1.17}$$

so that the required solution to the initial value problem is

$$u = A\cos(7t) + B\sin(7t) = \left(6/7\right)\sin(7t). \tag{1.18}$$

In (1.17) we used that $u'(0) = +6$ rather than $u'(0) = -6$ so that the solution $u = u(t)$ in (1.18) gives, as required by the question, the displacement of the mass *above* the equilibrium position at any time t subsequent to the initial excitation in which it was imparted a velocity of 6 meters per second *upwards*.

4. Find (a real-valued representation of) the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' - 3xy' + 13y = 0\,. \tag{1.19}$$

11 points

Solution

The differential equation (1.19) defines a linear differential operator L_x, in terms of which (1.19) can be written $L_x[y]=0$. On a function $y_r = x^r$ one finds that

$$\begin{aligned} L_x[y_r] &= \left(r(r-1)-3r+13\right)x^r = \left(r^2-4r+13\right)x^r = \left((r-2)^2+3^2\right)x^r \\ &= \left(r-2+3i\right)\left(r-2-3i\right)x^r = \left[r-\left(2-3i\right)\right]\left[r-\left(2+3i\right)\right]x^r, \end{aligned} \tag{1.20}$$

so that complex solutions of (1.19) are clearly then

$$\begin{aligned} y_{2+3i} &= x^{2+3i} = x^2 e^{3i\ln x} = x^2\left(\cos(3\ln x)+i\sin(3\ln x)\right), \text{ and} \\ y_{2-3i} &= x^{2-3i} = x^2 e^{-3i\ln x} = x^2\left(\cos(3\ln x)-i\sin(3\ln x)\right). \end{aligned} \tag{1.21}$$

Independent complex linear combinations of these (linearly independent, complex-valued) solutions give, as required, the following real-representation of the general solution:

$$y = x^2\left(A\cos(3\ln x)+B\sin(3\ln x)\right). \tag{1.22}$$

5. Solve the following initial value problem:

$$y'' - 4y' + 29y = 0; \quad y(0)=1, \; y'(0)=7. \tag{1.23}$$

12 points

Solution

This linear homogeneous differential equation is associated with the following characteristic equation and characteristic exponents r:

$$\begin{aligned} &0 = r^2-4r+29 = r^2-4r+4+25 = (r-2)^2-(5i)^2 \\ &\Leftrightarrow \\ &r = 2\pm 5i. \end{aligned} \tag{1.24}$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$\begin{gathered} y = e^{2t}\left(C_1 \cos 5t + C_2 \sin 5t\right) \\ \text{and} \\ y' = e^{2t}\left(\left(2C_1 + 5C_2\right)\cos 5t + \left(-5C_1 + 2C_2\right)\sin 5t\right). \end{gathered} \tag{1.25}$$

Inserting $t = 0$ into(1.25), and using the initial data given in(1.23), one obtains

$$\begin{gathered} y(0) = C_1 = 1 \\ \text{and} \\ y'(0) = 2C_1 + 5C_2 = 7, \end{gathered} \tag{1.26}$$

the solution to which being $C_1 = C_2 = 1$. Thus the solution to the initial value problem is

$$y = e^{2t}\left(\cos 5t + \sin 5t\right). \tag{1.27}$$

6. Given that $y_1 = t^2 + t^3$ is a solution of

$$t\left(1+t\right)^2 y'' - \left(1+3t\right)\left(1+t\right) y' + \left(1+3t\right) y = 0, \tag{1.28}$$

find a second, linearly independent solution y_2 of (1.28) ***by making the D'Alembert ansatz*** $y_2 = y_1 v = \left(t^2 + t^3\right) v$. (This is "reduction of order".)

14 points

Solution

First we check that $y_1 = t^2 + t^3$ is a solution of(1.28): we have

$$\begin{aligned} & t\left(1+t\right)^2 y'' - \left(1+3t\right)\left(1+t\right) y' + \left(1+3t\right) y \Big|_{y=y_1=t^2+t^3} \\ &= t\left(1+t\right)^2\left(2+6t\right) - \left(1+3t\right)\left(1+t\right)\left(2t+3t^2\right) + \left(1+3t\right)\left(t^2+t^3\right) \\ &= 2t\left(1+t\right)^2\left(1+3t\right) - t\left(1+3t\right)\left(1+t\right)\left(2+3t\right) + t^2\left(1+3t\right)\left(1+t\right) \\ &= t\left(1+t\right)\left(1+3t\right)\left(2\left(1+t\right) - \left(2+3t\right) + t\right) = t\left(1+t\right)\left(1+3t\right)\cdot 0 = 0 \end{aligned} \tag{1.29}$$

as advertised. Using D'Alembert's ansatz $y_2 = y_1 v = \left(t^2 + t^3\right) v$ we find

$$\begin{aligned} y_2 &= \left(t^2+t^3\right)v, \\ y_2' &= \left(2t+3t^2\right)v+\left(t^2+t^3\right)v', \text{ and} \\ y_2'' &= (2+6t)v+\left(4t+6t^2\right)v'+\left(t^2+t^3\right)v'', \end{aligned} \tag{1.30}$$

or, as appropriately weighted,

$$\begin{aligned} (1+3t)\,y_2 &= (1+3t)\left(t^2+t^3\right)v, \\ -(1+3t)(1+t)\,y_2' &= -(1+3t)(1+t)\left(2t+3t^2\right)v-(1+3t)(1+t)\left(t^2+t^3\right)v', \text{ and} \\ t(1+t)^2\,y_2'' &= t(1+t)^2(2+6t)v+ \quad t(1+t)^2\left(4t+6t^2\right)v'+t(1+t)^2\left(t^2+t^3\right)v''. \end{aligned} \tag{1.31}$$

Thus we demand that

$$\begin{aligned} 0 &= \Big((1+3t)\left(t^2+t^3\right)-(1+3t)(1+t)\left(2t+3t^2\right)+t(1+t)^2(2+6t)\Big)v \\ &\quad +\Big(-(1+3t)(1+t)\left(t^2+t^3\right)+\ t(1+t)^2\left(4t+6t^2\right)\Big)v'+t(1+t)^2\left(t^2+t^3\right)v'' \\ &= \Big(t^2(1+t)(1+3t)-t(1+t)(1+3t)(2+3t)+2t(1+t)^2(1+3t)\Big)v \\ &\quad +\Big(-t^2(1+3t)(1+t)^2+\ 2t^2(1+t)^2(2+3t)\Big)v'+t^3(1+t)^2(1+t)v'' \\ &= t(1+t)(1+3t)\cdot 0v+t^2(1+t)^2\big(-(1+3t)+\ 2(2+3t)\big)v'+t^3(1+t)^2(1+t)v'' \\ &= t^2(1+t)^2(3+3t)v'+t^3(1+t)^2(1+t)v'' \\ &= 3t^2(1+t)^3v'+t^3(1+t)^3v'', \end{aligned} \tag{1.32}$$

or, equivalently, that

$$0 = 3u+tu'. \tag{1.33}$$

In (1.33) we introduced $u = v'$. Separating the last equation gives

$$d\log u = \frac{du}{u} = -3\frac{dt}{t} = -3d(\log t) = d(-3\log t) = d\left(\log t^{-3}\right) \tag{1.34}$$

so that we have at least one solution u of (1.33) is evidently

$$\frac{d}{dt}v = v' = u = t^{-3} = \frac{d}{dt}\left(-\frac{1}{2}t^{-2}\right). \qquad (1.35)$$

A solution v of the latter equation is evidently

$$v = -\frac{1}{2}t^{-2}, \qquad (1.36)$$

so that finally we have

$$y_2 = v\left(t^2 + t^3\right) = -\frac{1}{2}t^{-2}\left(t^2 + t^3\right) = -\frac{1}{2}\left(1 + t\right) \qquad (1.37)$$

is another solution of (1.28), one that is linearly independent of $y_1 = t^2 + t^3$. By linearity, we could also get the simpler solution

$$y_2 = 1 + t. \qquad (1.38)$$

7. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' + 3xy' + y = 0\,. \qquad (1.39)$$

16 points

Solution

The differential equation (1.39) defines a linear differential operator L_x, in terms of which (1.39) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \left(r(r-1) + 3r + 1\right)x^r = \left(r^2 + 2r + 1\right)x^r = \left(r + 1\right)^2 x^r, \qquad (1.40)$$

so that a solution of (1.39) is clearly then $y_{-1} = x^{-1}$. To find the general solution to this second order differential equation we need to find a second, linearly independent solution. Since the ansatz $y_r = x^r$ only produces solutions dependent upon $y_{-1} = x^{-1}$, we must use another ansatz. Fortunately the structure of(1.40), together with the fact that the differential operators $\frac{d}{dr}$ and L_x commute, suggest such an alternative ansatz: applying $\frac{d}{dr}$ to both sides of(1.40), and using the indicated commutivity of $\frac{d}{dr}$ and L_x, one obtains

$$L_x[\frac{d}{dr}y_r] = (r+1)^2 x^r \ln x + 2(r+1)^1 x^r \,, \tag{1.41}$$

so that $\frac{d}{dr}y_r\Big|_{r=-1} = x^r \ln x\Big|_{r=-1} = x^{-1}\ln x$ is clearly a second, linearly independent solution of (1.39). Thus the general solution to this linear homogeneous equation is

$$y = x^{-1}(A + B\ln x). \tag{1.42}$$

8. Find the first *two* ***nonzero*** terms (if there are that many) in the series representation of *each* of 2 linearly independent solutions of the equation

$$(3x+6x^2)y'' + (2-2x)y' + 2y = 0 \tag{1.43}$$

about the point $x_0 = 0$.

18 points

Solution

The point $x_0 = 0$ is a singular point, so that the required series solution may not be a Taylor series: insert $y = \sum_n a_n x^{n+r}$ (with the assumption that $a_n = 0$ for $n < 0$, and that the sum is over the integers, and that $a_0 \neq 0$) in (1.43) to obtain

$$
\begin{aligned}
0 &= \sum_n \left(3x+6x^2\right)\left(n+r\right)(n+r-1)a_n x^{n+r-2} + \left(2-2x\right)\left(n+r\right)a_n x^{n+r-1} + 2a_n x^{n+r} \\
&= \sum_n \left\{ \begin{array}{ll} 3\left(n+r\right)(n+r-1)a_n x^{n+r-1} & +6\left(n+r\right)(n+r-1)a_n x^{n+r} \\ \qquad +2\left(n+r\right)a_n x^{n+r-1} & -2\left(n+r\right)a_n x^{n+r} \\ & +2a_n x^{n+r} \end{array} \right\} \\
&= \sum_n \left(n+r\right)\left[3(n+r-1)+2\right]a_n x^{n+r-1} + \left[6\left(n+r\right)(n+r-1) - 2\left(n+r\right)+2\right]a_n x^{n+r} \\
&= \sum_n \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left[3\left(n+r\right)^2 - 4\left(n+r\right)+1\right]a_n \right\} x^{n+r} \\
&= \sum_n \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left(n+r-1\right)\left[3(n+r)-1\right]a_n \right\} x^{n+r} \\
&= \sum_{n=-2}^{-\infty} \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]\cancel{a_{n+1}} + 2\left(n+r-1\right)\left[3(n+r)-1\right]\cancel{a_n} \right\} x^{n+r} + \\
&\quad r\left(3r-1\right)a_{-1+1}x^{-1+r} + 2\left(r-2\right)\left(3n-4\right)\cancel{a_{-1}} + \\
&\quad \sum_{n=0}^{\infty} \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left(n+r-1\right)\left[3(n+r)-1\right]a_n \right\} x^{n+r} \\
&= \sum_{n=-2}^{-\infty} \left\{0+0\right\} x^{n+r} \\
&\quad + r\left(3r-1\right)a_0 x^{-1+r} + 0 + \sum_{n=0}^{\infty} \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left(n+r-1\right)\left[3(n+r)-1\right]a_n \right\} x^{n+r} \\
&= r\left(3r-1\right)a_0 x^{-1+r} + \sum_{n=0}^{\infty} \left\{ \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left(n+r-1\right)\left[3(n+r)-1\right]a_n \right\} x^{n+r} .
\end{aligned}
\tag{1.44}
$$

Since the functions $x \mapsto x^{n+r}$, $n = -1, 0, 1, 2, \ldots$, are linearly independent, (1.44) holding over an interval of x's demands that

$$
\begin{aligned}
&0 = r\left(3r-1\right)a_0, \\
&0 = \left(n+r+1\right)\left[3(n+r)+2\right]a_{n+1} + 2\left(n+r-1\right)\left[3(n+r)-1\right]a_n,\ n = 0, 1, \ldots .
\end{aligned}
\tag{1.45}
$$

Since by convention we choose $a_0 \neq 0$, we evidently require in (1.45) that

$$0 = r\left(3r-1\right) \tag{1.46}$$

which gives the roots

$$r = 0, \frac{1}{3}, \tag{1.47}$$

and which we must treat separately. For $r = 0$ (1.45) becomes

$$\begin{aligned} & n = 0,1,\ldots, \\ & 0 = (n+1)(3n+2)a_{n+1} + 2(n-1)(3n-1)a_n, \\ & \Leftrightarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.48) \\ & a_{n+1} = -\frac{2(n-1)(3n-1)}{(n+1)(3n+2)}a_n,\ n = 0,1,\ldots. \end{aligned}$$

Thus we have

$$\begin{aligned} a_1 = a_{0+1} &= -\frac{2(0-1)(3\cdot 0-1)}{(0+1)(3\cdot 0+2)}a_0 = -a_0, \\ a_2 = a_{1+1} &= -\frac{2(1-1)(3\cdot 1-1)}{(1+1)(3\cdot 1+2)}a_1 = 0, \end{aligned} \tag{1.49}$$

and then all the other a's are zero also by (1.48). Thus the "infinite" series terminates, and we get one solution of (1.43) as

$$y = \sum_n a_n x^{n+r} = a_0 x^r + a_1 x^{r+1} = a_0 x^0 - a_0 x^{0+1} = a_0 (1-x). \tag{1.50}$$

Starting over with $r = \frac{1}{3}$ we see that (1.45) becomes

$$\begin{aligned} & n = 0,1,\ldots, \\ & 0 = \left(n+\frac{1}{3}+1\right)\left[3(n+\frac{1}{3})+2\right]a_{n+1} + 2\left(n+\frac{1}{3}-1\right)\left[3(n+\frac{1}{3})-1\right]a_n \\ & \quad = (3n+4)(n+1)a_{n+1} + 2n(3n-2)a_n,\ n = 0,1,\ldots. \end{aligned} \tag{1.51}$$

$$\Leftrightarrow$$

$$a_{n+1} = -\frac{2n(3n-2)}{(3n+4)(n+1)}a_n,\ n = 0,1,\ldots.$$

Thus we have

$$a_1 = a_{0+1} = -\frac{2\cdot 0\cdot(3\cdot 0-2)}{(3\cdot 0+4)(0+1)}a_0 = 0, \tag{1.52}$$

and then all the other a's are zero again. Thus the "infinite" series terminates again, and we get

$$y = \sum_n a_n x^{n+r} = a_0 x^r = a_0 x^{1/3}. \tag{1.53}$$

We note then that the general solution to (1.43) is, from (1.50) and(1.53),

$$y = c_1 x^{1/3} + c_2 (1 - x) \tag{1.54}$$

for $x > 0$.

9. Find the ***general solution*** of the following linear but non-homogeneous differential equation *by the method of variation of parameters*. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the "D'Alembert-like" ansatz that leads to that formula. (Also, do not use the method of undetermined coefficients.)

$$y'' + 2y' + y = t^2 e^{-t} \tag{1.55}$$

22 points

Solution

The characteristic equation of the homogeneous version of the constant coefficient differential equation (1.55) is

$$0 = r^2 + 2r + 1 = (r+1)^2 \tag{1.56}$$

so that the general solution of the corresponding homogeneous equation is

$$y = Ae^{-t} + Bte^{-t}, \tag{1.57}$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (1.57), we have also an ansatz there for the solution of the non-homogeneous equation (1.55): with such an ansatz one immediately has

$$y' = -Ae^{-t} + B(-t+1)e^{-t} + \left(A'e^{-t} + B'te^{-t}\right). \tag{1.58}$$

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$A'e^{-t} + B'te^{-t} = 0, \tag{1.59}$$

so that then (1.58) becomes simply

$$y' = -Ae^{-t} + B\left(-t+1\right)e^{-t} . \tag{1.60}$$

Differentiating (1.60) gives

$$y'' = Ae^{-t} + B\left(t-2\right)e^{-t} - A'e^{-t} + B'\left(-t+1\right)e^{-t} . \tag{1.61}$$

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$\begin{aligned} y &= Ae^{-t} + Bte^{-t} \\ 2y' &= -2Ae^{-t} + B\left(-2t+2\right)e^{-t} \\ y'' &= Ae^{-t} + B\left(t-2\right)e^{-t} - A'e^{-t} + B'\left(-t+1\right)e^{-t} . \end{aligned} \tag{1.62}$$

and from which it is clear that the differential equation (1.55) demands that

$$y'' + 2y' + y = -A'e^{-t} + B'\left(-t+1\right)e^{-t} = t^2 e^{-t} . \tag{1.63}$$

Combining this with the “consistency ansatz” (1.59) we get the system of equations

$$\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & \left(-t+1\right)e^{-t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 e^{-t} \end{bmatrix} , \tag{1.64}$$

or, equivalently,

$$\begin{bmatrix} 1 & t \\ -1 & -t+1 \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} -t^3 \\ t^2 \end{bmatrix} . \tag{1.65}$$

Solutions to (1.65) include the pair $A = -\frac{1}{4}t^4,\ B = \frac{1}{3}t^3,$ so that a solution to (1.55) is, according to(1.57),

$$y = Ae^{-t} + Bte^{-t} = -\frac{1}{4}t^4 e^{-t} + \frac{1}{3}t^3 te^{-t} = \left(-\frac{1}{4} + \frac{1}{3}\right)t^4 e^{-t} = \frac{1}{12}t^4 e^{-t} , \tag{1.66}$$

and the general solution to (1.55) is

$$y = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{12} t^4 e^{-t}, \tag{1.67}$$

where c_1 and c_2 are (truly) constants now. This agrees with(1.11).

10. Solve the initial value problem obtained from combining the differential equation of problem 9 with the initial data $y(0) = 0,\ y'(0) = 0$. In order that errors don't "cascade", I will tell you that $y = Ae^{-t} + Bte^{-t} + \frac{1}{12} t^4 e^{-t}$, is the general solution of the differential equation of problem 9. (So now if you just write down this solution to 9 without very convincing work, you will get 0 points on problem 9.) Thus, I am only testing if you understand the correct principles needed to construct the solution to the initial value problem given the general solution to the associated differential equation.

20 points

Solution

From the information given we have

$$\begin{aligned} y(0) = 0 = Ae^{-t} + Bte^{-t} + \frac{1}{12} t^4 e^{-t}, \Big|_{t=0} &= A \\ y'(0) = 0 = \left(-A + B - Bt + \frac{1}{3} t^3 - \frac{1}{12} t^4 \right) e^{-t} \Big|_{t=0} &= -A + B, \end{aligned} \tag{1.68}$$

the solution to which being

$$A = B = 0\,. \tag{1.69}$$

and the solution sought is

$$y = Ae^{-t} + Bte^{-t} + \frac{1}{12} t^4 e^{-t} \Big|_{A=B=0} = \frac{1}{12} t^4 e^{-t}\,. \tag{1.70}$$