KEY

Math 334 Test 3
Instructor: Scott Glasgow
Sections: 3
Dates: April 7-9.

Instructions: Use your own paper, be very neat, be painfully clear (use lots of English sentences), use large fonts and, so, lots of paper, and enjoy!

One final, important point: check all your answers (in an alternate mode) before settling on them. I don't need to see this checking in your submitted papers (other than the instances I indicate below), but you should carefully do this.

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1. Find the Laplace transform of

$$f(t) := \int_0^t (t-\tau)^2 \cos(2\tau) d\tau.$$

4pts.

Solution:

Write

$$f(t) = t^2 * \cos(2t) = \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] * \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{2}{s^3}\frac{s}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right]$$

$$\Leftrightarrow$$

$$\mathcal{L}\left[f(t)\right] = \frac{2}{s^2(s^2+4)},$$

by the convolution and other theorems.

2. Solve the following I.V.P. in terms of a convolution via the Laplace transform method.

$$y^{(4)} + 5y^{(2)} + 4y = g(t);\ y(0) - 1 = y^{(1)}(0) = y^{(2)}(0) = y^{(3)}(0) = 0.$$

15pts.

Solution:

Transforming the differential equation with data gives

$$s^4Y - s^3 + 5(s^2Y - s) + 4Y = G$$

$$\Leftrightarrow$$

$$Y = \frac{s^3 + 5s}{s^4 + 5s^2 + 4} + G\frac{1}{s^4 + 5s^2 + 4}$$

$$=: F + GH$$

$$\Leftrightarrow$$

$$y = f + g * h,$$

where then

$$f(t) = \mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[\frac{s^3 + 5s}{s^4 + 5s^2 + 4}\right](t) = \mathcal{L}^{-1}\left[s\frac{s^2 + 5}{(s^2 + 1)(s^2 + 4)}\right](t) = \mathcal{L}^{-1}\left[s\left(\frac{\frac{4}{3}}{s^2 + 1} + \frac{\frac{1}{-3}}{s^2 + 4}\right)\right](t)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3}\frac{s}{s^2 + 1^2} - \frac{1}{3}\frac{s}{s^2 + 2^2}\right](t) = \frac{4}{3}\cos t - \frac{1}{3}\cos 2t$$

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^4 + 5s^2 + 4}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2 + 1)(s^2 + 4)}\right](t) = \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{1}{-3}}{s^2 + 4}\right](t)$$

$$= \mathcal{L}^{-1}\left[\frac{1}{3}\frac{1}{s^2 + 1^2} - \frac{1}{6}\frac{2}{s^2 + 2^2}\right](t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t,$$

and whence

$$y(t) = f(t) + (g * h)(t)$$

$$= \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \int_0^t g(t - \tau)\left(\frac{1}{3}\sin\tau - \frac{1}{6}\sin 2\tau\right)d\tau.$$

3. For the system of ordinary differential equations

$$x_1' = x_1 - 2x_2,\ x_2' = 3x_1 - 4x_2, \qquad (1.1)$$

do the following:

a) Write the system of 2 first order equations as a single second order equation for x_1 (by eliminating x_2 between the equations).

4pts

Solution:

By isolating x_2 in, say, the first of these two equations, we get

$$2x_2 = -x_1' + x_1 \Rightarrow 2x_2' = -x_1'' + x_1', \tag{1.2}$$

and then, inserting these expressions into (a multiple of) the second equation,

$$-x_1'' + x_1' = 6x_1 - 4\left(-x_1' + x_1\right) \Leftrightarrow 0 = x_1'' + 3x_1' + 2x_1.$$

b) Find the general solution of the second order ODE obtained in a).

2pts

Solution:

By the usual exponential ansatz $x_1 = e^{rt}$ we obtain the characteristic equation

$$0 = r^2 + 3r + 2 = (r+1)(r+2) \Leftrightarrow r = -1, -2, \tag{1.3}$$

whence the general solution to this homogeneous linear equation is

$$x_1 = c_1 e^{-t} + c_2 e^{-2t}.$$

c) Find the ***corresponding*** general solution of x_2: the constants here are ***not*** independent of those chosen in b).

4pts

Solution:

We are not at liberty here to introduce new arbitrary constants, but must obtain x_2 consistent with $x_1 = c_1 e^{-t} + c_2 e^{-2t}$. We can do this from, say, (1.2), wherein we find

$$x_2 = \frac{1}{2}\left(-x_1' + x_1\right) = \frac{1}{2}\left(c_1 e^{-t} + 2c_2 e^{-2t} + c_1 e^{-t} + c_2 e^{-2t}\right) = \frac{1}{2}\left(2c_1 e^{-t} + 3c_2 e^{-2t}\right)$$
$$= c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t}.$$

Of course, in preparation for comparing our answers with d) we can record that the answers to b) and c) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-2t} \\ c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} e^{-2t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \tilde{c}_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}. \tag{1.4}$$

d) Find the general solution of the system "as a system", i.e. by finding eigenvectors, etc.

10pts

e) Check your answers to b, c) and d) by making sure that the answers to b) and c) are consistent with the answer to d). (That is, use d) to check b) and c), or vice versa.)

10pts

Solution:

The system (1.1) can be written as a vector ODE as follows

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underline{x}' = A\underline{x} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\underline{x} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (1.5)$$

Here the exponential ansatz $\underline{x}(t) = e^{rt}\underline{\xi}$ leads to the vector equation

$$(A - rI)\underline{\xi} = \underline{0} \Leftrightarrow \underline{\xi} = \underline{0}$$

unless r is a root of the characteristic equation

$$0 = \det(A - rI) = \det\begin{bmatrix} 1-r & -2 \\ 3 & -4-r \end{bmatrix} = r^2 - (-3)r - 4 + 6 = r^2 + 3r + 2 = (r+2)(r+1)$$

$$\Leftrightarrow r = -1, -2,$$

and which we note is identical to (1.3). So then the general solution to (1.5) is of the form

$$\underline{x}(t) = c_1 e^{-t}\underline{\xi}_{-1} + c_2 e^{-2t}\underline{\xi}_{-2}$$

where

$$\underline{\xi}_{-1} \in Nul\left[A - (-1)I\right] = Nul\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} = Nul\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \Leftarrow \underline{\xi}_{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\underline{\xi}_{-2} \in Nul\left[A - (-2)I\right] = Nul\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} = Nul\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\} \Leftarrow \underline{\xi}_{-2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

and which then is of the same form as (1.4), accomplishing e).

4. Given the system of equations

$$\underline{x}' = A\underline{x} = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}\underline{x}, \ \alpha \in \mathbb{R},$$

a) Determine the eigenvalues and corresponding eigenvectors of A.

10pts

Solution:

An eigenvector $\underline{\xi}$ of A is a nonzero solution of $A\underline{\xi} = r\underline{\xi} \Leftrightarrow (A - rI)\underline{\xi} = \underline{0}$, whence the associated eigenvalue r satisfies the characteristic equation (of A), which is

$$0=\det\left(A-rI\right)=\det\begin{bmatrix}\alpha-r & 1\\ -1 & \alpha-r\end{bmatrix}=\left(\alpha-r\right)^2+1\Leftrightarrow\alpha-r=\mp i\Leftrightarrow r=\alpha\pm i.$$

Thus $A\underline{\xi}_{\alpha\pm i}=\left(\alpha\pm i\right)\underline{\xi}_{\alpha\pm i}$, where

$$\underline{\xi}_{\alpha\pm i}\in Nul[A-\left(\alpha\pm i\right)I]=Nul\begin{bmatrix}\alpha-\left(\alpha\pm i\right) & 1\\ -1 & \alpha-\left(\alpha\pm i\right)\end{bmatrix}=Nul\begin{bmatrix}\mp i & 1\\ -1 & \mp i\end{bmatrix}=Nul\begin{bmatrix}\mp i & 1\\ 0 & 0\end{bmatrix}$$

$$=Span\left\{\begin{bmatrix}1\\ \pm i\end{bmatrix}\right\}\Leftarrow\underline{\xi}_{\alpha\pm i}=\begin{bmatrix}1\\ \pm i\end{bmatrix}.$$

b) Find a real-valued representation of the general solution for $\underline{x}$ in terms of α.

10pts

Solution:

One complex valued solution is

$$\underline{x}(t)=\underline{\xi}_{\alpha+i}e^{(\alpha+i)t}=\begin{bmatrix}1\\ i\end{bmatrix}e^{\alpha t}e^{it}=e^{\alpha t}\begin{bmatrix}1\\ i\end{bmatrix}\left(\cos t+i\sin t\right)=e^{\alpha t}\begin{bmatrix}\cos t+i\sin t\\ -\sin t+i\cos t\end{bmatrix}$$

$$=e^{\alpha t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+ie^{\alpha t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix},$$

whence, by the usual arguments,

$$\underline{x}(t)=c_1e^{\alpha t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+c_2e^{\alpha t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

is a real-valued representation of the general solution for the system.

c) Find an associated fundamental matrix of solutions $\Phi(t)$ with $\Phi(0)=I$.

4pts

Solution:

From b) evidently

$$\Psi(t)=e^{\alpha t}\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

is a fundamental matrix of solutions. And since $\Psi(0)=I$, $\Psi(t)$ also happens to be the particular fundamental matrix $\Phi(t)$ that we were asked to find.

d) Verify that $\Phi(t)$ is a fundamental matrix of solutions by showing that it satisfies $\Phi'(t)=A\Phi(t)$.

10pts

<u>Solution:</u>

We have

$$\Psi(t)=\begin{bmatrix}\cos(t)e^{\alpha t} & \sin(t)e^{\alpha t}\\ -\sin(t)e^{\alpha t} & \cos(t)e^{\alpha t}\end{bmatrix},$$

so that

$$\begin{aligned}\Psi'(t)&=\begin{bmatrix}\left(\alpha\cos(t)-\sin(t)\right)e^{\alpha t} & \left(\alpha\sin(t)+\cos(t)\right)e^{\alpha t}\\ \left(-\alpha\sin(t)-\cos(t)\right)e^{\alpha t} & \left(\alpha\cos(t)-\sin(t)\right)e^{\alpha t}\end{bmatrix}\\ &=e^{\alpha t}\begin{bmatrix}\alpha\cos(t)-\sin(t) & \alpha\sin(t)+\cos(t)\\ -\alpha\sin(t)-\cos(t) & \alpha\cos(t)-\sin(t)\end{bmatrix}\end{aligned},$$

while

$$A\Psi(t)=\begin{bmatrix}\alpha & 1\\ -1 & \alpha\end{bmatrix}\begin{bmatrix}\cos(t)e^{\alpha t} & \sin(t)e^{\alpha t}\\ -\sin(t)e^{\alpha t} & \cos(t)e^{\alpha t}\end{bmatrix}=\begin{bmatrix}\alpha\cos(t)e^{\alpha t}-\sin(t)e^{\alpha t} & \alpha\sin(t)e^{\alpha t}+\cos(t)e^{\alpha t}\\ -\cos(t)e^{\alpha t}-\alpha\sin(t)e^{\alpha t} & -\sin(t)e^{\alpha t}+\alpha\cos(t)e^{\alpha t}\end{bmatrix}$$

$$=e^{\alpha t}\begin{bmatrix}\alpha\cos(t)-\sin(t) & \alpha\sin(t)+\cos(t)\\ -\alpha\sin(t)-\cos(t) & \alpha\cos(t)-\sin(t)e^{\alpha t}\end{bmatrix}=\Psi'(t)$$

from above.

5. Find the solution of the following initial value problem by means of the Laplace transform:

$$2y''+4\beta y'+4\beta^2 y=2\beta\delta(t-a);\ y(0)=y'(0)-\pi\beta=0.$$

Here β is real and $a>0$.

15pts

<u>Solution:</u>

Transforming the equation with data one gets

$$2\left(s^2Y-s\cdot 0-\pi\beta\right)+4\beta\left(sY-0\right)+4\beta^2Y=2\beta e^{-as}$$

$$\Leftrightarrow$$

$$\begin{aligned}Y&=\beta\left(\pi+e^{-as}\right)\frac{1}{s^2+2\beta s+2\beta^2}\\ &=\left(\pi+e^{-as}\right)\frac{\beta}{(s+\beta)^2+\beta^2}\end{aligned}$$

$$\Leftrightarrow$$

$$y(t)=\pi e^{-\beta t}\sin(\beta t)+u_a(t)e^{-\beta(t-a)}\sin(\beta(t-a)).$$

6. Find the general solution to the system

$$\underline{x}' = A\underline{x} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \underline{x}. \tag{1.6}$$

15pts

Solution:

Here the exponential ansatz $\underline{x}(t) = e^{rt}\underline{\xi}$ leads to the vector equation

$$(A - rI)\underline{\xi} = \underline{0} \Leftrightarrow \underline{\xi} = \underline{0}$$

unless r is a root of the characteristic equation

$$0 = \det(A - rI) = \det\begin{bmatrix} 4-r & -1 \\ 1 & 2-r \end{bmatrix} = r^2 - (6)r + 8 + 1 = r^2 - 6r + 9 = (r-3)^2$$

$$\Leftrightarrow r = 3, 3.$$

So then a solution to (1.6) is

$$\underline{x}(t) = e^{3t}\underline{\xi}_3$$

where

$$\underline{\xi}_3 \in Nul\left[A - 3I\right] = Nul\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = Nul\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \Leftarrow \underline{\xi}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The ansatz $\underline{x}(t) = te^{3t}\underline{\xi} + e^{3t}\underline{\eta}$ leads in (1.6) to the vector equations

$$(A - 3I)\underline{\xi} = \underline{0}, (A - 3I)\underline{\eta} = \underline{\xi},$$

a solution to the first clearly being $\underline{\xi} = \underline{\xi}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the second, by writing explicitly

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\underline{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\underline{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

the vector $\underline{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, say. So the general solution to (1.6) can be expressed as

$$\begin{aligned} \underline{x}(t) &= c_1 e^{3t}\underline{\xi}_3 + c_2(te^{3t}\underline{\xi}_3 + e^{3t}\underline{\eta}) \\ &= c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(te^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \\ &= e^{3t}\left(c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = e^{3t}\begin{bmatrix} c_1 + c_2 + tc_2 \\ c_1 + tc_2 \end{bmatrix}. \end{aligned}$$

7. Directly from the definition, compute the Laplace transform of $f(t) = \cos(at)$. Do note use Euler, or otherwise "enter the complex plane".

10pts

Solution:

From the definition, by integration by parts twice, and assuming $s > 0$, we get

$$\mathcal{L}\left[f\right](s) = \int_0^t e^{-st}\cos(at)dt = -\frac{1}{s}\int_0^t \cos(at)de^{-st} = -\frac{1}{s}\left\{\cos(at)e^{-st}\Big|_0^\infty - \int_0^t e^{-st}d\cos(at)\right\}$$

$$= -\frac{1}{s}\left\{-1 + a\int_0^t e^{-st}\sin(at)dt\right\} = -\frac{1}{s}\left\{-1 - \frac{a}{s}\int_0^t \sin(at)de^{-st}\right\}$$

$$= -\frac{1}{s}\left\{-1 - \frac{a}{s}\left[\sin(at)e^{-st}\Big|_0^\infty - \int_0^t e^{-st}d\sin(at)\right]\right\} = -\frac{1}{s}\left\{-1 - \frac{a}{s}\left[0 - a\int_0^t e^{-st}\cos(at)dt\right]\right\}$$

$$= \frac{1}{s} - \frac{a^2}{s^2}\int_0^t e^{-st}\cos(at)dt = \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}\left[f\right](s)$$

$$\Leftrightarrow$$

$$\mathcal{L}\left[f\right](s) = \frac{\frac{1}{s}}{1 + \frac{a^2}{s^2}} = \frac{s}{s^2 + a^2}.$$

8. Find a linear, homogeneous system of 2 ODE's giving

$$\underline{x}^{(1)}(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix} \text{ and } \underline{x}^{(2)}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

as two of its solutions. That is, find $P = P(t)$ for which both $x^{(1)}(t)$ and $x^{(2)}(t)$ satisfy $\underline{x}' = P(t)\underline{x}$.

10pts

Solution:

Since the two solutions are clearly independent (as vector-valued functions over any interval, or just as a one-parameter family of vectors for most values of the parameter),

$$\Psi(t) = \left[\underline{x}^{(1)}(t) \quad \underline{x}^{(2)}(t)\right] = \begin{bmatrix} t^2 & e^t \\ 2t & e^t \end{bmatrix}$$

is clearly a fundamental matrix of solutions for the equation $\underline{x}' = P(t)\underline{x}$, and, so, $\Psi(t)$ satisfies the equation

$$\Psi'(t) = P(t)\Psi(t) \Leftrightarrow$$

$$\begin{aligned}
P(t) &= \Psi'(t)\Psi^{-1}(t) \\
&= \begin{bmatrix} 2t & e^t \\ 2 & e^t \end{bmatrix} \frac{1}{t^2e^t - 2te^t} \begin{bmatrix} e^t & -e^t \\ -2t & t^2 \end{bmatrix} \\
&= \frac{e^{-t}}{t^2 - 2t} \begin{bmatrix} 2te^t - 2te^t & -2te^t + t^2e^t \\ 2e^t - 2te^t & -2e^t + t^2e^t \end{bmatrix} = \frac{1}{t^2 - 2t} \begin{bmatrix} 0 & t^2 - 2t \\ -2(t-1) & t^2 - 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ \dfrac{-2(t-1)}{t^2 - 2t} & \dfrac{t^2 - 2}{t^2 - 2t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-2(t-1)}{t(t-2)} & \dfrac{t^2 - 2}{t(t-2)} \end{bmatrix}.
\end{aligned}$$