KEY

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Math 343 Midterm 3
Instructor: Scott Glasgow
Sections: 1, 6 and 8
Dates: December 12th and 16th, 2005

Instructions: As usual your work that I ultimately see and grade should be a logical work of art. As you certainly know by now, I have had to make a multiple choice section of the test in order to use the testing center—the written part is worth 250 points out of the total. As usual, USE YOUR OWN PAPER, except of course for this multiple choice section. Again I have included some tutorials, which you can ignore if you wish.

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Multiple Choice Section

In this section choose only one "best" answer—there should only be one answer that is always correct. Each multiple choice problem is worth 2 points, as discussed in the review session.

A. A linear mapping $T : V \to W$ being one-to-one is equivalent to
   a) T being invertible on all of the co-domain W.
   b) The kernel of T being the zero vector.
   c) The rank of T being the dimension of the co-domain W.
   d) The nullity of T being the dimension of the co-domain W.

The CORRECT ANSWER IS b).

B. Which of the following is *not* a similarity invariant of a square matrix A:
   a) A's determinant.
   b) A's invertibility.
   c) A's rank.
   d) None of the above.

The CORRECT ANSWER IS d).

C. The definition of matrix multiplication given by our mathematical forefathers was chosen to
   a) correspond to composition of the associated linear mappings.
   b) make life very difficult for 343 students.
   c) be commutative.
   d) make the process of taking a determinant be a linear mapping from the vector space of square matrices to the reals.

The CORRECT ANSWER IS a).

D. A $n \times n$ matrix A is invertible if and only if
   a) $A\mathbf{x} = \mathbf{0}$ has many solutions.
   b) the reduced echelon form of A is the zero matrix.
   c) A can be expressed as the sum of elementary matrices
   d) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$

The CORRECT ANSWER IS d).

Written Section

1. Let an inner-product be defined by

$$\langle f, g \rangle := \int_0^1 f(t)g(t)t(1-t)dt \tag{0.1}$$

and let the functions f_0, f_1, and f_2 be defined over the interval $[0,1]$ by

$$\begin{aligned} &f_0(t) = 1, \\ &f_1(t) = 2t - 1, \text{ and} \\ &f_2(t) = 5t^2 - 5t + 1. \end{aligned} \tag{0.2}$$

Find the three inner products:

$$\langle f_0, f_1 \rangle, \ \langle f_0, f_2 \rangle, \text{ and } \langle f_1, f_2 \rangle. \tag{0.3}$$

Note that for each nonnegative integer n, $\int_0^1 t^n dt = \frac{1}{n+1}$. Note then that the calculation of the three inner products certainly involves fractions (since $\frac{1}{n+1}$ is a fraction for nonnegative integer n). Nevertheless this problem has been engineered to give integer results for the three inner products—if you do not ultimately get 3 integers, try again!

30 Points

<u>Solutions:</u>

$$\langle f_0, f_1 \rangle = \int_0^1 f_0(t) f_1(t) t(1-t)dt = \int_0^1 1\left(2t-1\right)t(1-t)dt = \int_0^1 \left(2t-1\right)(t-t^2)dt$$

$$= \int_0^1 \left(2t^2 - t - 2t^3 + t^2\right)dt = \int_0^1 \left(-2t^3 + 3t^2 - t\right)dt = -2\frac{1}{4} + 3\frac{1}{3} - \frac{1}{2} = -\frac{1}{2} + 1 - \frac{1}{2} = 0.$$

$$\langle f_0, f_2 \rangle = \int_0^1 f_0(t) f_2(t) t(1-t)dt = \int_0^1 1\left(5t^2 - 5t + 1\right)t(1-t)dt = \int_0^1 \left(5t^2 - 5t + 1\right)(t-t^2)dt$$

$$= \int_0^1 \left(5t^3 - 5t^2 + t - 5t^4 + 5t^3 - t^2\right)dt = \int_0^1 \left(-5t^4 + 10t^3 - 6t^2 + t\right)dt = -5\frac{1}{5} + 10\frac{1}{4} - 6\frac{1}{3} + \frac{1}{2}$$

$$= -1 + \frac{5}{2} - 2 + \frac{1}{2} = \frac{6}{2} - 3 = 0, \text{ and}$$

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(t) f_2(t) t(1-t) dt = \int_0^1 (2t-1)\left(5t^2-5t+1\right) t(1-t) dt$$

$$= \int_0^1 \left(10t^3 - 10t^2 + 2t - 5t^2 + 5t - 1\right)(t-t^2) dt = \int_0^1 \left(10t^3 - 15t^2 + 7t - 1\right)(t-t^2) dt$$

$$= \int_0^1 \left(10t^4 - 15t^3 + 7t^2 - t - 10t^5 + 15t^4 - 7t^3 + t^2\right) dt = \int_0^1 \left(-10t^5 + 25t^4 - 22t^3 + 8t^2 - t\right) dt$$

$$= -10\frac{1}{6} + 25\frac{1}{5} - 22\frac{1}{4} + 8\frac{1}{3} - \frac{1}{2} = -\frac{5}{3} + 5 - \frac{11}{2} + \frac{8}{3} - \frac{1}{2} = 1 + 5 - 6 = 0.$$

2. a) By using the Gram-Schmidt process on the polynomials $\{1, t, t^2\}$ (in that order), find polynomials of zeroth, first, and second degree that are orthogonal with respect to the (real) innerproduct

$$\langle f, g \rangle := \int_0^1 f(t) g(t) t(1-t) dt. \qquad (0.4)$$

b) Check that the polynomials f_0, f_1, and f_2 that you generated in a) are, in fact, orthogonal with respect to the (unusual) innerproduct (0.4), i.e. check that $\langle f_0, f_1 \rangle = \langle f_0, f_2 \rangle = \langle f_1, f_2 \rangle = 0$: Yes, you will be graded on this checking (this part b)) as well as part a).

P.S. Make life easy on the grader (me) by "normalizing" as follows: i) make all polynomials f_0, f_1, and f_2 have integer coefficients, ii) make the leading order coefficient positive, and iii) eliminate common integer factors from all coefficients. For example (and in that same order) $2/7 - 4t/3 - 6t^2 \xrightarrow{i)} 6 - 28t - 126t^2 \xrightarrow{ii)} -6 + 28t + 126t^2 \xrightarrow{iii)} -3 + 14t + 63t^2$.

Note again that for each nonnegative integer n, $\int_0^1 t^n dt = \frac{1}{n+1}$.

<u>**Solution:**</u> Performing Gram-Schmidt on $\{1, t, t^2\}$ (in the usual order) produces the desired orthogonal set: Label these original (non-orthogonal) polynomials via

$$g_0(t) = 1,\ g_1(t) = t,\ g_2(t) = t^2.$$

Then, for all nonzero scalars α and β, the following polynomials have the required degrees and are orthogonal w.r.t. the innerproduct (0.4):

$$f_0 := g_0,\ f_1 := \alpha\left(g_1 - \frac{\langle g_1, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0\right),\ f_2 := \beta\left(g_2 - \frac{\langle g_2, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0 - \frac{\langle g_2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1\right). \qquad (0.5)$$

Evidently we must calculate several innerproducts. The first ones are

$$\langle g_1, f_0\rangle = \langle g_1, g_0\rangle = \int_0^1 g_1(t)g_0(t)t(1-t)dt = \int_0^1 t\cdot 1\cdot t(1-t)dt$$

$$= \int_0^1 (t^2 - t^3)dt = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \text{ and}$$

$$\langle f_0, f_0\rangle = \langle g_0, g_0\rangle = \int_0^1 g_0(t)g_0(t)t(1-t)dt = \int_0^1 1\cdot 1\cdot t(1-t)dt$$

$$= \int_0^1 (t - t^2)dt = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Since $\langle g_1, f_0\rangle / \langle f_0, f_0\rangle = 1/2$, we fulfill the "normalization" criteria when we pick α to be 2:

$$f_1(t) := 2\left(g_1(t) - \frac{\langle g_1, f_0\rangle}{\langle f_0, f_0\rangle} g_0(t)\right) = 2\left(g_1(t) - \frac{1}{2} g_0(t)\right) = 2g_1(t) - g_0(t) = 2t - 1. \quad (0.6)$$

The second set of inner products are then

$$\langle g_2, f_0\rangle = \langle g_2, g_0\rangle = \int_0^1 g_2(t)g_0(t)t(1-t)dt = \int_0^1 t^2\cdot 1\cdot t(1-t)dt$$

$$= \int_0^1 (t^3 - t^4)dt = \frac{1}{4} - \frac{1}{5} = \frac{1}{20},$$

$$\langle g_2, f_1\rangle = \langle g_2, 2g_1 - g_0\rangle = 2\langle g_2, g_1\rangle - \langle g_2, g_0\rangle = 2\int_0^1 g_2(t)g_1(t)t(1-t)dt - \frac{1}{20}$$

$$= 2\int_0^1 t^2\cdot t\cdot t(1-t)dt - \frac{1}{20} = 2\int_0^1 (t^4 - t^5)dt - \frac{1}{20} = 2\left(\frac{1}{5} - \frac{1}{6}\right) - \frac{1}{20}$$

$$= 2\frac{1}{30} - \frac{1}{20} = \frac{1}{60}, \text{ and}$$

$$\langle f_1, f_1\rangle = \langle 2g_1 - g_0, 2g_1 - g_0\rangle = 4\langle g_1, g_1\rangle - 4\langle g_1, g_0\rangle + \langle g_0, g_0\rangle = 4\langle g_1, g_1\rangle - 4\frac{1}{12} + \frac{1}{6}$$

$$= 4\langle g_1 t, g_1/t\rangle - \frac{1}{3} + \frac{1}{6} = 4\langle g_2, g_0\rangle - \frac{1}{3} + \frac{1}{6} = 4\frac{1}{20} - \frac{1}{6} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}.$$

Thus the last polynomial is

$$f_2 := \beta\left(g_2 - \frac{\langle g_2, f_0\rangle}{\langle f_0, f_0\rangle} f_0 - \frac{\langle g_2, f_1\rangle}{\langle f_1, f_1\rangle} f_1\right) = \beta\left(g_2 - \frac{\frac{1}{20}}{\frac{1}{6}} g_0 - \frac{\frac{1}{60}}{\frac{1}{30}}(2g_1 - g_0)\right)$$

$$= \beta\left(g_2 - \frac{3}{10} g_0 - \frac{1}{2}(2g_1 - g_0)\right) = \beta\left(g_2 - \frac{3}{10} g_0 - g_1 + \frac{1}{2} g_0\right)$$

$$= \beta\left(g_2 - g_1 + \frac{1}{5} g_0\right) = \frac{\beta}{5}(5g_2 - 5g_1 + g_0),$$

and we fulfill the normalization criteria by picking $\beta = 5$: in summary

$$f_0(t) = g_0(t) = 1,$$
$$f_1(t) = 2g_1(t) - g_0(t) = 2t - 1, \text{ and}$$
$$f_2(t) = 5g_2(t) - 5g_1(t) + g_0(t) = 5t^2 - 5t + 1.$$

Check: We checked the orthogonality of these polynomials in question 1.

50 Points

3. Find solutions to the following 3 systems of equations by row reducing an augmented matrix: the first "system" is
$$10\alpha_0 = 3, \tag{0.7}$$
the second is
$$\begin{aligned} 10\alpha_1 + 5\beta_1 &= 3 \\ 5\alpha_1 + 3\beta_1 &= 2, \end{aligned} \tag{0.8}$$
and the third is
$$\begin{aligned} 10\alpha_2 + 5\beta_2 + 3\gamma_2 &= 3 \\ 5\alpha_2 + 3\beta_2 + 2\gamma_2 &= 2 \\ 21\alpha_2 + 14\beta_2 + 10\gamma_2 &= 10. \end{aligned} \tag{0.9}$$
Finally check your answers by plugging them back into the equations and seeing if you get true statements.

30 Points

Solutions:

a) The first equation has an augmented matrix $\left[10|3\right] \sim \left[1|3/10\right]$ so that the solution is $\alpha_0 = 3/10$ (very gratuitous but there you go). This is checked by noting that indeed $10(3/10) = 3$.

b) Row reduction on the second system's augmented matrix gives
$$\left[\begin{array}{cc|c} 10 & 5 & 3 \\ 5 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -1 & -1 \\ 5 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 5 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 1 \end{array}\right]$$
so that the solution is $\alpha_1 = -1/5$ and $\beta_1 = 1$. This is checked by noting that indeed
$$10(-1/5) + 5 \cdot 1 = 3$$
$$5(-1/5) + 3 \cdot 1 = 2.$$

c) Row reduction on the third system's augmented matrix gives

$$\left[\begin{array}{ccc|c} 5 & 3 & 2 & 2 \\ 10 & 5 & 3 & 3 \\ 21 & 14 & 10 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 5 & 3 & 2 & 2 \\ 0 & -1 & -1 & -1 \\ 1 & 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 5 & 3 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -8 & -8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so that the solution is $\alpha_2 = 0$, $\beta_2 = 0$, and $\gamma_2 = 1$. This is checked by noting that

$$\begin{aligned} 10\cdot 0 + 5\cdot 0 + 3\cdot 1 &= 3 \\ 5\cdot 0 + 3\cdot 0 + 2\cdot 1 &= 2 \\ 21\cdot 0 + 14\cdot 0 + 10\cdot 1 &= 10. \end{aligned}$$

4. Let the (square of the) distance $d^2(f,g)$ between two functions f and g be defined through

$$d^2(f,g) := \|f-g\|^2 := \langle f-g, f-g\rangle := \int_0^1 \left[f(t) - g(t)\right]^2 t(1-t)dt, \qquad (0.10)$$

i.e. let the (square of the) distance $d^2(f,g)$ between two functions f and g be defined as the “natural” notion of distance associated with the (unusual) innerproduct (0.4). Now find (real) polynomials $h_0(t) = \alpha_0$, $h_1(t) = \alpha_1 + \beta_1 t$, and $h_2(t) = \alpha_2 + \beta_2 t + \gamma_2 t^2$ of zeroth, first, and second degree that minimize their (0.10)-distances to $F(t) = t^2$ on the interval $[0,1]$ by finding the minima of the three functions $E_0(\alpha_0)$, $E_1(\alpha_1,\beta_1)$, and $E_2(\alpha_2,\beta_2,\gamma_2)$ defined through

$$\begin{aligned} E_{0,F}(\alpha_0) &:= d^2(h_0,F) = \int_0^1 \left[h_0(t) - F(t)\right]^2 t(1-t)dt = \int_0^1 \left(\alpha_0 - t^2\right)^2 t(1-t)dt, \\ E_{1,F}(\alpha_1,\beta_1) &:= d^2(h_1,F) = \int_0^1 \left[h_1(t) - F(t)\right]^2 t(1-t)dt = \int_0^1 \left(\alpha_1 + \beta_1 t - t^2\right)^2 t(1-t)dt, \text{ and} \\ E_{2,F}(\alpha_2,\beta_2,\gamma_2) &:= d^2(h_2,F) = \int_0^1 \left[h_2(t) - F(t)\right]^2 t(1-t)dt = \int_0^1 \left(\alpha_2 + \beta_2 t + \gamma_2 t^2 - t^2\right)^2 t(1-t)dt. \end{aligned}$$

(0.11)

Tutorial In case you have forgotten some of the relevant calculus and/or the relevant theory I provided in class, let me give you this tutorial: suppose I wanted to find the first degree polynomial $H_1(t) = A_1 + B_1 t$ minimizing its (squared) distance $d^2(H_1,G)$ to a function $G(t)=t^3$. Then I would attempt to minimize the function $E_{1,G}(A_1,B_1)$ of two variables A_1 and B_1 defined by

$$E_{1,G}(A_1,B_1):=d^2\left(H_1,G\right)=\int_0^1\left[H_1(t)-G(t)\right]^2 t(1-t)dt=\int_0^1\left(A_1+B_1t-t^3\right)^2 t(1-t)dt \quad (0.12)$$

Now one way to proceed would be to perform the integration (0.12) with the parameters A_1 and B_1 left indeterminate, and then afterwards perform the differentiations with respect to those two variables A_1 and B_1. Alternatively, and somewhat more efficiently, I can perform these differentiations before performing the integration (0.12) as follows:

$$\begin{aligned}
0&=\frac{d}{dA_1}E_{1,G}(A_1,B_1)=\frac{d}{dA_1}\int_0^1\left(A_1+B_1t-t^3\right)^2 t(1-t)dt=\int_0^1\frac{d}{dA_1}\left(A_1+B_1t-t^3\right)^2 t(1-t)dt\\
&=\int_0^1 2\left(A_1+B_1t-t^3\right)\frac{d}{dA_1}\left(A_1+B_1t-t^3\right)t(1-t)dt=\int_0^1 2\left(A_1+B_1t-t^3\right)\cdot 1\cdot t(1-t)dt\\
&=2\int_0^1\left(A_1+B_1t-t^3\right)(t-t^2)dt=2\left\{A_1\int_0^1(t-t^2)dt+B_1\int_0^1(t^2-t^3)dt-\int_0^1(t^4-t^5)dt\right\}\\
&=2\left\{A_1\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_0^1+B_1\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1-\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1\right\}=2\left\{A_1\left(\frac{1}{2}-\frac{1}{3}\right)+B_1\left(\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{5}-\frac{1}{6}\right)\right\}\\
&=2\left\{A_1\frac{1}{6}+B_1\frac{1}{12}-\frac{1}{30}\right\}\Leftrightarrow 10A_1+5B_1=2,
\end{aligned}$$

and

$$\begin{aligned}
0&=\frac{d}{dB_1}E_{1,G}(A_1,B_1)=\frac{d}{dB_1}\int_0^1\left(A_1+B_1t-t^3\right)^2 t(1-t)dt=\int_0^1\frac{d}{dB_1}\left(A_1+B_1t-t^3\right)^2 t(1-t)dt\\
&=\int_0^1 2\left(A_1+B_1t-t^3\right)\frac{d}{dB_1}\left(A_1+B_1t-t^3\right)t(1-t)dt=\int_0^1 2\left(A_1+B_1t-t^3\right)\cdot t\cdot t(1-t)dt\\
&=2\int_0^1\left(A_1+B_1t-t^3\right)(t^2-t^3)dt=2\left\{A_1\int_0^1(t^2-t^3)dt+B_1\int_0^1(t^3-t^4)dt-\int_0^1(t^5-t^6)dt\right\}\\
&=2\left\{A_1\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1+B_1\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1-\left[\frac{t^6}{6}-\frac{t^7}{7}\right]_0^1\right\}=2\left\{A_1\left(\frac{1}{3}-\frac{1}{4}\right)+B_1\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)\right\}\\
&=2\left\{A_1\frac{1}{12}+B_1\frac{1}{20}-\frac{1}{42}\right\}\Leftrightarrow 35A_1+21B_1=10.
\end{aligned}$$

(0.13)

Now the two equations

$$\begin{aligned} 10A_1 + 5B_1 &= 2 \\ 35A_1 + 21B_1 &= 10 \end{aligned} \tag{0.14}$$

(which were obtained in (0.13) by differentiating the distance function) have the solution

$$A_1 = -\frac{8}{35},\ B_1 = \frac{6}{7} \tag{0.15}$$

so that $H_1(t) = A_1 + B_1 t = -\frac{8}{35} + \frac{6}{7}t$ is the polynomial of first degree that is closest to $G(t) = t^3$ in the indicated notion of distance. Now you try it! (for the 3 problems given you, not this example again!) Note that while fractions will appear in your three problems, they should not be as bad as those indicated in my example. In fact the largest denominator (after canceling common factors) should be the number 10 (not 35 as in my example).

Solutions:

a) For the zeroth degree polynomial the single requisite differentiation gives

$$\begin{aligned} 0 &= \frac{d}{d\alpha_0}E_{0,F}(\alpha_0) = \frac{d}{d\alpha_0}\int_0^1 \left(\alpha_0 - t^2\right)^2 t(1-t)dt = \int_0^1 \frac{d}{d\alpha_0}\left(\alpha_0 - t^2\right)^2 t(1-t)dt \\ &= \int_0^1 2\left(\alpha_0 - t^2\right)\frac{d}{d\alpha_0}\left(\alpha_0 - t^2\right)t(1-t)dt = 2\int_0^1 \left(\alpha_0 - t^2\right)\cdot 1\cdot t(1-t)dt = 2\int_0^1 \left(\alpha_0 - t^2\right)(t-t^2)dt \\ &= 2\left\{\alpha_0\int_0^1 (t-t^2)dt - \int_0^1 (t^3 - t^4)dt\right\} = 2\left\{\alpha_0\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 - \left[\frac{t^4}{4} - \frac{t^5}{5}\right]_0^1\right\} \\ &= 2\left\{\alpha_0\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right)\right\} = 2\left\{\alpha_0\frac{1}{6} - \frac{1}{20}\right\} \Leftrightarrow 10\alpha_0 = 3 \Leftrightarrow \alpha_0 = \frac{3}{10} \end{aligned} \tag{0.16}$$

so that that $h_0(t) = \alpha_0 = \frac{3}{10}$ is the polynomial of zeroth degree that is closest to $F(t) = t^2$ in the indicated notion of distance.

b) For the first degree polynomial the two requisite differentiations are

$$0=\frac{d}{d\alpha_1}E_{1,F}(\alpha_1,\beta_1)=\frac{d}{d\alpha_1}\int_0^1\left(\alpha_1+\beta_1 t-t^2\right)^2 t(1-t)dt=\int_0^1\frac{d}{d\alpha_1}\left(\alpha_1+\beta_1 t-t^2\right)^2 t(1-t)dt$$
$$=\int_0^1 2\left(\alpha_1+\beta_1 t-t^2\right)\frac{d}{d\alpha_1}\left(\alpha_1+\beta_1 t-t^2\right)(t-t^2)dt=2\int_0^1\left(\alpha_1+\beta_1 t-t^2\right)\cdot 1\cdot(t-t^2)dt$$
$$=2\left\{\alpha_1\int_0^1(t-t^2)dt+\beta_1\int_0^1(t^2-t^3)dt-\int_0^1(t^3-t^4)dt\right\}=2\left\{\alpha_1\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_0^1+\beta_1\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1-\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1\right\}$$
$$=2\left\{\alpha_1\left(\frac{1}{2}-\frac{1}{3}\right)+\beta_1\left(\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)\right\}=2\left\{\frac{1}{6}\alpha_1+\frac{1}{12}\beta_1-\frac{1}{20}\right\}\Leftrightarrow 10\alpha_1+5\beta_1=3$$

$$0=\frac{d}{d\beta_1}E_{1,F}(\alpha_1,\beta_1)=\frac{d}{d\beta_1}\int_0^1\left(\alpha_1+\beta_1 t-t^2\right)^2 t(1-t)dt=\int_0^1\frac{d}{d\beta_1}\left(\alpha_1+\beta_1 t-t^2\right)^2 t(1-t)dt$$
$$=\int_0^1 2\left(\alpha_1+\beta_1 t-t^2\right)\frac{d}{d\beta_1}\left(\alpha_1+\beta_1 t-t^2\right)(t-t^2)dt=2\int_0^1\left(\alpha_1+\beta_1 t-t^2\right)\cdot t\cdot(t-t^2)dt$$
$$=2\left\{\alpha_1\int_0^1(t^2-t^3)dt+\beta_1\int_0^1(t^3-t^4)dt-\int_0^1(t^4-t^5)dt\right\}=2\left\{\alpha_1\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1+\beta_1\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1-\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1\right\}$$
$$=2\left\{\alpha_1\left(\frac{1}{3}-\frac{1}{4}\right)+\beta_1\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{6}\right)\right\}=2\left\{\frac{1}{12}\alpha_1+\frac{1}{20}\beta_1-\frac{1}{30}\right\}\Leftrightarrow 5\alpha_1+3\beta_1=2$$
(0.17)

In the previous problem we found that the solution to the generated equations

$$\begin{aligned}10\alpha_1+5\beta_1&=3,\\5\alpha_1+3\beta_1&=2\end{aligned}\qquad(0.18)$$

is $\alpha_1=-\frac{1}{5}$ and $\beta_1=1$so that that $h_1(t)=\alpha_1+\beta_1 t=-\frac{1}{5}+t$ is the polynomial of first degree that is closest to $F(t)=t^2$ in the indicated notion of distance.

c) For the second degree polynomial the three requisite differentiations are

$$0=\frac{d}{d\alpha_2}E_{2,F}(\alpha_2,\beta_2,\gamma_2)=\frac{d}{d\alpha_2}\int_0^1\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)^2 t(1-t)dt$$

$$=\int_0^1\frac{d}{d\alpha_2}\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)^2 t(1-t)dt$$

$$=\int_0^1 2\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)\frac{d}{d\alpha_2}\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)t(1-t)dt$$

$$=\int_0^1 2\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)\cdot 1\cdot t(1-t)dt=2\left\{\int_0^1\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)(t-t^2)dt\right\}$$

$$=2\left\{\alpha_2\int_0^1(t-t^2)dt+\beta_2\int_0^1(t^2-t^3)dt+\gamma_2\int_0^1(t^3-t^4)dt-\int_0^1(t^3-t^4)dt\right\}$$

$$=2\left\{\alpha_2\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_0^1+\beta_2\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1+\gamma_2\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1-\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1\right\}$$

$$=2\left\{\alpha_2\left(\frac{1}{2}-\frac{1}{3}\right)+\beta_2\left(\frac{1}{3}-\frac{1}{4}\right)+\gamma_2\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)\right\}=2\left\{\frac{1}{6}\alpha_2+\frac{1}{12}\beta_2+\frac{1}{20}\gamma_2-\frac{1}{20}\right\}$$

$$=2\left\{\alpha_2\left(\frac{1}{2}-\frac{1}{3}\right)+\beta_2\left(\frac{1}{3}-\frac{1}{4}\right)+\gamma_2\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)\right\}=2\left\{\frac{1}{6}\alpha_2+\frac{1}{12}\beta_2+\frac{1}{20}\gamma_2-\frac{1}{20}\right\}$$

$$\Leftrightarrow 10\alpha_2+5\beta_2+3\gamma_2=3$$

and

$$0=\frac{d}{d\beta_2}E_{2,F}(\alpha_2,\beta_2,\gamma_2)=\frac{d}{d\beta_2}\int_0^1\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)^2 t(1-t)dt$$

$$=\int_0^1\frac{d}{d\beta_2}\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)^2 t(1-t)dt$$

$$=\int_0^1 2\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)\frac{d}{d\beta_2}\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)t(1-t)dt$$

$$=\int_0^1 2\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)\cdot t\cdot t(1-t)dt=2\left\{\int_0^1\left(\alpha_2+\beta_2t+\gamma_2t^2-t^2\right)(t^2-t^3)dt\right\}$$

$$=2\left\{\alpha_2\int_0^1(t^2-t^3)dt+\beta_2\int_0^1(t^3-t^4)dt+\gamma_2\int_0^1(t^4-t^5)dt-\int_0^1(t^4-t^5)dt\right\}$$

$$=2\left\{\alpha_2\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1+\beta_2\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1+\gamma_2\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1-\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1\right\}$$

$$=2\left\{\frac{1}{12}\alpha_2+\frac{1}{20}\beta_2+\frac{1}{30}\gamma_2-\frac{1}{30}\right\}\Leftrightarrow 5\alpha_2+3\beta_2+2\gamma_2=2$$

and

$$
\begin{aligned}
0 &= \frac{d}{d\gamma_2} E_{2,F}(\alpha_2,\beta_2,\gamma_2) = \frac{d}{d\gamma_2}\int_0^1 \left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)^2 t(1-t)dt \\
&= \int_0^1 \frac{d}{d\gamma_2}\left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)^2 t(1-t)dt \\
&= \int_0^1 2\left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)\frac{d}{d\gamma_2}\left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)t(1-t)dt \\
&= \int_0^1 2\left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)\cdot t^2\cdot t(1-t)dt = 2\left\{\int_0^1\left(\alpha_2+\beta_2 t+\gamma_2 t^2 - t^2\right)(t^3-t^4)dt\right\} \\
&= 2\left\{\alpha_2\int_0^1 (t^3-t^4)dt+\beta_2\int_0^1 (t^4-t^5)dt+\gamma_2\int_0^1 (t^5-t^6)dt-\int_0^1 (t^5-t^6)dt\right\} \\
&= 2\left\{\alpha_2\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1+\beta_2\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1+\gamma_2\left[\frac{t^6}{6}-\frac{t^7}{7}\right]_0^1-\left[\frac{t^6}{6}-\frac{t^7}{7}\right]_0^1\right\} \\
&= 2\left\{\frac{1}{20}\alpha_2+\frac{1}{30}\beta_2+\frac{1}{42}\gamma_2-\frac{1}{42}\right\} \Leftrightarrow 21\alpha_2+14\beta_2+10\gamma_2=10 .
\end{aligned}
\tag{0.19}
$$

Again, in the previous problem we found that the solution to the equations

$$
\begin{aligned}
10\alpha_2+5\beta_2+3\gamma_2 &= 3 \\
5\alpha_2+3\beta_2+2\gamma_2 &= 2 \\
21\alpha_2+14\beta_2+10\gamma_2 &= 10
\end{aligned}
\tag{0.20}
$$

is $\alpha_2=0$, $\beta_2=0$ and $\gamma_2=1$ so that $h_2(t)=\alpha_2+\beta_2 t+\gamma_2 t^2=t^2=F(t)$ is, unsurprisingly, the polynomial of second degree that is closest to $F(t)=t^2$ in the indicated notion of distance.

60 Points

5. Demonstrate a method different than that used in problem 4) (a patently more linear-algebraic one) to find (real) polynomials h_0, h_1, and h_2 of zeroth, first, and second degree that minimize their distance to $F(t)=t^2$ on the interval $[0,1]$, where the distance is, as in problem 4), given by

$$
d^2(f,g):=\|f-g\|^2:=\langle f-g, f-g\rangle:=\int_0^1\left[f(t)-g(t)\right]^2 t(1-t)dt . \tag{0.21}
$$

For example, if you were successful in generating an orthogonal basis of P_2 in problem 2), you might use the associated projection theorem. On the other hand, if you were not successful there (as indicated, say, by your checking in part b) there), or if you would just like to try something different, then you might perform the following alternate process: as in problem 4) write

$$h_0(t) = \alpha_0, \qquad h_1(t) = \alpha_1 + \beta_1 t, \ \text{ and } \qquad h_2(t) = \alpha_2 + \beta_2 t + \gamma_2 t^2,$$

but then solve the following three “ $A^T A\mathbf{x} = A^T \mathbf{b}$ -like” systems of equations for the 6 coefficients α_0, α_1, β_1, α_2, β_2, γ_2:

System I: $\langle 1, \alpha_0 \rangle = \langle 1, t^2 \rangle$

System II:
$$\begin{aligned} \langle 1, \alpha_1 + \beta_1 t \rangle &= \langle 1, t^2 \rangle \\ \langle t, \alpha_1 + \beta_1 t \rangle &= \langle t, t^2 \rangle \end{aligned}$$

System III:
$$\begin{aligned} \langle 1, \alpha_2 + \beta_2 t + \gamma_2 t^2 \rangle &= \langle 1, t^2 \rangle \\ \langle t, \alpha_2 + \beta_2 t + \gamma_2 t^2 \rangle &= \langle t, t^2 \rangle \ . \\ \langle t^2, \alpha_2 + \beta_2 t + \gamma_2 t^2 \rangle &= \langle t^2, t^2 \rangle \end{aligned}$$

Obviously your answers to problem 4) and this problem (whichever of the two methods you choose here) should be the same.

In order to make your life much easier, I will tell you the results of the following integrations (some of which may be irrelevant):

$$\begin{aligned}
&\int_0^1 t \cdot 1 \cdot t(1-t)dt = \frac{1}{12} = \frac{1}{2^2 \cdot 3}, \ \int_0^1 t^2 \cdot 1 \cdot t(1-t)dt = \frac{1}{20} = \frac{1}{2^2 \cdot 5}, \\
&\int_0^1 t \cdot t^2 \cdot t(1-t)dt = \frac{1}{30} = \frac{1}{2 \cdot 3 \cdot 5}, \int_0^1 t^2 \cdot t^2 \cdot t(1-t)dt = \frac{1}{42} = \frac{1}{2 \cdot 3 \cdot 7}, \\
&\int_0^1 1 \cdot 1 \cdot t(1-t)dt = \frac{1}{6} = \frac{1}{2 \cdot 3}, \ \int_0^1 t^2 (2t-1) t(1-t)dt = \frac{1}{60} = \frac{1}{2^2 \cdot 3 \cdot 5}, \\
&\int_0^1 (2t-1)^2 t(1-t)dt = \frac{1}{2 \cdot 3 \cdot 5}, \ \int_0^1 t^2 (5t^2 - 5t + 1) t(1-t)dt = \frac{1}{420} = \frac{1}{2^2 \cdot 3 \cdot 5 \cdot 7}, \\
&\int_0^1 (5t^2 - 5t + 1)^2 t(1-t)dt = \frac{1}{84} = \frac{1}{2^2 \cdot 3 \cdot 7}.
\end{aligned} \tag{0.22}$$

80 Points

Solution: For the first method we have from the results of problem 2) (and a "huge" theorem about orthogonal projections) that the distances are minimized when we choose

$$h_0 = \frac{\langle F, f_0\rangle}{\langle f_0, f_0\rangle} f_0 = \frac{\int_0^1 F(t) f_0(t)t(1-t)dt}{\int_0^1 f_0(t) f_0(t)t(1-t)dt} f_0 = \frac{\int_0^1 t^2 \cdot 1 \cdot t(1-t)dt}{\int_0^1 1\cdot 1\cdot t(1-t)dt} f_0 = \frac{\frac{1}{2^2\cdot 5}}{\frac{1}{2\cdot 3}} f_0 = \frac{2\cdot 3}{2^2\cdot 5} f = \frac{3}{2\cdot 5} f_0$$

$$= \frac{3}{10} f_0,$$

$$h_1 - h_0 = \frac{\langle F, f_1\rangle}{\langle f_1, f_1\rangle} f_1 = \frac{\int_0^1 F(t) f_1(t)t(1-t)dt}{\int_0^1 f_1(t) f_1(t)t(1-t)dt} f_1 = \frac{\int_0^1 t^2(2t-1)t(1-t)dt}{\int_0^1 (2t-1)^2 t(1-t)dt} = \frac{\frac{1}{2^2\cdot 3\cdot 5}}{\frac{1}{2\cdot 3\cdot 5}} f_1 = \frac{2\cdot 3\cdot 5}{2^2\cdot 3\cdot 5} f_1$$

$$= \frac{1}{2} f_1, \text{ and}$$

$$h_2 - h_1 = \frac{\langle F, f_2\rangle}{\langle f_2, f_2\rangle} f_2 = \frac{\int_0^1 F(t) f_2(t)t(1-t)dt}{\int_0^1 f_2(t) f_2(t)t(1-t)dt} f_2 = \frac{\int_0^1 t^2(5t^2-5t+1)t(1-t)dt}{\int_0^1 (5t^2-5t+1)^2 t(1-t)dt} = \frac{\frac{1}{2^2\cdot 3\cdot 5\cdot 7}}{\frac{1}{2^2\cdot 3\cdot 7}} f_2$$

$$= \frac{2^2\cdot 3\cdot 7}{2^2\cdot 3\cdot 5\cdot 7} f_2 = \frac{1}{5} f_2.$$

Here we used table (0.22). Hence the "Fourier" approximations to F (of "degree" 0, 1, and 2) are

$$h_0 = \frac{3}{10} f_0,\ h_1 = \frac{3}{10} f_0 + \frac{1}{2} f_1,\ h_2 = \frac{3}{10} f_0 + \frac{1}{2} f_1 + \frac{1}{5} f_2.$$

In terms of the original monomials $g_0(t) = 1,\ g_1(t) = t,\ g_2(t) = t^2$ we have

$$h_0(t) = \frac{3}{10} f_0(t) = \frac{3}{10} g_0(t) = \frac{3}{10},$$

$$h_1(t) = \frac{3}{10} g_0(t) + \frac{1}{2} f_1(t) = \frac{3}{10} g_0 v + \frac{1}{2}\left(2g_1(t) - g_0(t)\right) = g_1(t) - \frac{1}{5} g_0(t) = t - \frac{1}{5}$$

$$h_2(t) = g_1(t) - \frac{1}{5} g_0(t) + \frac{1}{5} f_2(t) = g_1(t) - \frac{1}{5} g_0(t) + \frac{1}{5}\left(5g_2(t) - 5g_1(t) + g_0(t)\right) = g_2(t) = t^2,$$

the last result as expected.

For the second method we use table (0.22) (and some simple manipulations) to get that

$$\langle 1,1\rangle=\frac{1}{2\cdot 3},\langle 1,t\rangle=\langle t,1\rangle=\frac{1}{2^2\cdot 3},\ \langle 1,t^2\rangle=\langle t^2,1\rangle=\frac{1}{2^2\cdot 5},\ \langle t,t\rangle=\langle 1,t^2\rangle=\frac{1}{2^2\cdot 5},$$

$$\langle t,t^2\rangle=\langle t^2,t\rangle=\frac{1}{2\cdot 3\cdot 5},\langle t^2,t^2\rangle=\frac{1}{2\cdot 3\cdot 7}.$$

Inserting these into systems I, II, and III one gets

System I: $$\langle 1,\alpha_0\rangle=\langle 1,t^2\rangle\Leftrightarrow \alpha_0\langle 1,1\rangle=\langle 1,t^2\rangle\Leftrightarrow \alpha_0=\frac{\langle 1,t^2\rangle}{\langle 1,1\rangle}=\frac{\frac{1}{2^2\cdot 5}}{\frac{1}{2\cdot 3}}=\frac{2\cdot 3}{2^2\cdot 5}=\frac{3}{2\cdot 5}=\frac{3}{10}.$$

System II: $$\begin{matrix}\langle 1,\alpha_1+\beta_1 t\rangle=\langle 1,t^2\rangle\\ \langle t,\alpha_1+\beta_1 t\rangle=\langle t,t^2\rangle\end{matrix}\Leftrightarrow\begin{matrix}\langle 1,1\rangle\alpha_1+\langle 1,t\rangle\beta_1=\frac{1}{2^2\cdot 5}\\ \langle t,1\rangle\alpha_1+\langle t,t\rangle\beta_1=\frac{1}{2\cdot 3\cdot 5}\end{matrix}\Leftrightarrow$$

$$\begin{matrix}\frac{1}{2\cdot 3}\alpha_1+\frac{1}{2^2\cdot 3}\beta_1=\frac{1}{2^2\cdot 5}\\ \frac{1}{2^2\cdot 3}\alpha_1+\frac{1}{2^2\cdot 5}\beta_1=\frac{1}{2\cdot 3\cdot 5}\end{matrix}\Leftrightarrow\begin{matrix}\frac{2^2\cdot 3\cdot 5}{2\cdot 3}\alpha_1+\frac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\beta_1=\frac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\\ \frac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\alpha_1+\frac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\beta_1=\frac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\end{matrix}\Leftrightarrow\begin{matrix}10\alpha_1+5\beta_1=3\\ 5\alpha_1+3\beta_1=2\end{matrix}$$

$$\Leftrightarrow\begin{matrix}\beta_1=1\\ \alpha_1=\frac{-1}{5}\end{matrix},$$

the last generated by row reduction in a previous problem, and

System III: $$\begin{matrix}\langle 1,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle 1,t^2\rangle\\ \langle t,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t,t^2\rangle\\ \langle t^2,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t^2,t^2\rangle\end{matrix}\Leftrightarrow\begin{matrix}\langle 1,1\rangle\alpha_2+\langle 1,t\rangle\beta_2+\langle 1,t^2\rangle\gamma_2=\frac{1}{2^2\cdot 5}\\ \langle t,1\rangle\alpha_2+\langle t,t\rangle\beta_2+\langle t,t^2\rangle\gamma_2=\frac{1}{2\cdot 3\cdot 5}\\ \langle t^2,1\rangle\alpha_2+\langle t^2,t\rangle\beta_2+\langle t^2,t^2\rangle\gamma_2=\frac{1}{2\cdot 3\cdot 7}\end{matrix}$$

$$\begin{matrix}\frac{1}{2\cdot 3}\alpha_2+\frac{1}{2^2\cdot 3}\beta_2+\frac{1}{2^2\cdot 5}\gamma_2=\frac{1}{2^2\cdot 5}\\ \frac{1}{2^2\cdot 3}\alpha_2+\frac{1}{2^2\cdot 5}\beta_2+\frac{1}{2\cdot 3\cdot 5}\gamma_2=\frac{1}{2\cdot 3\cdot 5}\\ \frac{1}{2^2\cdot 5}\alpha_2+\frac{1}{2\cdot 3\cdot 5}\beta_2+\frac{1}{2\cdot 3\cdot 7}\gamma_2=\frac{1}{2\cdot 3\cdot 7}\end{matrix}\Leftrightarrow\begin{matrix}\frac{2^2\cdot 3\cdot 5}{2\cdot 3}\alpha_2+\frac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\beta_2+\frac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\gamma_2=\frac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\\ \frac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\alpha_2+\frac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\beta_2+\frac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\gamma_2=\frac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\\ \frac{2^2\cdot 3\cdot 5\cdot 7}{2^2\cdot 5}\alpha_2+\frac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 5}\beta_2+\frac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 7}\gamma_2=\frac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 7}\end{matrix}$$

$$\begin{aligned} 10\alpha_2 + 5\beta_2 + 3\gamma_2 &= 3 \\ \Leftrightarrow \quad 5\alpha_2 + 3\beta_2 + 2\gamma_2 &= 2 \\ 21\alpha_2 + 14\beta_2 + 10\gamma_2 &= 10 \end{aligned} \quad \Leftrightarrow \begin{aligned} \gamma_2 &= 1 \\ \alpha_2 &= 0, \\ \beta_2 &= 0 \end{aligned}$$

again the solution given (and checked) in a previous problem.

Thus, via this method, our "closest" polynomials are

$$h_0(t) = \alpha_0 = \frac{3}{10}, \qquad h_1(t) = \frac{-1}{5} + 1t = t - \frac{1}{5}, \text{ and}$$

$$h_2(t) = \alpha_2 + \beta_2 t + \gamma_2 t^2 = 0 + 0t + 1t^2 = t^2,$$

just as calculated in the first method (or the method in problem 4)).