KEY

Math 334 Midterm III
Fall 2008
sections 001 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering paper. Write your name, course, and section number on the blue book, or on your own pile of papers. Again, do not write this or any other type of information on this exam.

Good Practice: A differential equation has the property that one can check whether a given function satisfies it. So check your solutions! If one doesn’t work, try again, or at least note that your proposed solution doesn’t work. You will get much more partial credit if you know whether or not you answer is correct.

1. Compute the Laplace transform $\mathcal{L}[f](s)$ of $f(t)=e^{at}\cos t$ directly from the definition. For which values of s does your formula hold? Assume here that a is a real number.

12 points

Solution

By definition of the Laplace transform, and by using integration by parts, and provided $s>a$, we get

$$\begin{aligned}\mathcal{L}[f](s)&=\int_0^{+\infty}e^{-st}f(t)dt=\int_0^{+\infty}e^{-st}e^{at}\cos t dt=\int_0^{+\infty}e^{-(s-a)t}\cos t dt=\frac{-1}{s-a}\int_0^{+\infty}\cos t de^{-(s-a)t}\\&=\frac{-1}{s-a}\left\{\cos te^{-(s-a)t}\Big|_0^{+\infty}-\int_0^{+\infty}e^{-(s-a)t}d\cos t\right\}\\&=\frac{-1}{s-a}\left\{-1+\int_0^{+\infty}e^{-(s-a)t}\sin t dt\right\}=\frac{-1}{s-a}\left\{-1-\frac{1}{s-a}\int_0^{+\infty}\sin t de^{-(s-a)t}\right\}\\&=\frac{-1}{s-a}\left\{-1-\frac{1}{s-a}\left\{\sin te^{-(s-a)t}\Big|_0^{+\infty}-\int_0^{+\infty}e^{-(s-a)t}d\sin t\right\}\right\}\\&=\frac{-1}{s-a}\left\{-1+\frac{1}{s-a}\int_0^{+\infty}\cos te^{-(s-a)t}dt\right\}=\frac{-1}{s-a}\left\{-1+\frac{1}{s-a}\mathcal{L}[f](s)\right\}.\end{aligned}\qquad(1.1)$$

Note that $s>a$ is also necessary for the integral to converge. Solving the last equation algebraically for $\mathcal{L}[f](s)$ we get

$$\mathcal{L}[f](s)=\frac{\frac{-1}{s-a}\{-1\}}{1-\frac{-1}{s-a}\left\{\frac{1}{s-a}\right\}}=\frac{\frac{1}{s-a}}{1+\frac{1}{(s-a)^2}}=\frac{s-a}{(s-a)^2+1}.\qquad(1.2)$$

2. Solve the following IVP. Using the Laplace transform and knowing how to deal with piecewise defined functions in this transform should make things easier. What is the value of $y(10)$? Do not express this value in terms of an abstract formula, but rather as a concrete number. To compute this value concretely, you will need to know that the function $\sin(t)$ is 2π periodic.

$$y''+4y=f(t)=\begin{cases} 0,\ t<0 \\ t,\ 0\le t<\pi \\ 2\pi-t,\ \pi\le t<2\pi \\ 0, t\ge 2\pi \end{cases},\quad y(0)=y'(0)=0. \tag{1.3}$$

25 points

Solution

Taking the Laplace transform of (1.3) we get

$$\begin{gathered} \left(s^2+2^2\right)\mathcal{L}\left[y\right](s)=\mathcal{L}\left[y''+4y\right](s)=\mathcal{L}\left[f\right](s) \\ \Leftrightarrow \\ \mathcal{L}\left[y\right](s)=\frac{1}{s^2+2^2}\mathcal{L}\left[f\right](s). \end{gathered}, \tag{1.4}$$

To compute $\mathcal{L}\left[f\right](s)$ we rewrite f in (1.3) as:

$$\begin{aligned} f(t) &= t\left(u_0(t)-u_\pi(t)\right)+\left(2\pi-t\right)\left(u_\pi(t)-u_{2\pi}(t)\right) \\ &= t\ u_0(t)+\left(2\pi-2t\right)u_{2\pi}(t)+\left(t-2\pi\right)u_{2\pi}(t) \\ &= \left(t-0\right)\ u_0(t)-2\left(t-\pi\right)u_{2\pi}(t)+\left(t-2\pi\right)u_{4\pi}(t). \end{aligned} \tag{1.5}$$

So, according to the relevant theorem, and given $\mathcal{L}\left[g\right](s)=\frac{1}{s^2}$ when $g(t)=t\ u_0(t)$ (as per the table provided you), then

$$\begin{aligned} \mathcal{L}\left[f\right](s) &= \mathcal{L}\left[g\right](s)-2e^{-\pi s}\mathcal{L}\left[g\right](s)+e^{-2\pi s}\mathcal{L}\left[g\right](s) \\ &= \mathcal{L}\left[g\right](s)\left(1-2e^{-\pi s}+e^{-2\pi s}\right)=\frac{1}{s^2}\left(1-2e^{-\pi s}+e^{-2\pi s}\right). \end{aligned} \tag{1.6}$$

Thus, explicitly, (1.4) is

$$\mathcal{L}\left[y\right](s)=\frac{1}{s^2+2^2}\mathcal{L}\left[f\right](s)=\frac{1}{s^2+2^2}\frac{1}{s^2}\left(1-2e^{-\pi s}+e^{-2\pi s}\right). \tag{1.7}$$

and we already know then that the solution is of the form

$$y(t) = h(t)u_0(t) - 2h(t-\pi)u_\pi(t) + h(t-2\pi)u_{2\pi}(t), \tag{1.8}$$

where then we need only find $h(t)$, whose Laplace transform is

$$\begin{aligned} \mathcal{L}[h](s) &= \frac{1}{s^2+2^2}\cdot\frac{1}{s^2} = \frac{1}{s^2+2^2}\cdot\frac{1}{-2^2} + \frac{1}{0+2^2}\cdot\frac{1}{s^2} \\ &= \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+2^2}\right). \end{aligned} \tag{1.9}$$

Thus, according to the table, or memorized formulae, we have that the solution is given by (1.8) and by

$$h(t) = \frac{1}{4}\left(t - \frac{1}{2}\sin(2t)\right). \tag{1.10}$$

Note that for any $t \geq 2\pi$, including $t = 10$, we have

$$\begin{aligned} y(t) &= h(t)u_0(t) - 2h(t-\pi)u_\pi(t) + h(t-2\pi)u_{2\pi}(t), \\ &= h(t) - 2h(t-\pi) + h(t-2\pi) \\ &= \frac{1}{4}\left(t - \frac{1}{2}\sin(2t) - 2\left(t-\pi-\frac{1}{2}\sin\left(2\left(t-\pi\right)\right)\right) + t - 2\pi - \frac{1}{2}\sin\left(2\left(t-2\pi\right)\right)\right) \\ &= \frac{1}{4}\left(t - \frac{1}{2}\sin(2t) - 2\left(t-\pi-\frac{1}{2}\sin\left(2t\right)\right) + t - 2\pi - \frac{1}{2}\sin\left(2t\right)\right) \\ &= 0. \end{aligned} \tag{1.11}$$

3. Solve the following initial value problem in terms of the convolution integral:

$$\left(D-1\right)\left(D+1\right)y = g(t), \quad y(0) = y'(0) = 0. \tag{1.12}$$

Here I expect you to write the solution $y = y(t)$ of (1.12) as

$$y(t) = \left(y_\delta * g\right)(t) = \int_0^t y_\delta(t-\tau)g(\tau)d\tau, \tag{1.13}$$

so that the problem is effectively only to determine the function $y_\delta(t)$ in (1.13). Recall $\left(D-1\right)\left(D+1\right)y = \left(D^2-1^2\right)y = y'' - y$, but realize the representation in (1.12) may be at least as useful as writing $y'' - y$ on the left-hand side. (Hint: Use the Laplace transform and the convolution theorem. Note the point value—this is straightforward if you recall the relevant theory.)

9 points

Solution

By the Laplace Transform we have from (1.12) that

$$(s-1)(s+1)\mathcal{L}[y](s) = \mathcal{L}\big[(D-1)(D+1)y\big](s) = \mathcal{L}[g](s)$$

$$\Leftrightarrow$$

$$\begin{aligned}\mathcal{L}[y](s) &= \frac{1}{(s-1)(s+1)}\mathcal{L}[g](s) = \left\{\frac{1}{(s-1)(1+1)} + \frac{1}{(-1-1)(s+1)}\right\}\mathcal{L}[g](s) \quad (1.14)\\ &= \mathcal{L}\left[\frac{1}{2}\left(e^{t}-e^{-t}\right)\right](s)\mathcal{L}[g](s) = \mathcal{L}[\sinh](s)\mathcal{L}[g](s)\\ &= \mathcal{L}[\sinh * g]\end{aligned}$$

so that evidently in (1.13) we have

$$y_{\delta}(t) = \sinh(t). \tag{1.15}$$

4. Suppose that both

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ \sin t \end{bmatrix} \text{ and } \mathbf{x}_2(t) = \begin{bmatrix} \sin t \\ 1 \end{bmatrix} \tag{1.16}$$

solve the system

$$\mathbf{x}' = \begin{bmatrix} -\tan t & \sec t \\ \sec t & -\tan t \end{bmatrix}\mathbf{x} \tag{1.17}$$

and that

$$\mathbf{x}_p(t) = \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix} \tag{1.18}$$

solves the system

$$\mathbf{x}' = \begin{bmatrix} -\tan t & \sec t \\ \sec t & -\tan t \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{1.19}$$

Then what is the solution of the IVP

$$\mathbf{x}' = \begin{bmatrix} -\tan t & \sec t \\ \sec t & -\tan t \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}? \tag{1.20}$$

This problem tests your general understanding of the structure of solutions of linear systems without requiring you to regurgitate a theorem (with all of the intricacies of its articulation). Note the point value—this problem is straightforward if you recall the theory.

12 points

Solution

According to the information given, the general solution of (1.20) can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} 1 & \sin t \\ \sin t & 1 \end{bmatrix} \mathbf{c} \ + \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix}, \tag{1.21}$$

Choosing $t = 0$ and using the initial data in(1.20), (1.21) becomes

$$\begin{aligned} &\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{c} \ + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{c} \ + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &\Leftrightarrow \\ &\mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{1.22}$$

So (1.21) becomes

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} 1 & \sin t \\ \sin t & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ + \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix} = \begin{bmatrix} \sin t \\ 1 \end{bmatrix} + \begin{bmatrix} \cos t \\ -\cos t \end{bmatrix} = \begin{bmatrix} \sin t + \cos t \\ 1 - \cos t \end{bmatrix}. \tag{1.23}$$

5. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ to the system

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{x}, \tag{1.24}$$

the one that has the property that $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

21 points

Solution

First we find a representation of the general solution of the system(1.24). This can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_+ e^{\lambda_+ t} + c_2 \boldsymbol{\xi}_- e^{\lambda_- t}, \tag{1.25}$$

provided the $\boldsymbol{\xi}$'s are independent eigenvectors associated with the eigenvalues λ_+ and λ_- of the matrix in (1.24). To determine this we note

$$\begin{bmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = \lambda^2 - 1 + 2 = \lambda^2 + 1^2 \tag{1.26}$$

$$\Leftrightarrow$$

$$\lambda = \pm i = +i, -i =: \lambda_+, \lambda_-.$$

So

$$\mathbf{0} = \begin{bmatrix} -1-(\pm i) & 1 \\ -2 & 1-(\pm i) \end{bmatrix} \boldsymbol{\xi}_\pm = \begin{bmatrix} -1\mp i & 1 \\ -2 & 1\mp i \end{bmatrix} \boldsymbol{\xi}_\pm \Leftarrow \boldsymbol{\xi}_\pm = \begin{bmatrix} 1 \\ 1\pm i \end{bmatrix}. \tag{1.27}$$

Thus, explicitly, (1.25) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{-it}. \tag{1.28}$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) := \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{it} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}, \tag{1.29}$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}. \tag{1.30}$$

A fundamental matrix of solutions $\Psi = \Psi(t)$, one not necessarily having the desired property, can be found from the above general solution (1.30) as in

$$\Psi(t) = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}. \tag{1.31}$$

The desired fundamental matrix $\Phi = \Phi(t)$ can be obtained from $\Psi = \Psi(t)$ via

$$\begin{aligned} \Phi = \Phi(t) = \Psi(t)\Psi^{-1}(0) &= \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix}. \end{aligned} \tag{1.32}$$

6. Solve the initial value problem given by the system of problem 5 and the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.33}$$

6 points

Solution

Using the fundamental matrix of problem 5 we have

$$\begin{aligned} \mathbf{x}(t) = \Phi(t)\mathbf{x}(0) &= \begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1(\cos t - \sin t) + 1(\sin t) \\ 1(-2\sin t) + 1(\cos t + \sin t) \end{bmatrix} \\ &= \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}. \end{aligned} \tag{1.34}$$

7. Calculate

$$e^{\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\pi}. \tag{1.35}$$

6 points

Solution

We have, from problem 5 and the general theory,

$$e^{\begin{bmatrix}-1 & 1\\ -2 & 1\end{bmatrix}\pi} = \Phi(\pi) = \begin{bmatrix}\cos\pi - \sin\pi & \sin\pi \\ -2\sin\pi & \cos\pi + \sin\pi\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = -\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \quad (1.36)$$

8. Find a representation of the general solution of the system

$$\mathbf{x}' = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\mathbf{x}\,. \quad (1.37)$$

21 points

Solution

The matrix in (1.37) has a repeated eigenvalue with only one eigenvector. Hence the general solution is of the form

$$\mathbf{x} = \mathbf{x}(t) = c_1\boldsymbol{\xi}e^{\lambda t} + c_2\left(\boldsymbol{\xi}t + \boldsymbol{\eta}\right)e^{\lambda t} \quad (1.38)$$

where λ is the sole eigenvalue, $\boldsymbol{\xi}$ is one of its eigenvectors, and $\boldsymbol{\eta}$ is an associated pseudo eigenvector: First

$$\begin{bmatrix}1-\lambda & 1\\ -1 & -1-\lambda\end{bmatrix}\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det\begin{bmatrix}1-\lambda & 1\\ -1 & -1-\lambda\end{bmatrix} = \lambda^2 - 1 + 1 = \lambda^2 \quad (1.39)$$

$$\Leftrightarrow$$

$$\lambda = 0, 0.$$

So

$$\mathbf{0} = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\boldsymbol{\xi} \Leftarrow \boldsymbol{\xi} = \begin{bmatrix}1\\ -1\end{bmatrix},$$

and (1.40)

$$\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\boldsymbol{\eta} = \boldsymbol{\xi} = \begin{bmatrix}1\\ -1\end{bmatrix} \Leftrightarrow \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{\eta} = \begin{bmatrix}1\\ 0\end{bmatrix} \Leftrightarrow \boldsymbol{\eta} = \begin{bmatrix}\eta_1\\ 1-\eta_1\end{bmatrix}.$$

Thus, explicitly, (1.38) is

$$\begin{aligned}\mathbf{x} = \mathbf{x}(t) &= c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{0t} + c_2\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}\eta_1\\1-\eta_1\end{bmatrix}\right)e^{0t}\\ &= c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}\eta_1\\1-\eta_1\end{bmatrix}\right).\end{aligned} \tag{1.41}$$

9. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ for the system of problem 8 that satisfies $\Phi(0) = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

15 points

<u>Solution</u>

From (1.41) we have a fundamental matrix of solutions

$$\Psi(t) = \begin{bmatrix}1 & t+\eta_1\\-1 & -t+1-\eta_1\end{bmatrix}, \tag{1.42}$$

whence the one desired is

$$\begin{aligned}\Phi(t) = \Psi(t)\Psi^{-1}(0) = \Psi(t) &= \begin{bmatrix}1 & t+\eta_1\\-1 & -t+1-\eta_1\end{bmatrix}\begin{bmatrix}1 & \eta_1\\-1 & 1-\eta_1\end{bmatrix}^{-1}\\ &= \begin{bmatrix}1 & t+\eta_1\\-1 & -t+1-\eta_1\end{bmatrix}\frac{1}{1-\eta_1+\eta_1}\begin{bmatrix}1-\eta_1 & -\eta_1\\1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & t+\eta_1\\-1 & -t+1-\eta_1\end{bmatrix}\begin{bmatrix}1-\eta_1 & -\eta_1\\1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1\cdot(1-\eta_1)+(t+\eta_1)\cdot 1 & 1\cdot(-\eta_1)+(t+\eta_1)\cdot 1\\-1\cdot(1-\eta_1)+(-t+1-\eta_1)\cdot 1 & -1\cdot(-\eta_1)+(-t+1-\eta_1)\cdot 1\end{bmatrix}\\ &= \begin{bmatrix}1+t & t\\-t & 1-t\end{bmatrix}.\end{aligned} \tag{1.43}$$

10. Solve the initial value problem obtained from combining the differential equation of problem 8 with the initial data

$$\mathbf{x}(0) = \begin{pmatrix}2\\-2\end{pmatrix}. \tag{1.44}$$

6 points

Solution

From problem 9 we have

$$\begin{aligned}\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) &= \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 2\begin{pmatrix} 1\cdot(1+t)-1\cdot t \\ 1\cdot(-t)-1\cdot(1-t) \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \mathbf{x}(0).\end{aligned} \tag{1.45}$$