

Math 311-1 Final  
Winter 1998

Theory Section

Instructions

- Attempt no more than *five* questions
- Time allowed: *Three* hours
- Calculators are allowed
- Books and notes are not allowed.
- A list of important formulae and theorems is attached.
- If you think that the questions contain errors that make it nearly impossible to provide an answer, and that by correcting the mistake the questions may be answered properly, you may modify the question and proceed to provide an answer to the modified question. In such event, you must clearly document the error and your modification. Note however that if the suspected error is not really a mistake on the instructor's part, marks will be deducted.
- Each question carries equal points.
- This section accounts for 80% of the final grade.
- If you answer more than five questions, only the first five will be graded.

1. (a) [7] Derive the *open* Trapezoidal rule (*excluding* the error term)

$$\int_a^b f(x)dx = \frac{3h}{2}[f(a+h) + f(a+2h)] + \frac{3}{4}h^3 f''(\xi)$$

where  $h = (b - a)/3$  and  $a < \xi < b$ , by considering an appropriate polynomial approximation to  $f(x)$ .

- (b) [7] Let  $T(a, b)$  and  $T(a, \frac{a+b}{2}) + T(\frac{a+b}{2}, b)$  be the single and double applications of the *closed* Trapezoidal rule to  $\int_a^b f(x)dx$ . Derive the relationship between

$$\left| T(a, b) - T(a, \frac{a+b}{2}) + T(\frac{a+b}{2}, b) \right|$$

and

$$\left| \int_a^b f(x)dx - T(a, \frac{a+b}{2}) + T(\frac{a+b}{2}, b) \right|.$$

- (c) [4] Explain how the the above relationship may be used in an adaptive quadrature strategy.
- (d) [2] Is the procedure more efficient than the composite Simpson's method in achieving a certain accuracy?
2. (a) [8] Approximate

$$\int_2^3 \frac{x}{\sqrt{x^2 - 4}} dx$$

using the 3-point Gauss quadrature.

- (b) [4] Compute the relative error of the above result. Discuss why the numerical result is not very accurate.
- (c) [8] Determine constants  $a, b, c$  that will produce a quadrature formula

$$\int_{-1}^1 xf(x)dx = af(-c) + bf(c)$$

that has the highest degree of precision. State the degree of precision of the formula.

3. (a) [10] Use Romberg integration to complete the following table for

$$\int_0^1 t^5 dt$$

$k$	$h_k$	$R_{k,1}$	$R_{k,2}$	$R_{k,3}$
1	1		—	—
2	1/2			—
3	1/4	0.192383		

- (b) [2] State which integration rule is related to the values in  $R_{k,2}$ ?  
 (c) [8] If the error term for the composite trapezoidal rule were

$$\int_a^b f(x)dx - T(a, b, h) = c_1h + c_2h^2 + c_3h^3 + \dots$$

instead of  $K_1h^2 + K_2h^4 + K_3h^6 + \dots$ , then what modification should be made to Romberg approximation formula?

4. Consider the system of linear equations  $Ax = b$  where  $A$  is an  $n \times n$  matrix and  $x, b$  are  $n$ -vectors. If the vector  $b$  is perturbed by an amount  $\delta b$  and  $\delta x$  is the corresponding change in  $x$ , show that

- (a) [8]

$$\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\delta b\|}{\|b\|},$$

where  $k(A) = \|A\| \cdot \|A^{-1}\|$  is the condition number of  $A$ .

- (b) [2] Show that  $k(A) \geq 1$ .  
 (c) [10] Consider the equations (with exact solution  $x = (1, 1)^T$ ):

$$\begin{aligned} 41x_1 + 40x_2 &= 81, \\ 40x_1 + 39x_2 &= 79. \end{aligned}$$

For  $b = (80.99, 79.01)^T$ , the solution is  $x = (1.79, 0.19)^T$  while for  $b = (80.98, 79.02)^T$ , the solution is  $x = (2.58, -0.62)^T$ . Explain carefully why the computed values given above differ so widely with such small changes in the vector  $b$ .

5. Let

$$A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -5 & 8 \\ 3 & -2 & 1 \end{pmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

- (a) [8] Find matrices  $P, L, U$  so that  $PA = LU$  with  $P$  being a permutation matrix,  $L$  a lower triangular matrix with 1's on its diagonal and nondiagonal entries  $|m_{ij}| \leq 1$ , and  $U$  an upper triangular matrix. Specify which pivoting strategy you are employing.  
 (b) [3] Use the above factorization to solve  $Ax = b$ .  
 (c) [4] Explain how the  $A = P^T LU$  factorization may be used efficiently to solve  $Ax = b$  with multiple right hand sides  $b = b_1, b_2, \dots, b_10$ .  
 (d) [5] Count the number of operations needed to perform a forward solve.

6. Consider the linear system

$$\begin{aligned} 10x_1 &- 5x_2 && & & & = & 6, \\ -5x_1 &+ 10x_2 &- 4x_3 && & & = & 25, \\ &- 4x_2 &+ 8x_3 &- x_4 & & & = & -11, \\ &&- x_3 &+ 5x_4 & & & = & -11 \end{aligned}$$

- (a) [2] Is the matrix corresponding to the linear system diagonally dominant? strictly diagonally dominant?
- (b) [6] Let the sequence  $x^{(k)}$  be defined by the iteration scheme

$$x^{(k)} = Tx^{(k-1)} + c, \quad k = 1, 2, \dots$$

and that  $x = Tx + c$ , show that

$$\|x - x^{(k)}\| \leq \|T\|^k \|x - x^{(0)}\|$$

- (c) [2] What condition must  $\|T\|$  satisfy in order to conclude that  $x^{(k)}$  converge to  $x$  as  $k$  tends to infinity?
- (d) [7] Find the 1-norm and the  $\infty$ -norm of the Jacobian iteration matrix. Is the Jacobi iteration convergent for any initial guess? Explain.
- (e) [3] Is the Gauss-Seidel process convergent for this problem? If so, does it converge faster than Jacobi iteration? Justify your answer.

7. Given the data

$x$	5000	6000	7000	8000	9200	10500
$f(x)$	21	22.5	23	22.25	19.5	14

- (a) [8] Form the normal equations *without scaling* for the least squares polynomial of degree two. (Do not solve the system.)
- (b) [4] Explain why one should perform some scaling on the data before setting up the normal equations.
- (c) [4] If the data are exponentially related in the form  $f(x) = bx^a$  for some constants  $a, b$ , describe how the normal equations are constructed.
- (d) [4] In the process of interpolating the data using a natural spline, we obtain a matrix equation. State the properties of the matrix.