Math 343 Midterm III KEY
Winter 2009
sections 001 and 004
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Please do NOT write on this booklet. No credit will be given for such work. Rather write in a blue book, or on your own paper. (Despite any markings here made by the testing center, the sole purpose of this booklet is actually only to communicate to you the relevant problems to work on. There is not enough room here to communicate in full sentences your solutions.)

1. Find the orthogonal projection of $\mathbf{u}=(6,1,5,2)$ onto the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1=(1,2,1,2)$, $\mathbf{v}_2=(2,3,2,1)$ and $\mathbf{v}_3=(1,3,3,1)$.

15 points

Solution

The projection $\mathbf{u}_{\|}$ is $\mathbf{u}=(6,1,5,2)$ iff $\mathbf{u}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, otherwise it is the vector $\mathbf{u}_{\|}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ nearest $\mathbf{u}$. Actually, in either case, we have

$$\mathbf{u}_{\|}=A\mathbf{x} \tag{1.1}$$

where $\mathbf{x}$ is *any* solution of

$$A^TA\mathbf{x}=A^T\mathbf{u}\,,^1 \tag{1.2}$$

and where

$$A=[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]=\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}. \tag{1.3}$$

We solve (1.2) by row reducing $\left[A^TA\middle|A^T\mathbf{u}\right]$ after calculating A^TA and $A^T\mathbf{u}$:

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[1] The vector $\mathbf{u}_{\|}=A\mathbf{x}$ is unique even if $\mathbf{x}$ solving (1.2) isn't: Let $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve(1.2). Then $\mathbf{d}=\mathbf{x}_1-\mathbf{x}_2$ satisfies $A^TA\mathbf{d}=\mathbf{0}$. Thus $A\mathbf{d}\in Nul(A^T)=(ColA)^{\perp}$, the latter by theorem. On the other hand we certainly have $A\mathbf{d}\in ColA$, by definition. Thus $A\mathbf{d}\in ColA\bigcap(ColA)^{\perp}=\{\mathbf{0}\}$, the latter theorem simply being that the only vector orthogonal to itself is $\mathbf{0}$. Thus, for $\mathbf{u}_{\|1}:=A\mathbf{x}_1$ and $\mathbf{u}_{\|2}:=A\mathbf{x}_2$, we get $\mathbf{u}_{\|1}-\mathbf{u}_{\|2}=A\mathbf{x}_1-A\mathbf{x}_2=A(\mathbf{x}_1-\mathbf{x}_2)=A\mathbf{d}=\mathbf{0}$, i.e. $\mathbf{u}_{\|1}=\mathbf{u}_{\|2}$.

$$A^TA=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}=\begin{bmatrix}10&12&12\\12&18&18\\12&18&20\end{bmatrix},\ A^T\mathbf{u}=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}6\\1\\5\\2\end{bmatrix}=\begin{bmatrix}17\\27\\26\end{bmatrix},$$

$$\left[A^TA\middle|A^T\mathbf{u}\right]=\left[\begin{array}{ccc|c}10&12&12&17\\12&18&18&27\\12&18&20&26\end{array}\right]\sim\left[\begin{array}{ccc|c}10&12&12&17\\2&6&6&10\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\10&12&12&17\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&18&18&33\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&6&11\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&0&14\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&6&10\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&0&13\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&0&0&-1\\0&3&0&7\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}6&0&0&-3\\0&6&0&14\\0&0&6&-3\end{array}\right].$$

(1.4)

Thus

$$\mathbf{u}_{\|}=A\mathbf{x}=\frac{1}{6}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}\begin{bmatrix}-3\\14\\-3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}22\\27\\16\\5\end{bmatrix}=\begin{bmatrix}11/3\\9/2\\8/3\\5/6\end{bmatrix}. \tag{1.5}$$

2. Use the Gram-Schmidt process, etc., to transform the following members of the Euclidean space $\mathbb{R}^3$ into an orthonormal set in that innerproduct space:

$$\mathbf{u}_1=(1,1,1),\mathbf{u}_2=(0,1,1),\ \text{and}\ \mathbf{u}_3=(0,0,1). \tag{1.6}$$

10 points

Solution

We produce an orthogonal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ from the original basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ via (a modified version of) Gram-Schmidt:

$$\begin{aligned}
&\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1) \\
&\mathbf{v}_2 \propto \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (0,1,1) - \frac{2}{3}(1,1,1) = \frac{1}{3}(-2,1,1) \propto (-2,1,1) = \mathbf{v}_2 \\
&\mathbf{v}_3 \propto \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{6}(-2,1,1) \\
&\quad = \frac{1}{6}(0,-3,3) \propto (0,-1,1) = \mathbf{v}_3.
\end{aligned} \tag{1.7}$$

Finally we normalize each vector to obtain the required orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$:

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}(1,1,1),\ \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(-2,1,1),\ \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}(0,-1,1). \tag{1.8}$$

3. Find the transition matrix $P_{B'B}$ from the $\mathbb{R}^3$ basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ to the $\mathbb{R}^3$ basis $B' = \{(1,2,4),(2,5,8),(3,6,13)\}$.

15 points

Solution

Let $(\mathbf{v})_B$ and $(\mathbf{v})_{B'}$ be coordinate vectors with respect to the bases B and B' for the vector $\mathbf{v}$. Then, by definition of coordinate vector,

$$B(\mathbf{v})_B = B'(\mathbf{v})_{B'} = \mathbf{v}, \tag{1.9}$$

where, abusing notation, B and B' here represent matrices containing the vectors of the bases B and B' as columns. Thus B in (1.9) is the identity matrix. According to (1.9) then,

$$(\mathbf{v})_{B'} = B'^{-1}(\mathbf{v})_B =: P_{B'B}(\mathbf{v})_B. \tag{1.10}$$

We calculate the inverse required in (1.10) by row reducing $[B'|I]$ to $[I|B'^{-1}] = [I|P_{B'B}]$:

$$\begin{aligned}
[B'|I] &= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 4 & 8 & 13 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 13 & 0 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 17 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] = [I|P_{B'B}].
\end{aligned} \tag{1.11}$$

4. Prove that if $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is an orthonormal basis for an innerproduct space V, then, for any vector $\mathbf{v}\in V$,

$$\mathbf{v}=\langle\mathbf{v},\mathbf{v}_1\rangle\mathbf{v}_1+\langle\mathbf{v},\mathbf{v}_2\rangle\mathbf{v}_2+\ldots+\langle\mathbf{v},\mathbf{v}_n\rangle\mathbf{v}_n\,. \tag{1.12}$$

10 points

Solution

By previous theorem, since $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is a basis for V, we immediately have that for any $\mathbf{v}\in V$ there exists one and only one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that

$$\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n\,. \tag{1.13}$$

Taking the innerproduct of both sides of (1.13) with $\mathbf{v}_j\in\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_j,\ldots,\mathbf{v}_n\}$, we get

$$\begin{aligned}\langle\mathbf{v},\mathbf{v}_j\rangle&=\langle c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_j\mathbf{v}_j+\ldots+c_n\mathbf{v}_n,\mathbf{v}_j\rangle\\&=c_1\langle\mathbf{v}_1,\mathbf{v}_j\rangle+c_2\langle\mathbf{v}_2,\mathbf{v}_j\rangle+\ldots+c_j\langle\mathbf{v}_j,\mathbf{v}_j\rangle+\ldots+c_n\langle\mathbf{v}_n,\mathbf{v}_j\rangle\\&=0+0+\ldots+c_j\langle\mathbf{v}_j,\mathbf{v}_j\rangle+\ldots+0=c_j,\end{aligned} \tag{1.14}$$

so that (1.12) follows.

5. Find the angle $\theta\in[0,\pi]$ between $\mathbf{u}=(2,4)$ and $\mathbf{v}=(1,-3)$ in the Euclidean innerproduct space $\mathbb{R}^2$.

5 points

Solution

By (good) definition,

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{-10}{\sqrt{10}\sqrt{20}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3}{4}\pi. \tag{1.15}$$

6. If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}. \tag{1.16}$$

If such a matrix P exists, what is the associated diagonalized form of A?

15 points

Solution

We seek an invertible matrix P with eigenvectors of A as its columns. First we calculate A's eigenvalues:

λ is an eigenvalue of A iff

$$\begin{aligned} 0 = \det(A-\lambda I) &= \det\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = -\lambda(2-\lambda)(3-\lambda) - 2\big(0 - 1(2-\lambda)\big) \\ &= (2-\lambda)\big(-\lambda(3-\lambda)+2\big) = (2-\lambda)\big(\lambda^2 - 3\lambda + 2\big) = (2-\lambda)(\lambda-2)(\lambda-1) \end{aligned} \tag{1.17}$$

$\Leftrightarrow$

$\lambda = 1, 2, 2.$

The associated eigenvectors are elements of the nontrivial nullspaces of $A - \lambda I$:

$$\mathbf{0} \neq \mathbf{x}_{\lambda=1} \in Nul\left(A-1\cdot I\right) = Nul\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Span\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}, \tag{1.18}$$

$$\mathbf{0} \neq \mathbf{x}_{\lambda=2} \in Nul\left(A-2\cdot I\right) = Nul\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Thus

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{1.19}$$

diagonalizes A to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.20}$$

7. Find a basis of the orthogonal complement of the column space of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}. \tag{1.21}$$

10 points

Solution

By theorem

$$\begin{aligned}(ColA)^{\perp} = NulA^{T} = Nul\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix} = Nul\begin{bmatrix}1 & 4 & 7\\ 0 & -3 & -6\\ 0 & -6 & -12\end{bmatrix} = Nul\begin{bmatrix}1 & 4 & 7\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}\\ = Nul\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix} = Span\left\{\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}\right\}.\end{aligned} \tag{1.22}$$

8. Let A be an $n\mathrm{x}n$ matrix. Prove that if

$$A\mathbf{x}\cdot A\mathbf{y} = \mathbf{x}\cdot\mathbf{y} \tag{1.23}$$

holds for every $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, then A is orthogonal.

15 points

Solution

$$\begin{aligned}\mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y} = A\mathbf{x}\cdot A\mathbf{y} = (A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y}\\ \Leftrightarrow\\ 0 = \mathbf{x}^T A^T A\mathbf{y} - \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\left(A^T A - I\right)\mathbf{y}.\end{aligned} \tag{1.24}$$

Since (1.24) holds for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, it holds for $\mathbf{x} = \left(A^T A - I\right)\mathbf{y}$, so that (1.24) becomes

$$\begin{aligned}0 = \left(\left(A^T A - I\right)\mathbf{y}\right)^T\left(A^T A - I\right)\mathbf{y} = \left\|\left(A^T A - I\right)\mathbf{y}\right\|^2\\ \Leftrightarrow\\ \mathbf{0} = \left(A^T A - I\right)\mathbf{y}\end{aligned} \tag{1.25}$$

for every $\mathbf{y}\in\mathbb{R}^n$. Thus (1.25) holds for $\mathbf{y}$ chosen as each of the rows of the matrix $A^T A - I$, showing that each of these rows is the zero vector (why?) and, so, showing that $A^T A - I$ is the zero matrix, i.e. $A^T A = I$, which, together with A square, means that A is an orthogonal matrix.

9. Prove the Best Approximation Theorem: If W is a finite-dimensional subspace of an innerproduct space V, and if $\mathbf{u}\in V$, then the orthogonal projection of $\mathbf{u}$ onto W, denoted $proj_W\mathbf{u}$ or, say, $\mathbf{u}_{\|}$, is the best approximation to $\mathbf{u}$ in the sense that

$$\left\|\mathbf{u}-\mathbf{u}_{||}\right\| < \left\|\mathbf{u}-\mathbf{w}\right\| \tag{1.26}$$

for every vector $\mathbf{w} \in W$ that is different from $\mathbf{u}_{||}$. (Hint: Recall $\mathbf{u}_{||}$ can be defined by demanding that $\mathbf{u}-\mathbf{u}_{||}$ is in the orthogonal complement of W.)

15 points

Solution

For every $\mathbf{w} \in W$, $\mathbf{u}-\mathbf{u}_{||}$ is, by definition of $\mathbf{u}_{||}$, orthogonal to $\mathbf{w}$, and, since $\mathbf{u}_{||} \in W$, $\mathbf{u}-\mathbf{u}_{||}$ is also orthogonal to $\mathbf{u}_{||}-\mathbf{w} \in W$. Thus, by Pythagoras,

$$\begin{gathered}
\left\|\mathbf{u}-\mathbf{w}\right\|^2 = \left\|\left(\mathbf{u}-\mathbf{u}_{||}\right)+\left(\mathbf{u}_{||}-\mathbf{w}\right)\right\|^2 = \left\|\mathbf{u}-\mathbf{u}_{||}\right\|^2 + \left\|\mathbf{u}_{||}-\mathbf{w}\right\|^2 \\
> \left\|\mathbf{u}-\mathbf{u}_{||}\right\|^2 \\
\Leftrightarrow \\
\left\|\mathbf{u}-\mathbf{u}_{||}\right\| < \left\|\mathbf{u}-\mathbf{w}\right\|,
\end{gathered} \tag{1.27}$$

where we used that $\mathbf{w}$ is different than $\mathbf{u}_{||}$.

10. Prove that if A is an orthogonal nxn matrix, then

$$\left\|A\mathbf{x}\right\| = \left\|\mathbf{x}\right\| \tag{1.28}$$

for every $\mathbf{x} \in \mathbb{R}^n$.

5 points

Solution

For every $\mathbf{x} \in \mathbb{R}^n$ we have

$$\begin{gathered}
\left\|A\mathbf{x}\right\|^2 = \left(A\mathbf{x}\right)^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = \left\|\mathbf{x}\right\|^2 \\
\Leftrightarrow \\
\left\|A\mathbf{x}\right\| = \left\|\mathbf{x}\right\|.
\end{gathered} \tag{1.29}$$

11. Find the least squares solution $\mathbf{x} \in \mathbb{R}^2$ to the following (inconsistent?) system of equations:

$$A\mathbf{x} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{b}. \tag{1.30}$$

5 points

Solution

The point value is small, because the matrix A has already been factored into QR, with Q containing orthonormal columns (but not "orthogonal", as per the parlance, since it is not square), and with R upper triangular and invertible (which, in this construction, implies and is implied by A's columns being independent). Thus, as we have discussed, the (possibly inconsistent) system $A\mathbf{x} = QR\mathbf{x} = \mathbf{b}$ gives rise to the following form for the normal equations:

$$\begin{aligned} &R^T R\mathbf{x} = R^T IR\mathbf{x} = R^T Q^T QR\mathbf{x} = \left(QR\right)^T QR\mathbf{x} = A^T A\mathbf{x} = A^T\mathbf{b} = R^T Q^T\mathbf{b} \\ &\quad \Leftrightarrow \\ &R\mathbf{x} = Q^T\mathbf{b}. \end{aligned} \tag{1.31}$$

the latter since R invertible implies R^T invertible. Thus the possibly inconsistent system (1.30) gives rise to the following system which a) is guaranteed to be consistent and which b) gives the least square solution of(1.30):

$$R\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = Q^T\mathbf{b}. \tag{1.32}$$

So then if we write $\mathbf{x} = \left(x_1, x_2\right)^T$, clearly (using back substitution)

$$x_2 = -1,\ x_1 = 1 - 1 \cdot x_2 = 1 - 1 \cdot \left(-1\right) = 2. \tag{1.33}$$

12. Find the linear function $l(x)=\alpha+\beta x$ that is closest to the quadratic polynomial $q(x)=1+6x^2$, given that the notion of distance is given by

$$\left(d\left(f_1,f_2\right)\right)^2=\|f_1-f_2\|^2_{\langle,\rangle}=\langle f_1-f_2,f_1-f_2\rangle, \tag{1.34}$$

where

$$\langle f_1,f_2\rangle=\int_0^1 f_1(x)\,f_2(x)\,dx. \tag{1.35}$$

Use Gram-Schmidt to generate an appropriate orthogonal basis, and the relevant extension of the orthogonal basis representation theorem (that we have termed "Fourier's theorem), to find $l(x)$.

You may find it convenient to know that

$$\int_0^1 1\cdot 1dx=1,\int_0^1 1\cdot xdx=\frac{1}{2},\int_0^1 1\cdot\left(1+6x^2\right)dx=3,\int_0^1 1\cdot(2x-1)\,dx=0,\int_0^1 x\cdot(2x-1)\,dx=\frac{1}{6},$$
$$\int_0^1 x\cdot xdx=\frac{1}{3},\int_0^1 x\cdot\left(1+6x^2\right)dx=2,\int_0^1(2x-1)\cdot(2x-1)\,dx=\frac{1}{3},\int_0^1(2x-1)\cdot\left(1+6x^2\right)dx=1,$$
$$\int_0^1\left(1+6x^2\right)\cdot\left(1+6x^2\right)dx=\frac{61}{5}. \tag{1.36}$$

15 points

Solution

A "big"/ambient inner product space V in which both $q(x)$ and $l(x)$ live is $P_2=Span\left\{1,x,x^2\right\}$, the inner product as in(1.35). "Fourier's theorem" is that if $\left\{\mathbf{v}_1,\ldots,\mathbf{v}_r,\mathbf{v}_{r+1},\ldots,\mathbf{v}_n\right\}$ is an orthogonal basis for an inner product space V, then

$$\mathbf{u}_{\|}:=\frac{\langle\mathbf{v}_1,\mathbf{u}\rangle}{\langle\mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1+\ldots\frac{\langle\mathbf{v}_r,\mathbf{u}\rangle}{\langle\mathbf{v}_r,\mathbf{v}_r\rangle}\mathbf{v}_r+0\mathbf{v}_{r+1}+\ldots+0\mathbf{v}_n=\frac{\langle\mathbf{v}_1,\mathbf{u}\rangle}{\langle\mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1+\ldots\frac{\langle\mathbf{v}_r,\mathbf{u}\rangle}{\langle\mathbf{v}_r,\mathbf{v}_r\rangle}\mathbf{v}_r \tag{1.37}$$

is the element of

$$\begin{aligned} W &= Span\left\{\mathbf{v}_1,\ldots,\mathbf{v}_r,0\mathbf{v}_{r+1},\ldots,0\mathbf{v}_n\right\} = Span\left\{\mathbf{v}_1,\ldots,\mathbf{v}_r\right\} \\ &\underset{\text{space}}{\subset} Span\left\{\mathbf{v}_1,\ldots,\mathbf{v}_r,\mathbf{v}_{r+1},\ldots,\mathbf{v}_n\right\} = V \end{aligned} \tag{1.38}$$

that is nearest $\mathbf{u} \in V$. Now the linear function $l\left(x\right) = \alpha + \beta x$ is in

$$W = Span\left\{1, x, 0x^2\right\} = Span\left\{1, x\right\} \underset{\text{space}}{\subset} Span\left\{1, x, x^2\right\} = V = P_2, \tag{1.39}$$

so it is enough to perform Gram-Shmidt on the basis $B = \left\{1, x\right\} =: \left\{\mathbf{u}_1, \mathbf{u}_2\right\}$ for W to use formula (1.37). We get an orthogonal basis $B_\perp = \left\{\mathbf{v}_1, \mathbf{v}_2\right\}$ via (the modified G-S algorithm)

$$\mathbf{v}_1 = \mathbf{u}_1 = 1,\ \mathbf{v}_2 \propto \mathbf{u}_2 - \frac{\left\langle \mathbf{v}_1, \mathbf{u}_2 \right\rangle}{\left\langle \mathbf{v}_1, \mathbf{v}_1 \right\rangle}\mathbf{v}_1 = x - \frac{\left\langle 1, x\right\rangle}{\left\langle 1,1\right\rangle}\cdot 1 = x - \frac{\int_0^1 1\cdot x dx}{\int_0^1 1\cdot 1 dx}\cdot 1 = x - \frac{\frac{1}{2}}{1}\cdot 1 \propto 2x - 1 = \mathbf{v}_2. \tag{1.40}$$

Thus (1.37) is

$$\begin{aligned} \left(1+6x^2\right)_{\|} &= \frac{\left\langle \mathbf{v}_1, 1+6x^2\right\rangle}{\left\langle \mathbf{v}_1, \mathbf{v}_1\right\rangle}\mathbf{v}_1 + \frac{\left\langle \mathbf{v}_2, 1+6x^2\right\rangle}{\left\langle \mathbf{v}_2, \mathbf{v}_2\right\rangle}\mathbf{v}_2 \\ &= \frac{\left\langle 1, 1+6x^2\right\rangle}{\left\langle 1,1\right\rangle}1 + \frac{\left\langle 2x-1, 1+6x^2\right\rangle}{\left\langle 2x-1, 2x-1\right\rangle}\left(2x-1\right) \\ &= \frac{\int_0^1 1\cdot\left(1+6x^2\right)dx}{\int_0^1 1\cdot 1dx} + \frac{\int_0^1 \left(2x-1\right)\cdot\left(1+6x^2\right)dx}{\int_0^1 \left(2x-1\right)\cdot\left(2x-1\right)dx}\left(2x-1\right) \\ &= \frac{3}{1} + \frac{1}{\frac{1}{3}}\left(2x-1\right) = 6x. \end{aligned} \tag{1.41}$$

13. Determine whether the proposed innerproduct below is actually an inner product by orthogonally diagonalizing the relevant matrix:

$$\left\langle \mathbf{u}, \mathbf{v}\right\rangle = \left\langle \left(u_1, u_2\right), \left(v_1, v_2\right)\right\rangle := u_1v_1 + 3u_2v_1 + 3u_1v_2 + u_2v_2 = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{u}^T A\mathbf{v}. \tag{1.42}$$

If it is a “good” innerproduct, explain why, and if not, explain why.

15 points

Solution

Clearly the innerproduct passes all tests/axioms except possibly positivity. The latter will occur iff the symmetric matrix A has positive eigenvalues: In such case we have there exists orthogonal matrix P and diagonal matrix D (with the positive eigenvalues of A on its diagonal) such that

$$A = PDP^{-1} = PDP^T = PD^{1/2}D^{1/2}P^T = \left(D^{1/2}P^T\right)^T D^{1/2}P^T =: R^T R \tag{1.43}$$

where then R has real values (because none of the eigenvalues is negative) and is invertible (because none of the eigenvalues is zero). Thus (1.42) becomes

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v} = \mathbf{u}^T R^T R\mathbf{v} = \left(R\mathbf{u}\right)^T \left(R\mathbf{v}\right). \tag{1.44}$$

In particular

$$\langle \mathbf{u}, \mathbf{u} \rangle = \left(R\mathbf{u}\right)^T \left(R\mathbf{u}\right) = \left\|R\mathbf{u}\right\|^2 \geq 0 \tag{1.45}$$

with equality iff $R\mathbf{u} = \mathbf{0}$, which holds iff $\mathbf{u} = \mathbf{0}$ since R is invertible. Here we have explained when the indicated form will be a good inner product, as required by the problem.

For the problem at hand we have λ is an eigenvalue of A iff

$$\begin{aligned} 0 &= \det\left(\lambda I - A\right) = \det\left(\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\right) = \det\begin{bmatrix}\lambda - 1 & -3\\ -3 & \lambda - 1\end{bmatrix} = \left(\lambda - 1\right)^2 - \left(-3\right)^2 \\ &= \left(\lambda - 1 + \left(-3\right)\right)\left(\lambda - 1 - \left(-3\right)\right) = \left(\lambda - 4\right)\left(\lambda + 2\right) \\ &\Leftrightarrow \\ \lambda &= -2 \text{ or } 4. \end{aligned} \tag{1.46}$$

Because of the negative eigenvalue, this proposed innerproduct is only a “poser”—it is not an (good) innerproduct.