KEY

Math 334 Final
Instructor: Scott Glasgow
Sections: 1 and 2
Dates: August 10, 2005

Instructions: Use your own paper, be very neat, be painfully clear (use lots of English sentences), use large fonts and, so, lots of paper, and enjoy! If you finish before the end of the period, check your answers. Do this as many times as possible.

1. Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0,\ t > 0,$$

find a second, linearly independent solution $y_2(t)$.

Solution

The variation of parameter-like ansatz $y = y_2 = y_1 u$ (which leads to a reduction of order), gives

$$\begin{aligned}
0 &= 2t^2 y'' + 3ty' - y = 2t^2 \left(y_1 u\right)'' + 3t\left(y_1 u\right)' - y_1 u \\
&= 2t^2 \left(t^{-1}u\right)'' + 3t\left(t^{-1}u\right)' - t^{-1}u = 2t^2\left(t^{-1}u' - t^{-2}u\right)' + 3t\left(t^{-1}u' - t^{-2}u\right) - t^{-1}u \\
&= 2t^2\left(t^{-1}u'' - 2t^{-2}u' + 2t^{-3}u\right) + 3u' - 3t^{-1}u - t^{-1}u \\
&= 2tu'' - 4u' + 4t^{-1}u + 3u' - 3t^{-1}u - t^{-1}u \\
&= 2tu'' - u' \\
&\Leftrightarrow
\end{aligned}$$

$$\frac{u''}{u'} = \frac{1}{2t} \Leftarrow \ln|u'| = \frac{1}{2}\ln\left(\frac{3}{2}\right)^2 t \Leftarrow u' = \frac{3}{2}t^{1/2} \Leftarrow u = t^{3/2}.$$

Thus $y_2 = y_1 u = t^{-1}t^{3/2} = t^{1/2}$ is a second, linearly independent solution.

22 Points

2. Find an integrating factor for the equation

$$\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$$

and then find the general solution of the resulting exact equation.

Solution

Defining $M = 3xy + y^2$ and $N = x^2 + xy$ one notes that $M_y = 3x + 2y \neq 2x + y = N_x$, so that the equation is not exact as is. But since

$$\frac{M_y - N_x}{N} = \frac{x + y}{x^2 + xy} = \frac{1}{x}$$

is a pure function of x, one knows that multiplication by the integrating factor $\mu = \mu(x)$ satisfying

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{1}{x}\mu \Leftrightarrow$$
$$\frac{d\mu}{\mu} = \frac{dx}{x} \Leftarrow \mu = x$$

will render the equation exact: for multiplication by $\mu = x$ gives the exact equation

$$d\Psi = \left(3x^2 y + xy^2\right)dx + \left(x^3 + x^2 y\right)dy = 0$$

with potential $\Psi = \Psi(x, y)$ satisfying

$$\Psi_x = 3x^2 y + xy^2 \Leftrightarrow \Psi = x^3 y + \frac{1}{2}x^2 y^2 + g(y)$$
$$\Psi_y = x^3 + x^2 y \Leftrightarrow \Psi = x^3 y + \frac{1}{2}x^2 y^2 + h(x).$$

Comparison of the two versions of $\Psi = \Psi(x, y)$ leads to a choice $\Psi(x, y) = x^3 y + \frac{1}{2}x^2 y^2$, and ultimately to the general solution $x^3 y + \frac{1}{2}x^2 y^2 = C$ with C an arbitrary constant (independent of x and y).

28 Points

3. Solve the initial value problem

$$y'' + 4y = \frac{1}{5}\left\{\left(t - 5\right)u_5(t) - \left(t - 10\right)u_{10}(t)\right\};\ y(0) = y'(0) = 0\,.$$

Solution

Laplace transformation gives

$$\left(s^2+2^2\right)\mathcal{L}[y](s)=\frac{1}{5}\left\{\frac{e^{-5s}}{s^2}-\frac{e^{-10s}}{s^2}\right\}=\frac{e^{-5s}-e^{-10s}}{5}\cdot\frac{1}{s^2}$$

$$\Leftrightarrow$$

$$\mathcal{L}[y](s)=\frac{e^{-5s}-e^{-10s}}{5}\frac{1}{s^2\left(s^2+2^2\right)}=\frac{e^{-5s}-e^{-10s}}{5}\left\{\frac{1}{s^2\left(0+2^2\right)}+\frac{1}{-2^2\left(s^2+2^2\right)}\right\}$$

$$=\frac{e^{-5s}-e^{-10s}}{5}\left\{\frac{1}{4}\frac{1}{s^2}-\frac{1}{8}\frac{2}{s^2+2^2}\right\}=\frac{e^{-5s}-e^{-10s}}{40}\left\{2\frac{1}{s^2}-\frac{2}{s^2+2^2}\right\}$$

$$=\frac{e^{-5s}-e^{-10s}}{40}\mathcal{L}[2t-\sin 2t](s)$$

$$\Leftrightarrow$$

$$y(t)=\frac{1}{40}\left\{[2(t-5)-\sin 2(t-5)]u_5(t)-[2(t-10)-\sin 2(t-10)]u_{10}(t)\right\}.$$

31 Points

4. Find a fundamental matrix of real-valued solutions $\Psi(t)$ for the system

$$\underline{x}'=A\underline{x}=\begin{bmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{bmatrix}\underline{x}.$$

<u>Solution</u>

The ansatz $\underline{x}(t)=\underline{\xi}e^{\lambda t}$ above leads to the equation

$$\left(A-\lambda I\right)\underline{\xi}=\underline{0}\Leftrightarrow\underline{\xi}=\underline{0},\text{ unless}$$

$$0=\det[A-\lambda I]=\det\begin{bmatrix}-\frac{1}{2}-\lambda & 1\\ -1 & -\frac{1}{2}-\lambda\end{bmatrix}=\left(\lambda+\frac{1}{2}\right)^2+1$$

$$\Leftrightarrow\lambda=\lambda_\pm=-\frac{1}{2}\pm i.$$

In the latter cases

$$\underline{\xi} = \underline{\xi}_{\pm} \in Nul\left[A - \left(-\frac{1}{2} \pm i\right)I\right] = Nul\begin{bmatrix} -\frac{1}{2} - \left(-\frac{1}{2} \pm i\right) & 1 \\ -1 & -\frac{1}{2} - \left(-\frac{1}{2} \pm i\right) \end{bmatrix},$$

$$= Nul\begin{bmatrix} \mp i & 1 \\ -1 & \mp i \end{bmatrix} = Span\left\{\begin{pmatrix} 1 \\ \pm i \end{pmatrix}\right\}$$

giving the nontrivial solutions

$$\underline{x}(t) = \underline{\xi}_{\pm} e^{\lambda_{\pm} t} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix} e^{\left(-\frac{1}{2} \pm i\right)t} = e^{-\frac{1}{2}t}\begin{bmatrix} 1 \\ \pm i \end{bmatrix} e^{\pm it} = e^{-\frac{1}{2}t}\begin{bmatrix} 1 \\ \pm i \end{bmatrix}\left(\cos t \pm i \sin t\right)$$

$$= e^{-\frac{1}{2}t}\begin{bmatrix} \cos t \pm i \sin t \\ \pm i \cos t - \sin t \end{bmatrix} = e^{-\frac{1}{2}t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \pm i e^{-\frac{1}{2}t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

and from which, via the usual arguments, we get a fundamental matrix of real-valued solutions of the form

$$\Psi\left(t\right) = e^{-\frac{1}{2}t}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

37 Points

5. Find a particular solution of

$$y'' + 4y = 3\csc t\,.$$

Solution

The general solution of the associated homogeneous equation is

$$y = c_1 \cos 2t + c_2 \sin 2t\,,$$

with c_1 and c_2 free parameters. Varying these parameters by writing $c_1 = u_1 = u_1(t)$, $c_2 = u_2 = u_2(t)$, and assuming in addition that

$$\cos 2t\ u_1' + \sin 2t\ u_2' = 0$$

we get

$$\left(\cos 2t\right)'\ u_1' + \left(\sin 2t\right)'\ u_2' = -2\sin t\ u_1' + 2\cos t\ u_2' = 3\csc t\,.$$

This with the assumption gives

$$\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin t & 2\cos t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3\csc t \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin t & 2\cos t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3\csc t \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 2\cos t & -\sin 2t \\ 2\sin t & \cos 2t \end{bmatrix} \begin{bmatrix} 0 \\ \csc t \end{bmatrix}$$

$$= \frac{3}{2} \begin{bmatrix} -\csc t \sin 2t \\ \cos 2t \csc t \end{bmatrix} = \frac{3}{2} \begin{bmatrix} -2\csc t \sin t \cos t \\ \left(1 - 2\sin^2 t\right)\csc t \end{bmatrix}$$

$$= \frac{3}{2} \begin{bmatrix} -2\cos t \\ \csc t - 2\sin t \end{bmatrix}$$

$$\Leftarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} -2\sin t \\ \ln\left|\csc t - \cot t\right| + 2\cos t \end{bmatrix},$$

so that a solution to the non-homogeneous equation is

$$y = -3\sin t \cos 2t + \left(\frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\cos t\right)\sin 2t$$

$$= \frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\left(\sin 2t \cos t - \sin t \cos 2t\right)$$

$$= \frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\sin t.$$

40 Points

6. Find the general solution of

$$\underline{x}' = A\underline{x} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \underline{x}.$$

<u>Solution</u>

The ansatz $\underline{x}(t) = \underline{\xi} e^{\lambda t}$ above leads to the equation

$$\left(A - \lambda I\right)\underline{\xi} = \underline{0} \Leftrightarrow \underline{\xi} = \underline{0}, \text{ unless}$$

$$0 = \det\left[A - \lambda I\right] = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \left(3-\lambda\right)\left(1-\lambda\right) + 1$$

$$= \lambda^2 - 4\lambda + 4 = \left(\lambda - 2\right)^2 \Leftrightarrow \lambda = 2, 2.$$

In the latter case

$$\underline{\xi} \in Nul\left[A-2I\right] = Nul\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} = Nul\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\},$$

giving the nontrivial solution

$$\underline{x}(t) = \underline{\xi}e^{\lambda t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}.$$

For a second, linearly independent solution the ansatz $\underline{x}(t) = t\underline{\xi}e^{2t} + \underline{\eta}e^{2t}$ leads to the equations

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\underline{\xi} = (A-2I)\underline{\xi} = \underline{0} \Leftarrow \underline{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\underline{\eta} = (A-2I)\underline{\eta} = \underline{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Leftarrow \underline{\eta} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

giving the nontrivial solution

$$\underline{x}(t) = t\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t}.$$

Thus the general solution can be expressed as

$$\underline{x}(t) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t}\right),$$

with C_1 and C_2 arbitrary parameters.

43 Points

7. Solve

$$\underline{x}' = A\underline{x} + \underline{g}(t), \underline{x}(0) = \underline{0}$$

via the Laplace transform, where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \underline{g}(t) = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

<u>Solution</u>

Transformation of the equation gives, with the trivial initial data,

$$sL[\underline{x}](s) = AL[\underline{x}](s) + L[\underline{g}](s) \Leftrightarrow$$

$$(sI - A)L[\underline{x}](s) = L[\underline{g}](s) \Leftrightarrow$$

$$L[\underline{x}](s) = (sI - A)^{-1} L[\underline{g}](s)$$

$$= \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} = \frac{1}{(s+2)^2 - 1^2} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix}$$

$$= \frac{1}{(s+3)(s+1)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{bmatrix} = \frac{1}{(s+3)(s+1)} \begin{bmatrix} 2\frac{s+2}{s+1} + \frac{3}{s^2} \\ \frac{2}{s+1} + 3\frac{s+2}{s^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\frac{s+2}{(s+3)(s+1)^2} + \frac{3}{s^2(s+3)(s+1)} \\ \frac{2}{(s+3)(s+1)^2} + 3\frac{s+2}{s^2(s+3)(s+1)} \end{bmatrix} = \begin{bmatrix} \frac{\frac{-2}{3}}{s+3} + \frac{1}{(s+1)^2} + \frac{2}{s+1} + \frac{1}{s^2} - \frac{\frac{4}{3}}{s} \\ \frac{\frac{2}{3}}{s+3} + \frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{2}{s^2} - \frac{\frac{5}{3}}{s} \end{bmatrix}$$

$$= L\begin{bmatrix} \frac{-2}{3}e^{-3t} + te^{-t} + 2e^{-t} + t - \frac{4}{3} \\ \frac{2}{3}e^{-3t} + te^{-t} + e^{-t} + 2t - \frac{5}{3} \end{bmatrix}(s) \Leftrightarrow$$

$$\underline{x}(t) = \begin{bmatrix} \frac{-2}{3}e^{-3t} + te^{-t} + 2e^{-t} + t - \frac{4}{3} \\ \frac{2}{3}e^{-3t} + te^{-t} + e^{-t} + 2t - \frac{5}{3} \end{bmatrix}.$$

47 Points

8. Solve

$$\underline{x}' = A\underline{x} + \underline{g}(t), \underline{x}(0) = \underline{0}$$

via variation of parameters, where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \underline{g}(t) = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

Solution

Making the ansatz $\underline{x} = \Phi(t)\underline{u}$

where

$$\Phi'(t) = A\Phi(t), \Phi(0) = I \Leftrightarrow$$
$$\Phi(t) = e^{At}$$

gives

$$\underline{x}' = \Phi'(t)\underline{u} + \Phi(t)\underline{u}' = A\Phi(t)\underline{u} + \Phi(t)\underline{u}' = A\Phi(t)\underline{u} + \underline{g}(t) = A\underline{x} + \underline{g}(t) \Rightarrow$$
$$\Phi(t)\underline{u}' = \underline{g}(t) \Leftrightarrow \underline{u}' = \Phi^{-1}(t)\underline{g}(t) = \Phi(-t)\underline{g}(t) = e^{-At}\underline{g}(t)$$
$$\Leftarrow \underline{u} = \int_0^t e^{-A\tau}\underline{g}(\tau)d\tau \Rightarrow \underline{u}(0) = \underline{0}.$$

So if we can write $A = T\Lambda T^{-1}$, and since then $e^{At} = Te^{\Lambda t}T^{-1}$, the latter gives

$$\underline{x} = \Phi(t)\underline{u} = e^{At}\underline{u} = e^{At}\int_0^t e^{-A\tau}\underline{g}(\tau)d\tau = Te^{\Lambda t}T^{-1}\int_0^t Te^{-\Lambda\tau}T^{-1}\underline{g}(\tau)d\tau$$
$$= Te^{\Lambda t}\int_0^t e^{-\Lambda\tau}T^{-1}\underline{g}(\tau)d\tau.$$

Here of course we choose T to be a matrix of eigenvectors, and Λ the matrix of associated eigenvalues: we form from vectors $\underline{\xi}$ satisfying $\underline{\xi} \in Nul\left[A - \lambda I\right]$(with $0 = \det\left[A - \lambda I\right] = \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = (\lambda+2)^2 - 1 = (\lambda+1)(\lambda+3) \Leftrightarrow \lambda = -1, -3$), the matrix T. Thus, since

$$\underline{\xi}_{-1} \in Nul\left[A - (-1)I\right] = Nul\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} = Nul\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

and

$$\underline{\xi}_{-3} \in Nul\left[A - (-3)I\right] = Nul\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} = Nul\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

we can choose $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Leftrightarrow T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ to get

$$\underline{x} = Te^{\Lambda t}\int_0^t e^{-\Lambda\tau}T^{-1}\underline{g}(\tau)d\tau = Te^{\Lambda t}\int_0^t e^{-\Lambda\tau}T^{-1}\underline{g}(\tau)d\tau$$

$$= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} e^{\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}t} \int_0^t e^{-\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}\tau} \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}2e^{-\tau}\\ 3\tau\end{bmatrix}d\tau$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}\int_0^t \begin{bmatrix}e^{\tau} & 0\\ 0 & e^{3\tau}\end{bmatrix}\begin{bmatrix}2e^{-\tau}+3\tau\\ 2e^{-\tau}-3\tau\end{bmatrix}d\tau$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}\int_0^t \begin{bmatrix}2+3\tau e^{\tau}\\ 2e^{2\tau}-3\tau e^{3\tau}\end{bmatrix}d\tau$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}\left.\begin{bmatrix}2\tau+3(\tau-1)e^{\tau}\\ e^{2\tau}-\left(\tau-\frac{1}{3}\right)e^{3\tau}\end{bmatrix}\right|_0^t$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}\begin{bmatrix}2t+3(t-1)e^{t}-\left(3(0-1)\right)\\ e^{2t}-\left(t-\frac{1}{3}\right)e^{3t}-\left(1-\left(0-\frac{1}{3}\right)\right)\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}\begin{bmatrix}3+2t+3(t-1)e^{t}\\ -\frac{4}{3}+e^{2t}-\left(t-\frac{1}{3}\right)e^{3t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}3e^{-t}+2te^{-t}+3(t-1)\\ -\frac{4}{3}e^{-3t}+e^{-t}-\left(t-\frac{1}{3}\right)\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}3e^{-t}+2te^{-t}+3(t-1)-\frac{4}{3}e^{-3t}+e^{-t}-\left(t-\frac{1}{3}\right)\\ 3e^{-t}+2te^{-t}+3(t-1)+\frac{4}{3}e^{-3t}-e^{-t}+\left(t-\frac{1}{3}\right)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4e^{-t}+2te^{-t}+2t-\frac{8}{3}-\frac{4}{3}e^{-3t}\\ 2e^{-t}+2te^{-t}+4t-\frac{10}{3}+\frac{4}{3}e^{-3t}\end{bmatrix}$$

$$= \begin{bmatrix}2e^{-t}+te^{-t}+t-\frac{4}{3}-\frac{2}{3}e^{-3t}\\ e^{-t}+te^{-t}+2t-\frac{5}{3}+\frac{2}{3}e^{-3t}\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}e^{-t}-\frac{2}{3}\begin{bmatrix}1\\ -1\end{bmatrix}e^{-3t}+\begin{bmatrix}1\\ 1\end{bmatrix}te^{-t}+\begin{bmatrix}1\\ 2\end{bmatrix}t-\frac{1}{3}\begin{bmatrix}4\\ 5\end{bmatrix}.$$

52 Points