Splines

Spline: Continuity at knots

$$S_0(t_0) = y_0$$

$$S_0(t_1) = S_1(t_1) = y_1$$

$$S_1(t_2) = S_2(t_2) = y_2$$

$$\vdots$$

$$S_{n-2}(t_{n-1}) = S_{n-1}(t_{n-1}) = y_{n-1}$$

$$S_{n-1}(t_n) = y_n$$

$$1 + 2(n-1) + 1 = 2n$$

Continuity for derivatives:

$S'_0(t_1) = S'_1(t_1)$	$S''_0(t_1) = S''_1(t_1)$
$S'_0(t_2) = S'_2(t_1)$	$S''_1(t_2) = S''_2(t_2)$
$\vdots$	$\vdots$
$S'_{n-2}(t_{n-1}) = S'_{n-1}(t_{n-1})$	$S''_{n-2}(t_{n-1}) = S''_{n-1}(t_{n-1})$
$n-1$	$n-1$

Total $4n-2$ conditions:
still have two degrees of freedom

Determination of spline functions

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3,\ x \in [t_i, t_{i+1}]$$

$$S''_i(x) = 2c_i + 6d_i x \text{ (LINEAR)}$$

Let

$$\left.\begin{array}{rcl} S''_i(t_i) & = & M_i \\ S''_i(t_{i+1}) & = & M_{i+1} \end{array}\right\} \text{unknowns}$$

then

$$S''_i(x) = M_i \left(\frac{t_{i+1} - x}{t_{i+1} - t_i} \right) + M_{i+1} \left(\frac{x - t_i}{t_{i+1} - t_i} \right)$$

Integrate twice and fix constants of integration using $S_i(t_i) = y_i, S_i(t_{i+1}) = y_{i+1}$:

$$S_i(x) \quad = \quad \frac{M_i}{6} \frac{(t_{i+1} - x)^3}{t_{i+1} - t_i} + \frac{M_{i+1}}{6} \frac{(x - t_i)^3}{t_{i+1} - t_i}$$

$$+\frac{6y_{i+1} - M_{i+1}(t_{i+1}-t_i)^2}{6(t_{i+1}-t_i)}(x-t_i)$$
$$+\frac{6y_i - M_i(t_{i+1}-t_i)^2}{6(t_{i+1}-t_i)}(t_{i+1}-x)$$

Thus if M_i, M_{i+1} are determined, we have $S_i(x)$.

Determination of M_i's
Use continuity of derivative of spline function to find M_i's
(continuity of second derivative already assumed)

$$S'_{i-1}(t_i) = S'_i(t_i), \quad i = 1, 2, ..., n-1$$

Since

$$\begin{aligned} S'_i(x) &= -\frac{M_i}{2}\frac{(t_{i+1}-x)^2}{t_{i+1}-t_i} + \frac{M_{i+1}}{2}\frac{(x-t_i)^2}{t_{i+1}-t_i} \\ &+\frac{6y_{i+1} - M_{i+1}(t_{i+1}-t_i)^2}{6(t_{i+1}-t_i)} \\ &-\frac{6y_i - M_i(t_{i+1}-t_i)^2}{6(t_{i+1}-t_i)} \end{aligned}$$

so

$$S'_i(t_i) = -\frac{t_{i+1}-t_i}{3}M_i - \frac{t_{i+1}-t_i}{6}M_{i+1} + \frac{y_{i+1}-y_i}{t_{i+1}-t_i}$$
$$S'_{i-1}(t_i) = \frac{t_i-t_{i-1}}{6}M_{i-1} + \frac{t_i-t_{i-1}}{3}M_i + \frac{y_i-y_{i-1}}{t_i-t_{i-1}}$$

Determination of M_i's (cont'd)
So for $i = 1, 2, ..., n-1$:

$$\frac{t_i-t_{i-1}}{6}M_{i-1} + \frac{t_{i+1}-t_{i-1}}{3}M_i + \frac{t_{i+1}-t_i}{6}M_{i+1}$$
$$= \frac{y_{i+1}-y_i}{t_{i+1}-t_i} - \frac{y_i-y_{i-1}}{t_i-t_{i-1}}$$

Get $n-1$ equations with $n+1$ unknowns.
Need end conditions:

natural	clamped	specified
$M_0 = 0$	$S'(t_0) = f'(t_0)$	$M_0 = f''(t_0)$
$M_n = 0$	$S'(t_n) = f'(t_n)$	$M_n = f''(t_n)$

Note. other end conditions possible:
e.g. *not-a-knot* condition

Tridiagonal systems from spline functions

For natural splines get a symmetric, diagonally dominant tridiagonal system:

$$A \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \\ v_{n-1} \end{bmatrix}$$

where

$$A = \begin{pmatrix} u_1 & \frac{1}{6}h_1 & & & \\ \frac{1}{6}h_1 & u_2 & \frac{1}{6}h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6}h_{n-3} & u_{n-2} & \frac{1}{6}h_{n-2} \\ & & & \frac{1}{6}h_{n-2} & u_{n-1} \end{pmatrix}$$

$$h_i = t_{i+1} - t_i, \quad u_i = \frac{1}{3}(t_{i+1} - t_{i-1})$$

$$b_i = (y_{i+1} - y_i)/h_i, \quad v_i = b_i - b_{i-1}$$

Natural Cubic Spline

x	1	1.5	2	3	5
$f(x)$	1	0.67	0.5	0.33	0.2857

With $M_0 = M_4 = 0$,

$$\begin{pmatrix} 0.3333 & 0.083330 & \\ 0.08333 & 0.5 & 0.1667 \\ & 0.1667 & 1 \end{pmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 0.3200 \\ 0.1700 \\ 0.1479 \end{bmatrix}$$

$$M_1 = 0.9238, \; M_2 = 0.1448, \; M_3 = 0.1237$$

In $[0, 1.5]$,

$$S(x) = 1.429 + 0.1868\,x - 0.9238\,x^2 + 0.3079\,x^3$$

In $[1.5, 2]$,
$t_i = 1.5, t_{i+1} = 2, y_i = 0.67, y_{i+1} = 0.5,$
$M_i = 0.9238, M_{i+1} = 0.1448$

$$S(x) = 3.345 - 3.645\,x + 1.630\,x^2 - 0.2597\,x^3$$

In $[2, 3]$,

$$S(x) = 1.295 - 0.5706\,x + 0.09347\,x^2 - 0.003513\,x^3$$

In $[3, 3.5]$,

$$S(x) = 1.479 - 0.7541\,x + 0.1546\,x^2 - 0.01031\,x^3$$