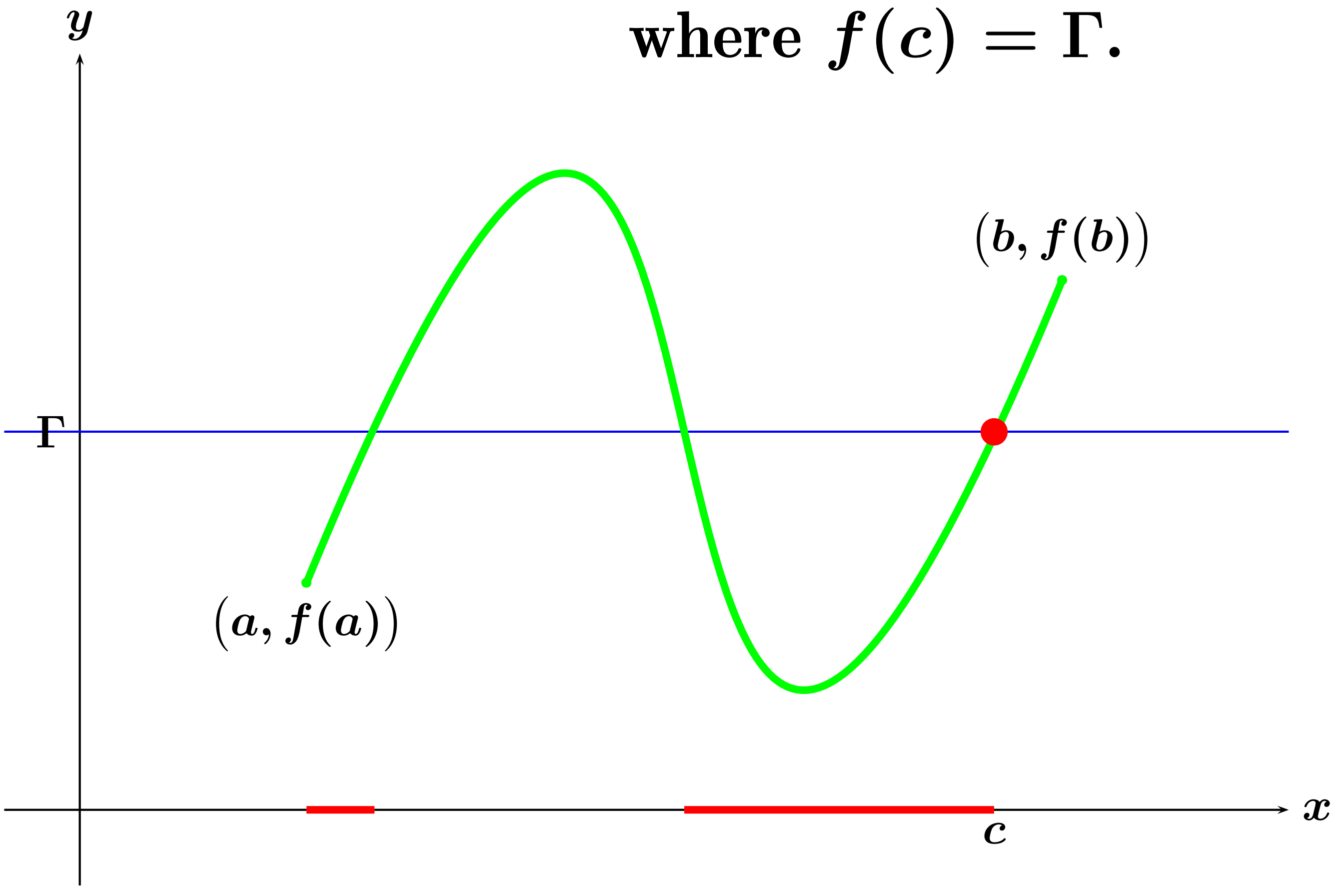
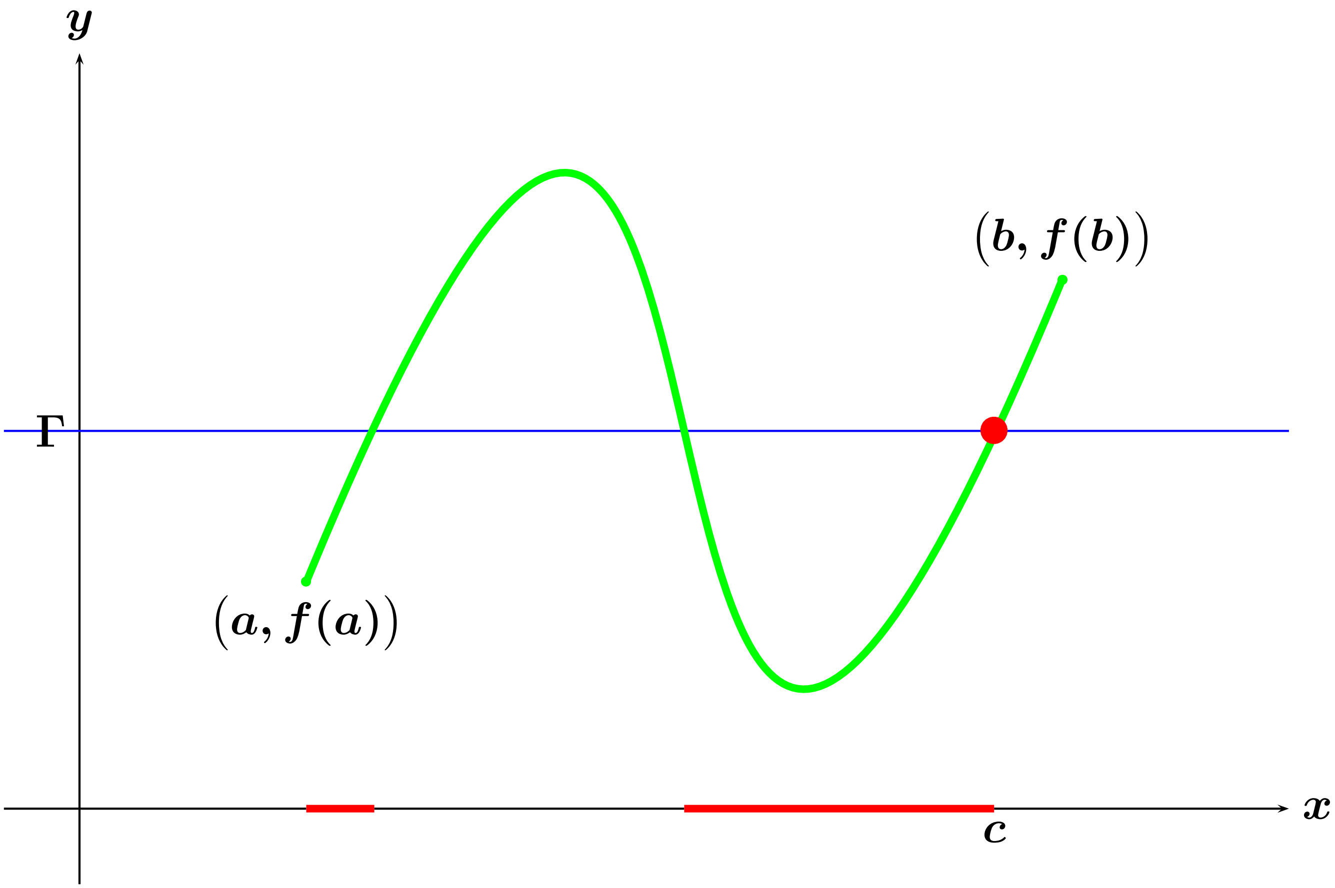
Intermediate Value Theorem:

If $f(x)$ is continuous for $a \leq x \leq b$, and if $f(a) < \Gamma < f(b)$, then there is at least one number c between a and b where $f(c) = \Gamma$.





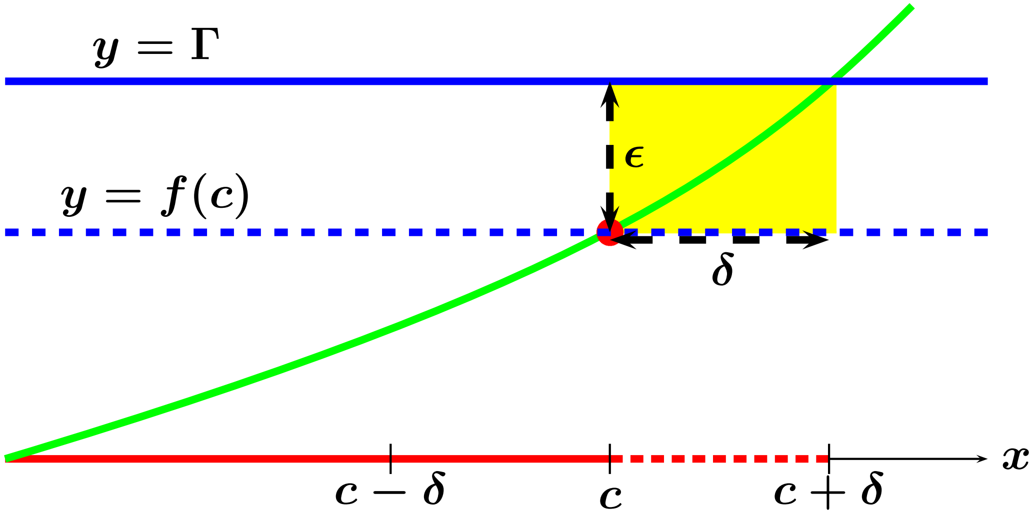
Proof:

Let S be the set of x's in $[a, b]$ having $f(x) \leq \Gamma$.

Let c be its least upper bound.

To show that $f(c) = \Gamma$, both $f(c) < \Gamma$ and $f(c) > \Gamma$ will be eliminated by contradictions in the following:

Suppose that $\boldsymbol{f(c)}$ were less than $\boldsymbol{\Gamma}$. Let $\boldsymbol{\epsilon}$ equal $\boldsymbol{\Gamma - f(c)}$, which would be $\boldsymbol{> 0}$.



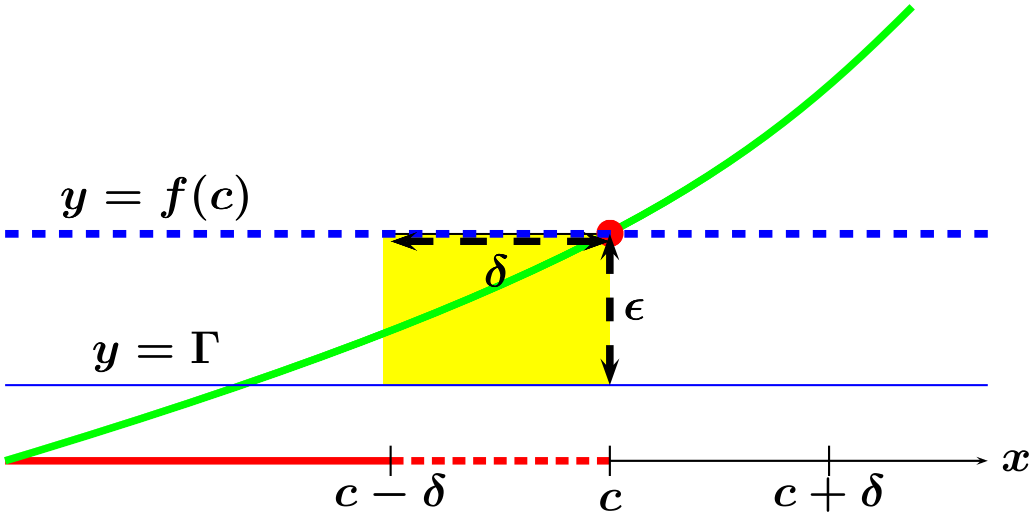
Continuity of $\boldsymbol{f}$ at $\boldsymbol{x = c}$ would require that there be a $\boldsymbol{\delta > 0}$ such that $\boldsymbol{c < x < c + \delta}$ would imply that $\boldsymbol{|f(x) - f(c)| < \Gamma - f(c)}$,

$$\begin{aligned} &\text{that}\quad \boldsymbol{f(x) - f(c)} && \boldsymbol{< \Gamma - f(c)}, \\ &\text{that}\quad \boldsymbol{f(x)} && \boldsymbol{< \Gamma}, \end{aligned}$$

that $\boldsymbol{x}$ belongs to the set $\boldsymbol{S}$, so that $\boldsymbol{x \le c}$.

(A Contradiction)

Suppose that
$\boldsymbol{f(c)}$ were greater than $\boldsymbol{\Gamma}$.
Let $\boldsymbol{\epsilon}$ be $\boldsymbol{f(c)-\Gamma}$,
which would now be $\boldsymbol{>0}$.



Continuity of $\boldsymbol{f}$ at $\boldsymbol{x = c}$ would require that there be a $\boldsymbol{\delta > 0}$ such that $\boldsymbol{c-\delta < x < c}$ would imply that $\boldsymbol{|f(x) - f(c)| < f(c) - \Gamma}$,

$$\boldsymbol{-\big(f(c)-\Gamma\big) < f(x) - f(c)},$$

$$\boldsymbol{\Gamma - f(c) < f(x) - f(c)},$$

and that $\qquad \boldsymbol{\Gamma \qquad < f(x)}$.

None of these $\boldsymbol{x}$'s belong to $\boldsymbol{S}$. Thus $\boldsymbol{c-\delta}$ is an upper bound, and $\boldsymbol{c}$ is not the least upper bound of $\boldsymbol{S}$. (Another Contradiction)

Intermediate Value Theorem:

If $f(x)$ is continuous for $a \le x \le b$, and if $f(a) < \Gamma < f(b)$, then there is at least one number c between a and b where $f(c) = \Gamma$.

Example: $f(x) = x^2$.

$f(1) = 1,\ f(2) = 4$

$1 < 3 < 4$

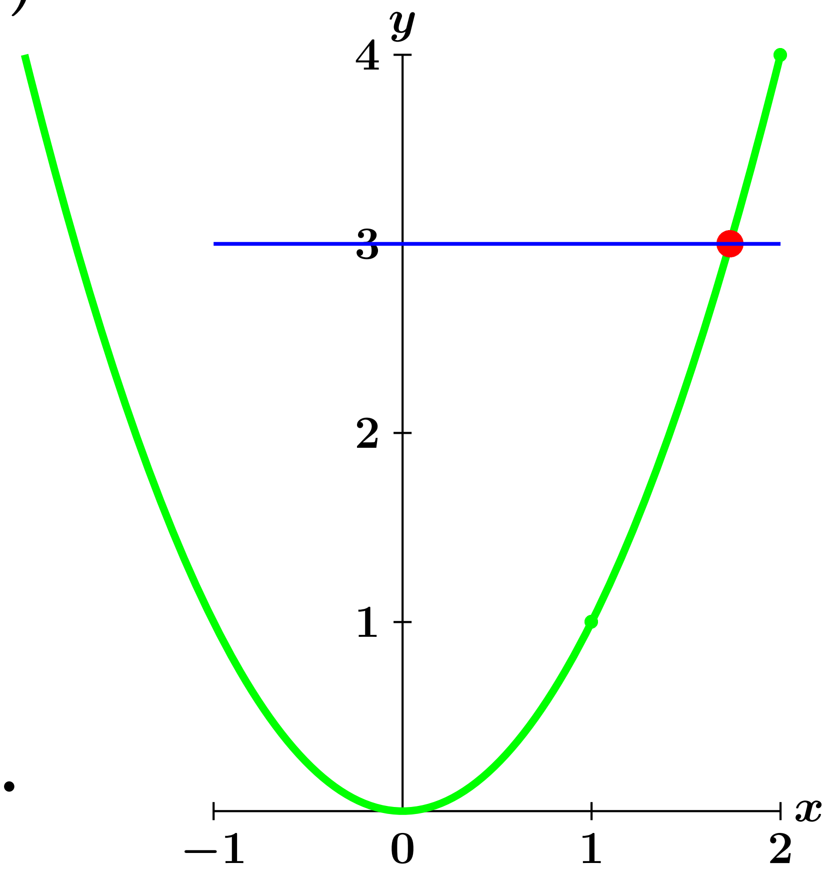
There is a number c satisfying

$f(c) = 3$

$c^2 = 3$

$c = \sqrt{3}$

Thus $\sqrt{3}$ exists as a real number.



A function is said to be increasing if $f(a) < f(b)$ for all a and b in its domain satisfying $a < b$.

This implies for any c and d that

$$f(c) < f(d) \quad \text{if } c < d,$$

because f is increasing,

$$f(c) > f(d) \quad \text{if } c > d,$$

also because f is increasing, and

$$f(c) = f(d) \quad \text{if } c = d,$$

because f is a function.

Since we have for any c and d that

$$
\begin{aligned}
f(c) &< f(d) \quad \text{if } c < d,\\
f(c) &> f(d) \quad \text{if } c > d,\\
f(c) &= f(d) \quad \text{if } c = d,
\end{aligned}
$$

to have $f(c) = f(d)$ would require

that neither $f(c) < f(d)$

nor $f(c) > f(d)$ could be true;

that neither $c < d$

nor $c > d$ could be true;

but that c must equal d.

For any increasing function f,

$f(c) = f(d)$ happens if, and only if, $c = d$.

Any increasing function f is one-to-one:

For each y in the range of f,

there is a unique x in the domain of f

such that $y = f(x)$.

Define $f^{-1}(y)$ to be this x.

For each y in the range of f,

there is a unique x in the domain of f

such that $y = f(x)$.

Define $f^{-1}(y)$ to be this x.

Clearly, $\boxed{f^{-1}\big(f(x)\big) \text{ equals } x}$

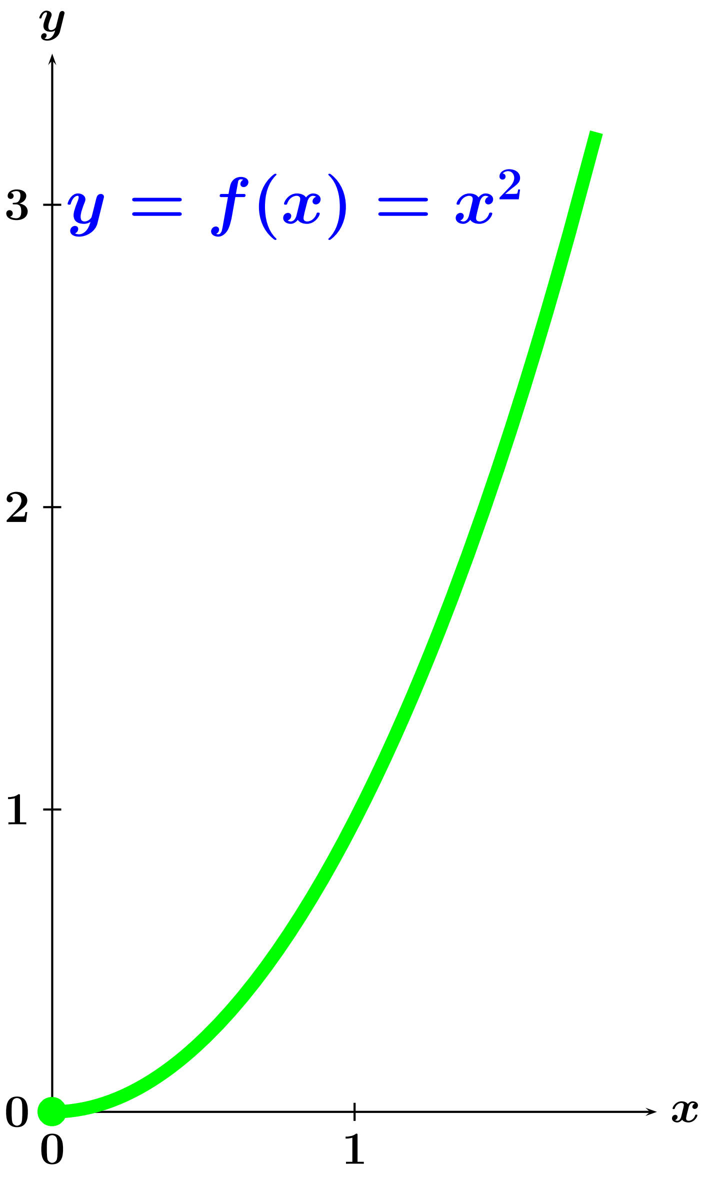
because $f^{-1}\big(f(x)\big) = f^{-1}(y) = x$,

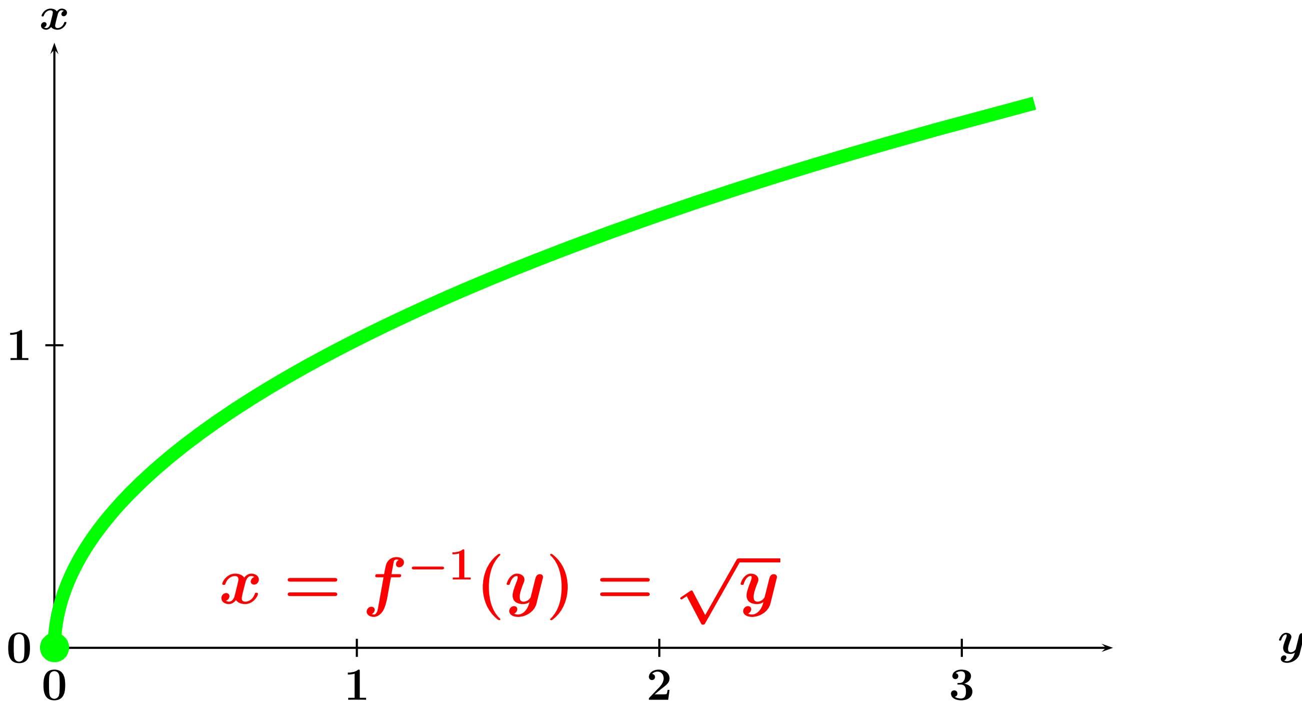
and $\boxed{f\big(f^{-1}(y)\big) \text{ equals } y}$

$$\text{because } f\big(f^{-1}(y)\big) = f\Big(f^{-1}\big(f(x)\big)\Big) = f(x) = y.$$

The inverse function of f is the function f^{-1}.

The inverse function of f^{-1} is the function f.


y
$y = f(x) = x^2$
3
2
1
0
0
1
x


x
$x = f^{-1}(y) = \sqrt{y}$
1
0
0
1
2
3
y

Since f is increasing, we have for any c and d in the domain of f that

$$f(c) < f(d) \quad \text{if } c < d,$$

$$f(c) > f(d) \quad \text{if } c > d,$$

$$f(c) = f(d) \quad \text{if } c = d,$$

to have $f(c) < f(d)$ would require

that neither $f(c) = f(d)$

nor $f(c) > f(d)$ could be true;

that neither $c = d$

nor $c > d$ could be true;

but that c must be less than d.

For any increasing function f,
$f(c) < f(d)$ happens if, and only if, $c < d$.

For any a and b in the domain of f^{-1}:

$$\begin{aligned} &a < b \\ \text{is equivalent to } &f\big(f^{-1}(a)\big) < f\big(f^{-1}(b)\big) \\ \text{and } &f^{-1}(a) < f^{-1}(b), \end{aligned}$$

so that the inverse function f^{-1} is also increasing.

Therefore:
Any increasing function f has an inverse function f^{-1},
which is also increasing.

Theorem: If $f(x)$ is defined, increasing and continuous
for all x on an interval $a \leq x \leq b$,
then the inverse function f^{-1}, defined above,
is also increasing and continuous.
Moreover, f^{-1} has $f(a) \leq y \leq f(b)$ as its domain,
and f^{-1} has $a \leq x \leq b$ as its range.

Proof: Obviously, $f^{-1}\big(f(a)\big)$ equals a and $f^{-1}\big(f(b)\big)$ equals b.

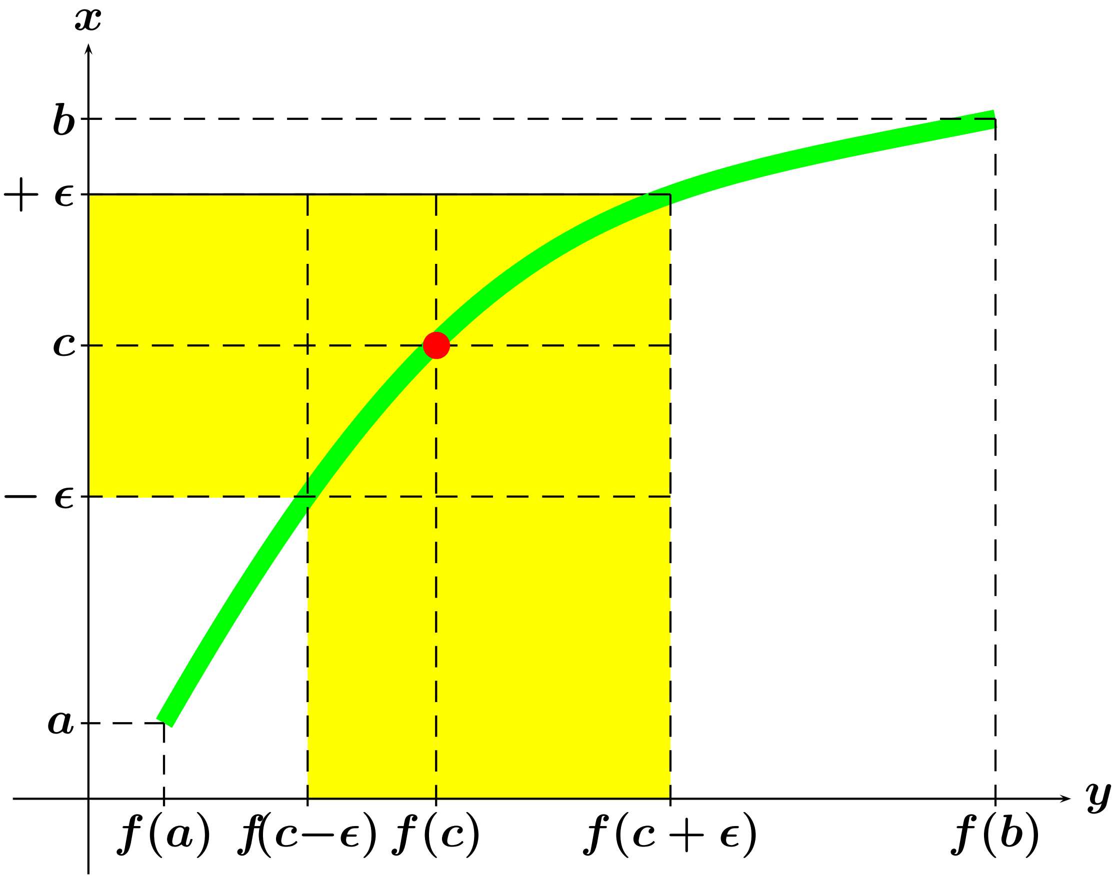
For any y between $f(a)$ and $f(b)$, $\qquad f(a) < y < f(b)$
the Intermediate Value Theorem assures the existence
of at least one c between a and b $\qquad a < c < b$
for which $f(c)$ equals this y.

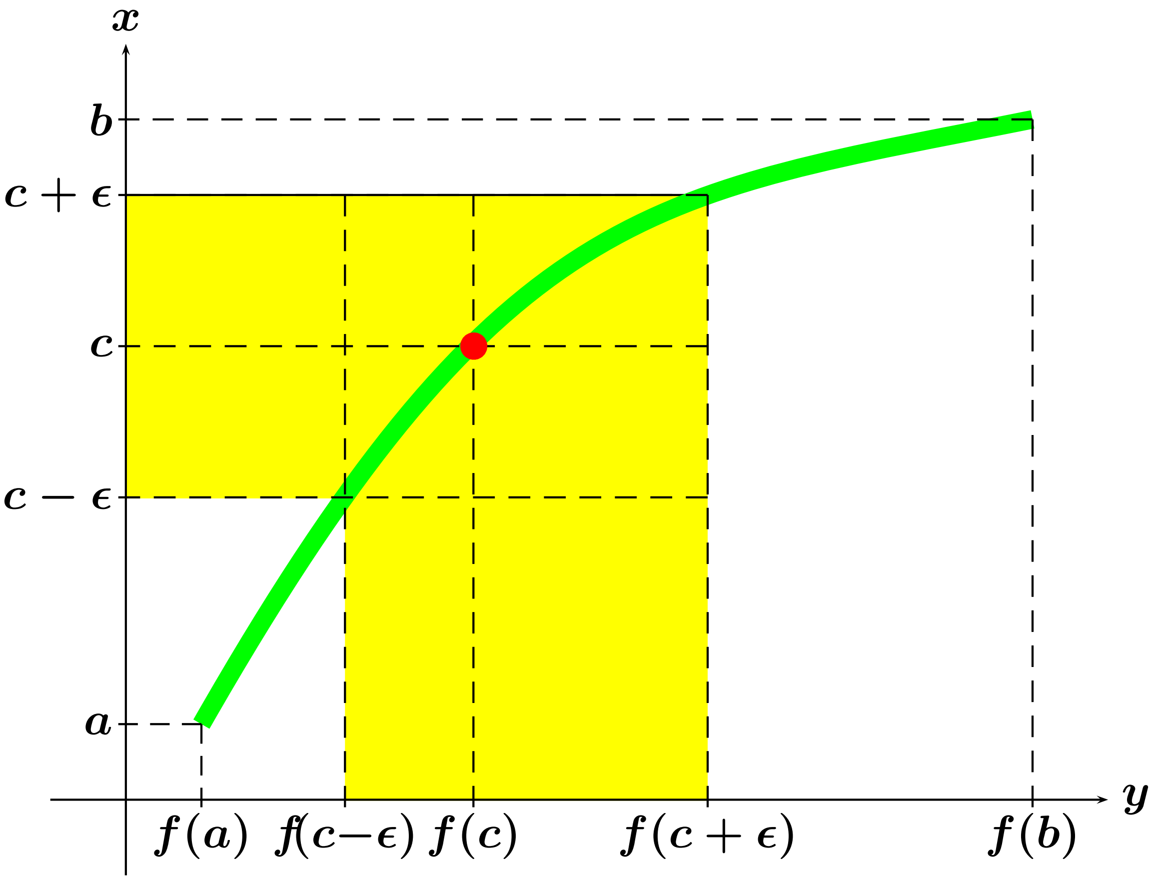
Since f is increasing, and thus one-to-one,
there can be only one such c where $f(c) = y$,
and $f^{-1}(y)$ equals this c.

$$\begin{array}{rrcl} \text{So far, we have} & a < c & = & f^{-1}(y) < b \\ \text{and} & f(a) < y & = & f(c) < f(b). \end{array}$$

To show the continuity of f^{-1} at $y = f(c)$,
consider WOLG any positive ϵ smaller than both $c - a$ and $b - c$.

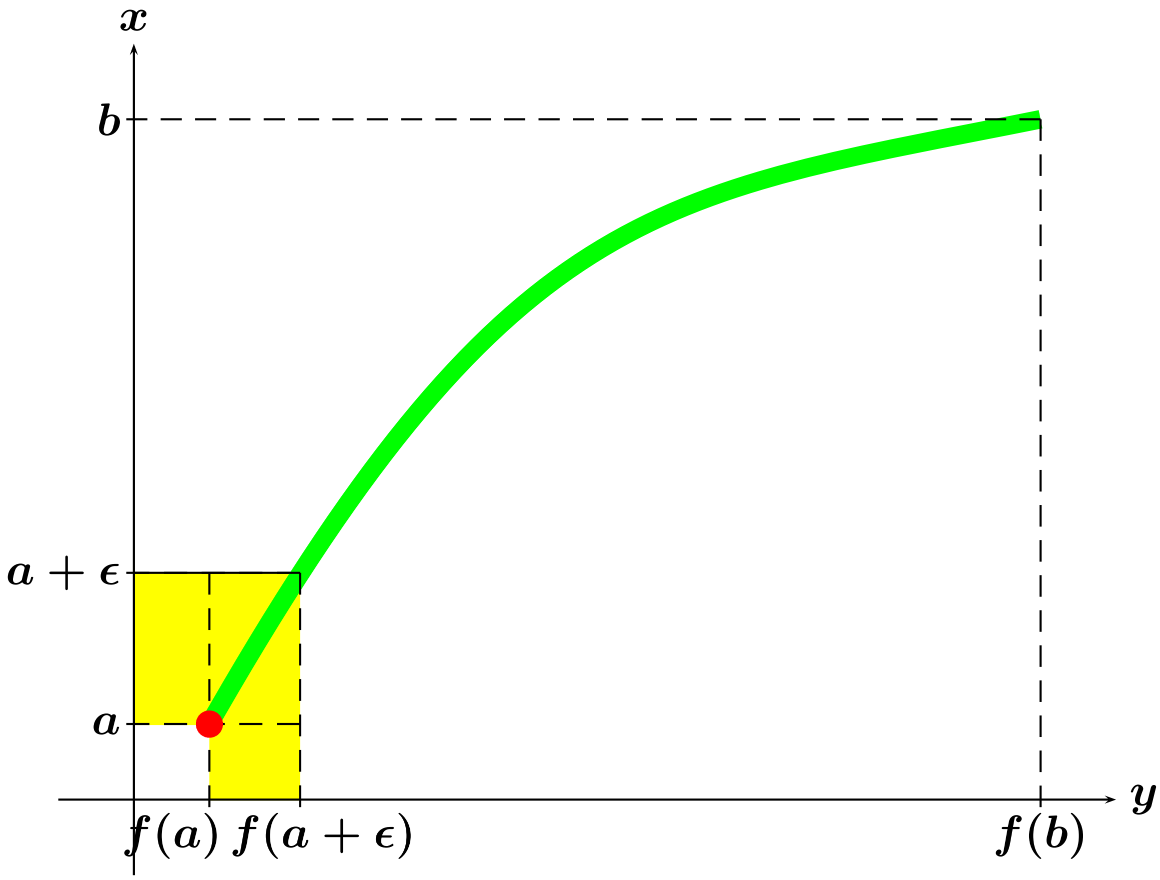
To show the continuity of f^{-1} at $y = f(c)$, consider WOLG any positive ϵ smaller than both $c - a$ and $b - c$.





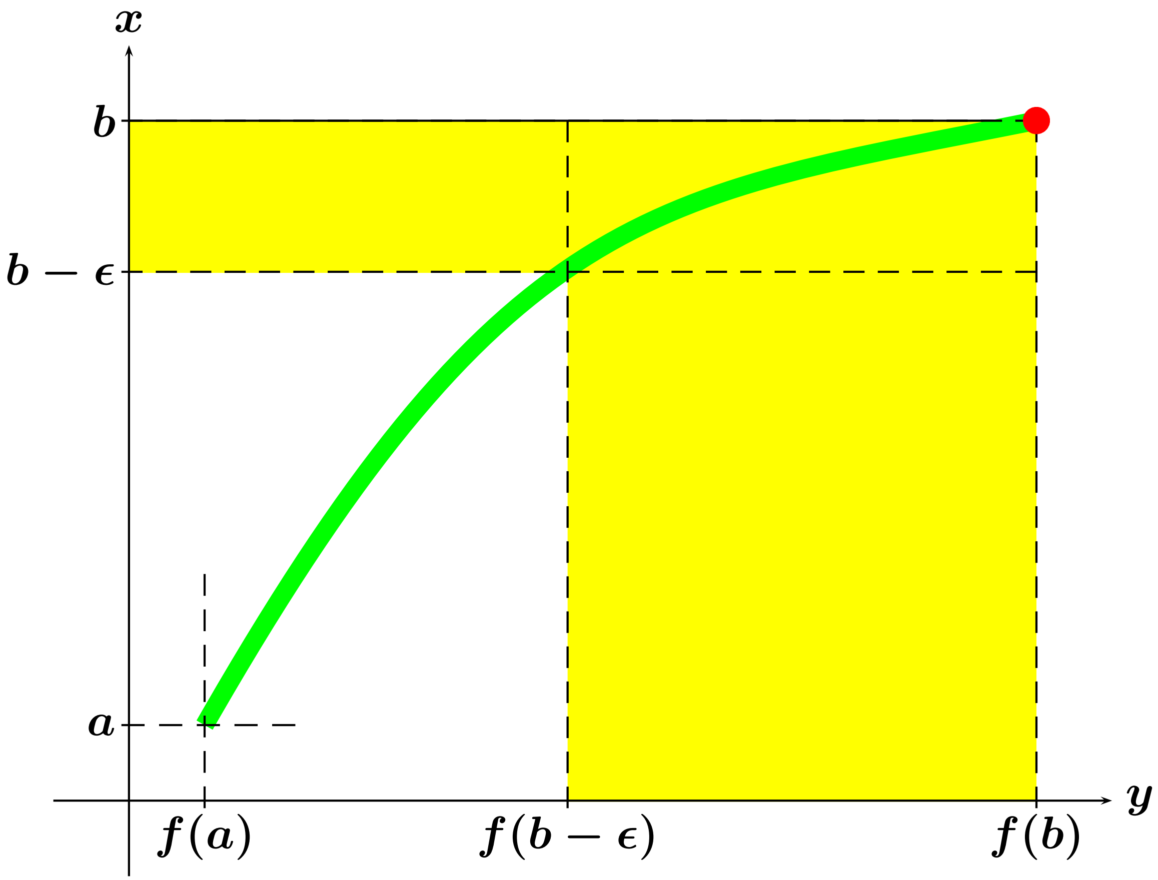
It would be sufficient to let δ be any positive number at least as small as the smaller of $f(c) - f(c-\epsilon)$ and $f(c+\epsilon) - f(c)$, i.e. $\delta < \min\big(f(c) - f(c-\epsilon), f(c+\epsilon) - f(c)\big)$.

To show the continuity of f^{-1} at $y = f(a)$ consider WOLG any positive ϵ smaller than $b - a$.



It would be sufficient to let δ be any positive number at least as small as $f(a + \epsilon) - f(a)$.

To show the continuity of f^{-1} at $y = f(b)$ consider WOLG any positive ϵ smaller than $b - a$.



It would be sufficient to let δ be any positive number at least as small as $f(b) - f(b - \epsilon)$.

Theorem:

If $f(x)$ is defined, increasing and continuous for all x on an interval, then the inverse function f^{-1}, defined above, is also increasing and continuous.

The domain of f^{-1} is the range of f.

The range of f^{-1} is the domain of f.

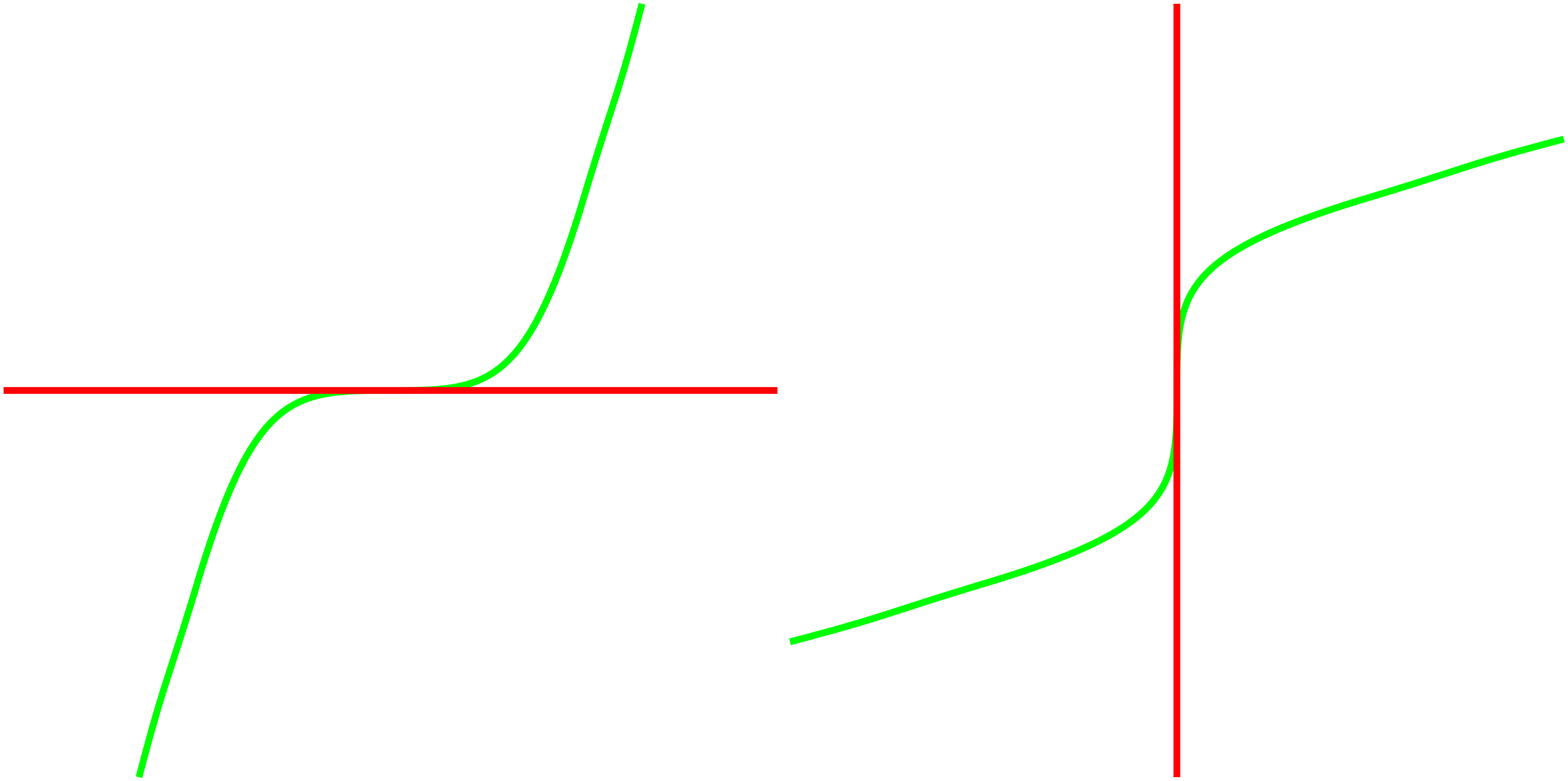
Theorem:

If $f(x)$ is defined, decreasing and continuous for all x on an interval, then the inverse function f^{-1}, defined above, is also decreasing and continuous.

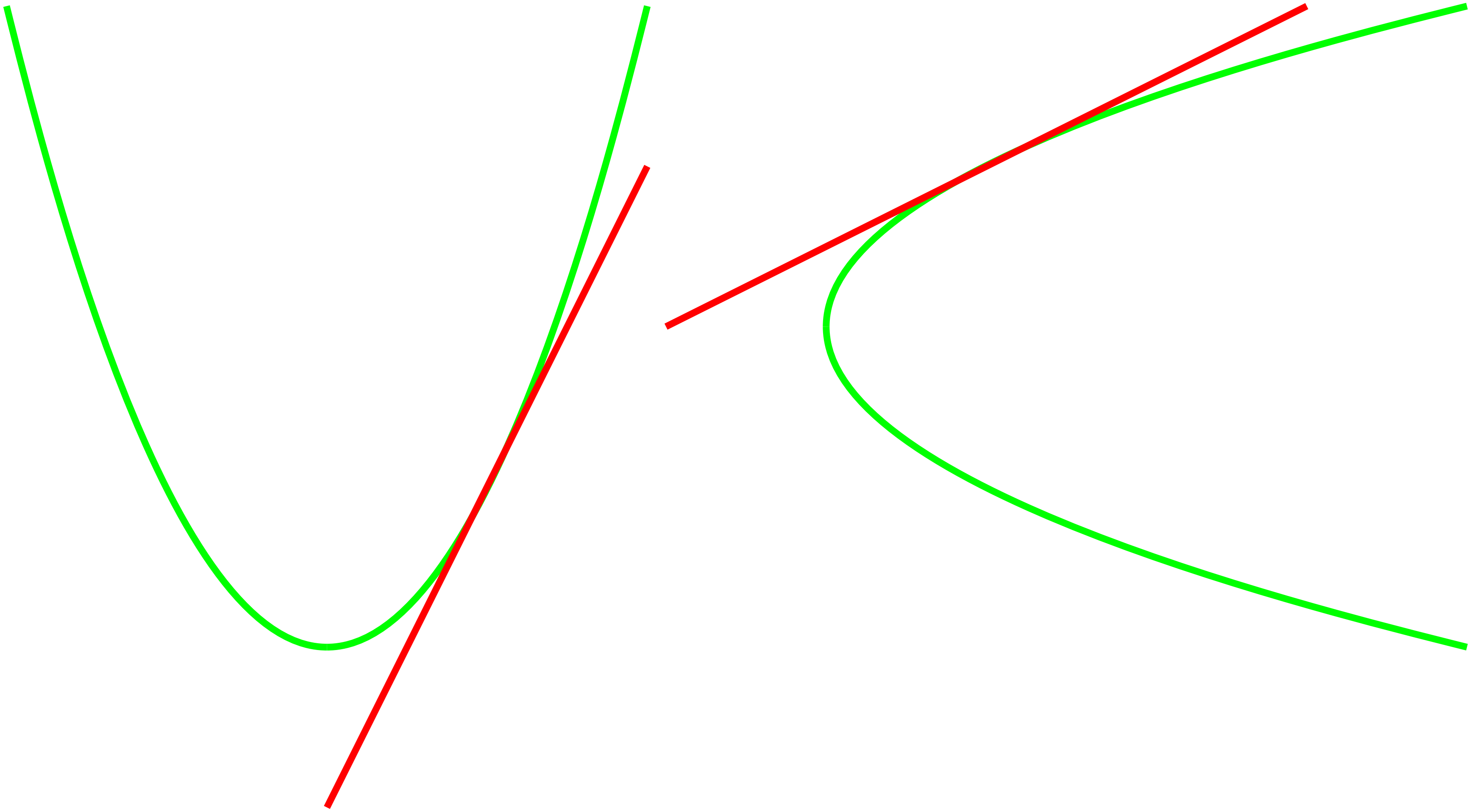
The domain of f^{-1} is the range of f.

The range of f^{-1} is the domain of f.

A vertical line tangent to the graph of f will correspond to a horizontal tangent for f^{-1}:



A diagonal tangent line for one graph will become a diagonal tangent line for the other graph, but with a reciprocal slope:



Theorem: With $y = f(x)$, if $f'(x)$ exists and is nonzero, then $\left(f^{-1}\right)'(y)$ exists and equals the reciprocal of $f'(x)$.

Proof:

$$
\begin{aligned}
\lim_{v\to y}\frac{f^{-1}(v)-f^{-1}(y)}{v-y} &= \lim_{u\to x}\frac{f^{-1}(f(u))-f^{-1}(f(x))}{f(u)-f(x)} \\
= \lim_{u\to x}\frac{u-x}{f(u)-f(x)} &= \lim_{u\to x}\frac{1}{\dfrac{f(u)-f(x)}{u-x}} \\
= \frac{1}{\displaystyle\lim_{u\to x}\frac{f(u)-f(x)}{u-x}} &= \frac{1}{f'(x)}
\end{aligned}
$$

Theorem: With $y = f(x)$, if $f'(x)$ exists and is nonzero, then $\left(f^{-1}\right)'(y)$ exists and equals the reciprocal of $f'(x)$.

Proof, using $f(u) - f(x) = \big(f'(x) + o(1)\big)(u - x)$:

$$u - x = \frac{f(u) - f(x)}{f'(x) + o(1)}$$

$$f^{-1}(v) - f^{-1}(y) = \frac{f(f^{-1}(v)) - f(f^{-1}(y))}{f'(x) + o(1)}$$

$$f^{-1}(v) - f^{-1}(y) = \frac{1}{f'(x) + o(1)}(v - y)$$

$$f^{-1}(v) - f^{-1}(y) = \left(\frac{1}{f'(x)} + o(1)\right)(v - y)$$

Theorem: With $y = f(x)$, if $f'(x)$ exists and is nonzero, then $\left(f^{-1}\right)'(y)$ exists and equals the reciprocal of $f'(x)$.

This can be interpreted notationally as:

$$\left(f^{-1}\right)'(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f'(x)} = \frac{1}{f'\big(f^{-1}(y)\big)}$$

Example:

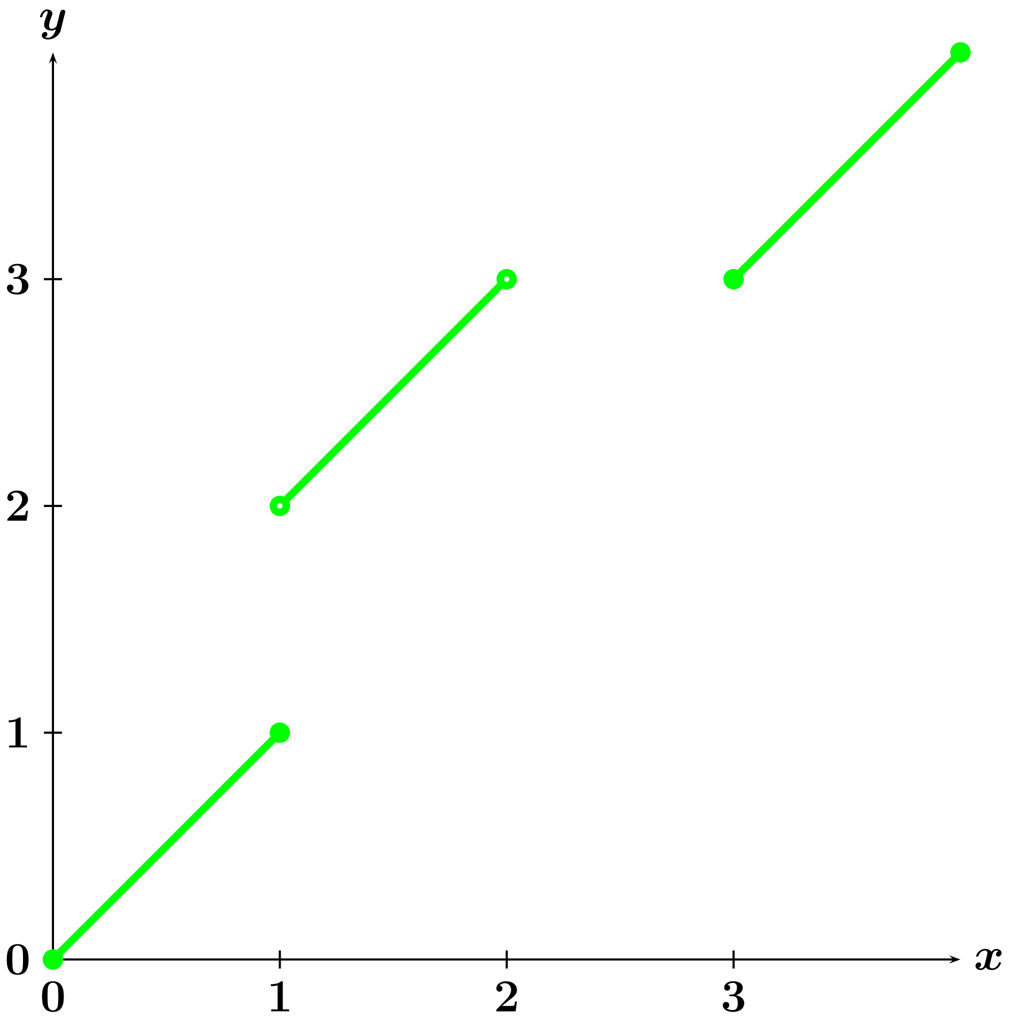
$$f(u) = u^{\frac{1}{n}}, \qquad f'(u) = \frac{1}{n}\ u^{\frac{1}{n}-1} = \frac{u^{\frac{1}{n}-1}}{n} = \frac{u^{\frac{1-n}{n}}}{n}$$

$$f^{-1}(y) = y^n, \qquad \left(f^{-1}\right)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)} = \frac{1}{\frac{\left(f^{-1}(y)\right)^{\frac{1-n}{n}}}{n}}$$

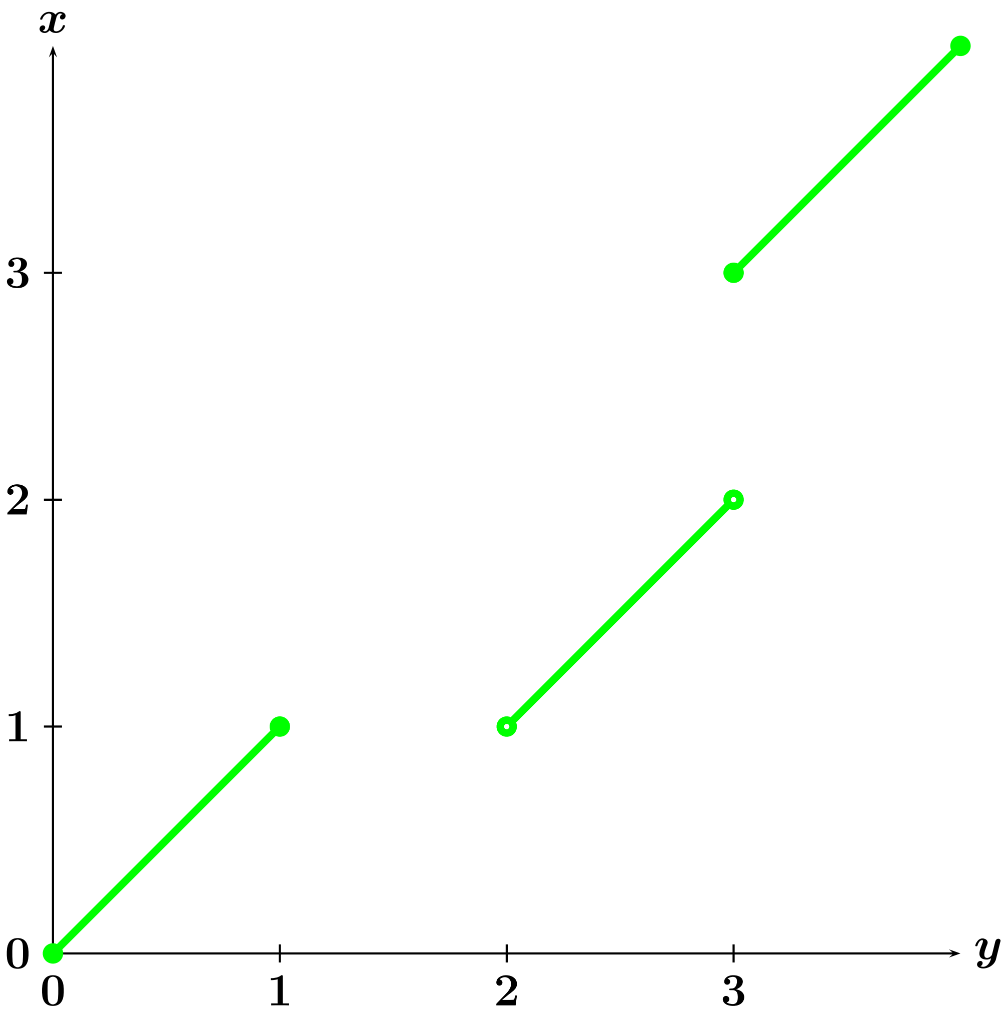
$$= \frac{n}{\left(f^{-1}(y)\right)^{\frac{1-n}{n}}} = \frac{n}{\left(y^n\right)^{\frac{1-n}{n}}} = \frac{n}{y^{1-n}} = ny^{n-1}.$$

Example:

$$y = f(x) = \begin{cases} x, \\ \quad \text{if } 0 \le x \le 1 \\ x+1, \\ \quad \text{if } 1 < x < 2 \\ \text{undefined}, \\ \quad \text{if } 2 \le x < 3 \\ x, \\ \quad \text{if } 3 \le x \le 4 \end{cases}$$



$$x = f^{-1}(y) = \begin{cases} y, & \text{if } 0 \leq y \leq 1 \\ \text{undefined}, & \text{if } 1 < y \leq 2 \\ y - 1, & \text{if } 2 < y < 3 \\ y, & \text{if } 3 \leq y \leq 4 \end{cases}$$



Example: For any positive integer n, and positive $2n+1$:

$$y = f(x) = x^{2n+1} \text{ for } -\infty < x < \infty$$

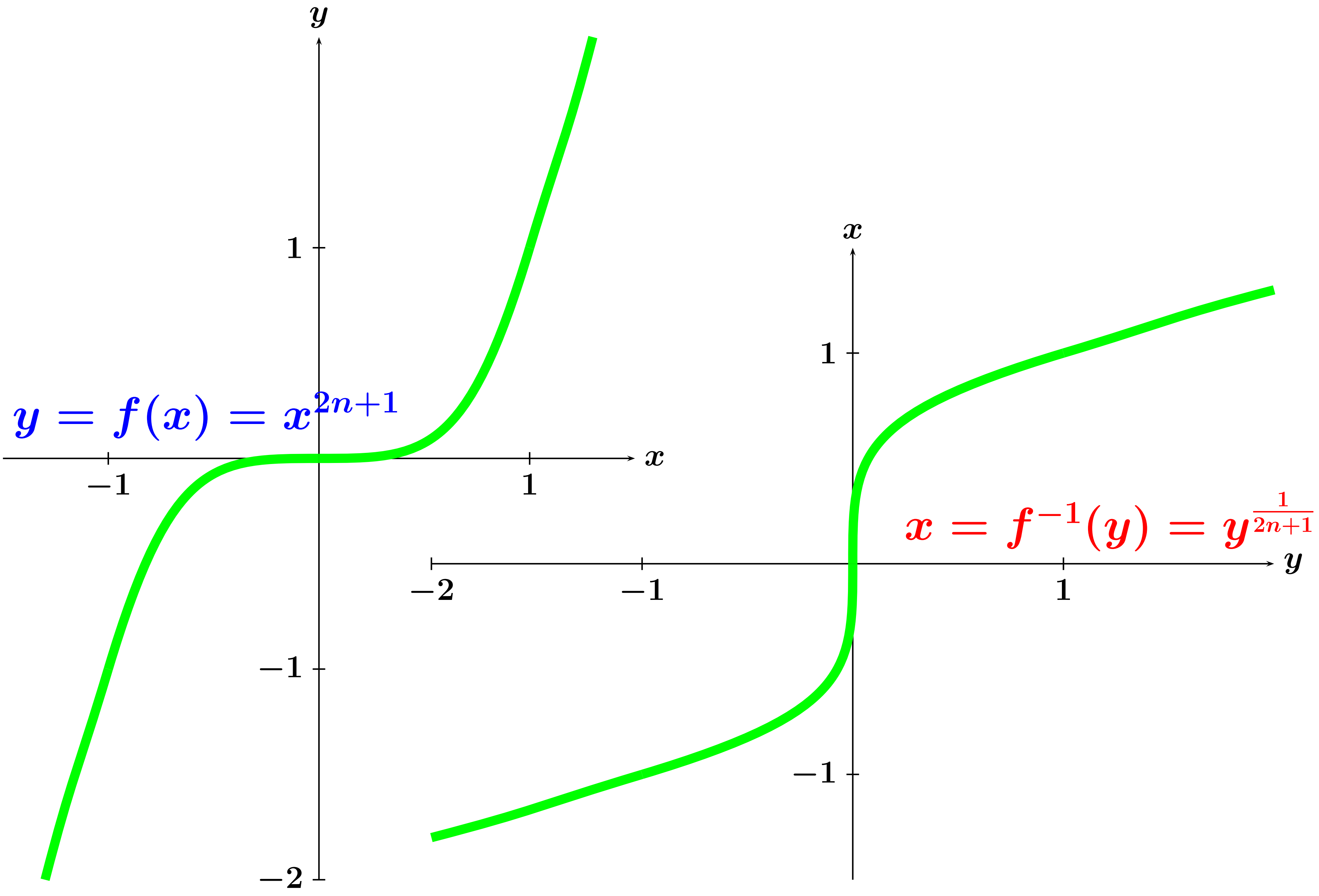
$$\frac{dy}{dx} = f'(x) = (2n+1)x^{2n}$$

$$x = f^{-1}(y) = y^{\frac{1}{2n+1}} \text{ for } -\infty < y < \infty$$

$$\frac{d}{dy}f^{-1}(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{(2n+1)x^{2n}} = \frac{x^{-2n}}{2n+1}$$

$$= \frac{\left(y^{\frac{1}{2n+1}}\right)^{-2n}}{2n+1} = \frac{y^{\left(\frac{-2n}{2n+1}\right)}}{2n+1} = \frac{y^{\left(\frac{1}{2n+1}-1\right)}}{2n+1}$$

$$\text{for } -\infty < y < 0, \quad 0 < y < \infty. \qquad (y \neq 0)$$


$y = f(x) = x^{2n+1}$
y
x
1
-1
-2
$x = f^{-1}(y) = y^{\frac{1}{2n+1}}$
x
y
1
-1
-2

Example: $y = f(x) = x^{2n}$ for $0 \le x < \infty$

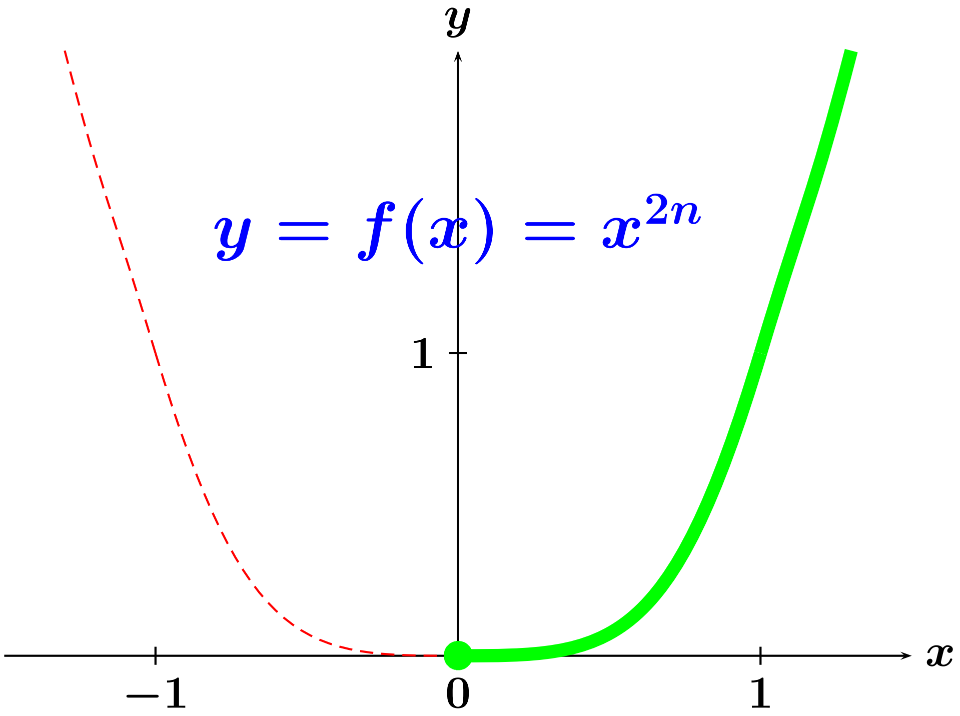
$$\frac{dy}{dx} = f'(x) = 2nx^{2n-1}$$

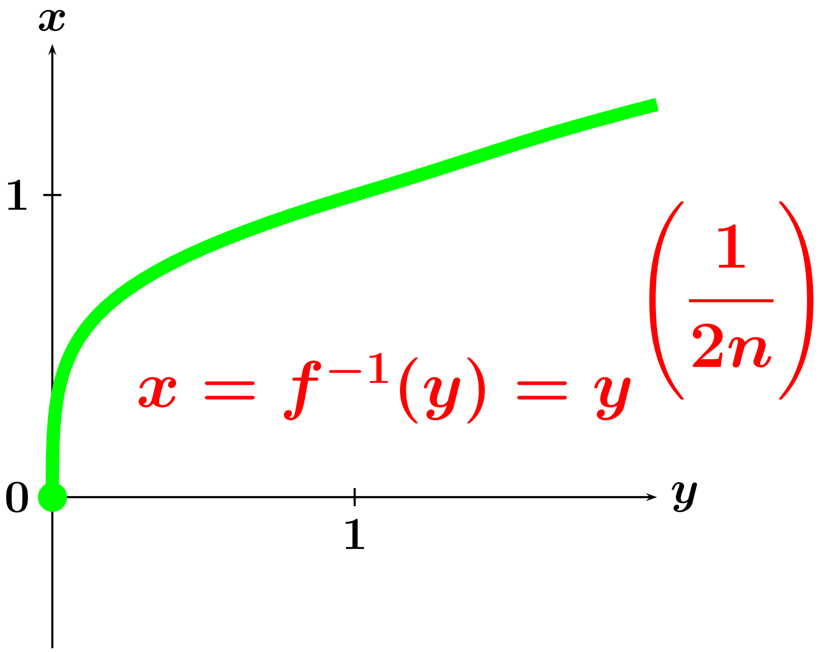
$$x = f^{-1}(y) = y^{\left(\frac{1}{2n}\right)} \text{ for } 0 \le y < \infty$$

$$\frac{d}{dy}f^{-1}(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2nx^{2n-1}} = \frac{x^{1-2n}}{2n}$$

$$= \frac{\left(y^{\left(\frac{1}{2n}\right)}\right)^{1-2n}}{2n} = \frac{y^{\left(\frac{1-2n}{2n}\right)}}{2n} = \frac{1}{2n}y^{\left(\frac{1}{2n}-1\right)}$$

for $0 < y < \infty$.


y
$y = f(x) = x^{2n}$
1
x
-1
0
1


x
1
$x = f^{-1}(y) = y^{\left(\frac{1}{2n}\right)}$
0
y
1

Example:

$$y = f(x) = x^2 \text{ for } 0 \leq x < \infty$$

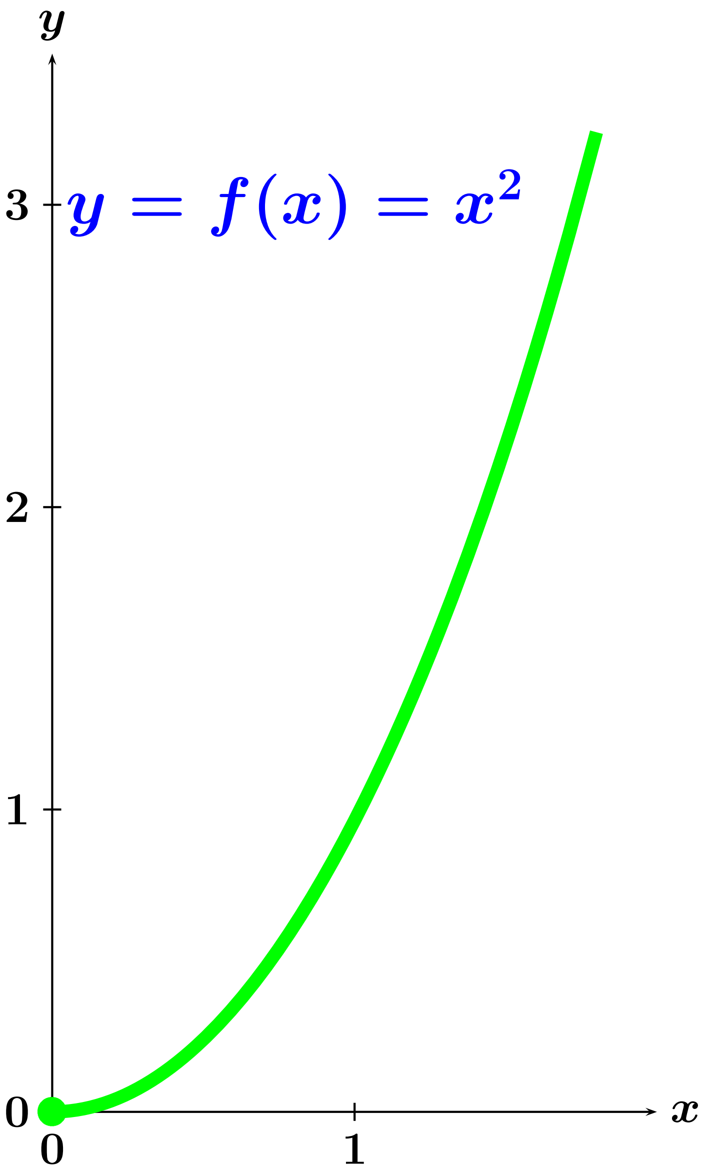
$$\frac{dy}{dx} = f'(x) = 2x \text{ for}$$

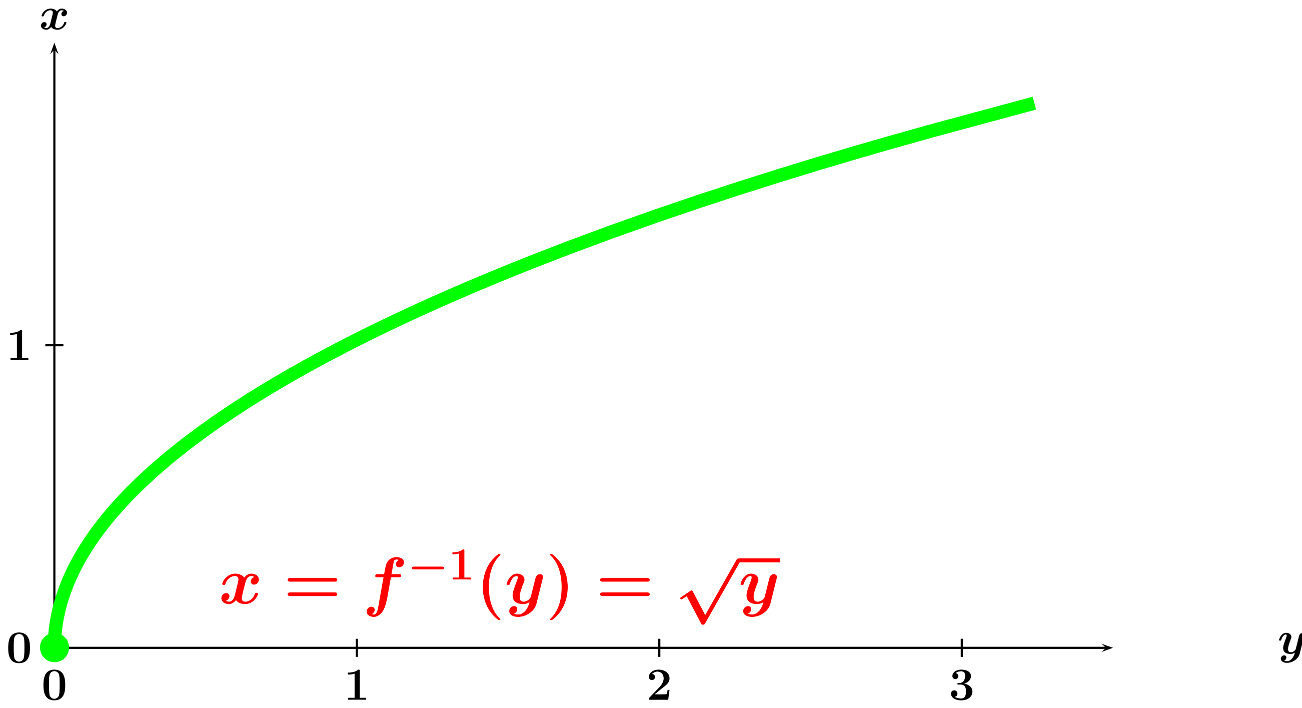
$$x = f^{-1}(y) = \sqrt{y} \text{ for } 0 \leq y < \infty$$

$$\frac{d}{dy}f^{-1}(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x}$$

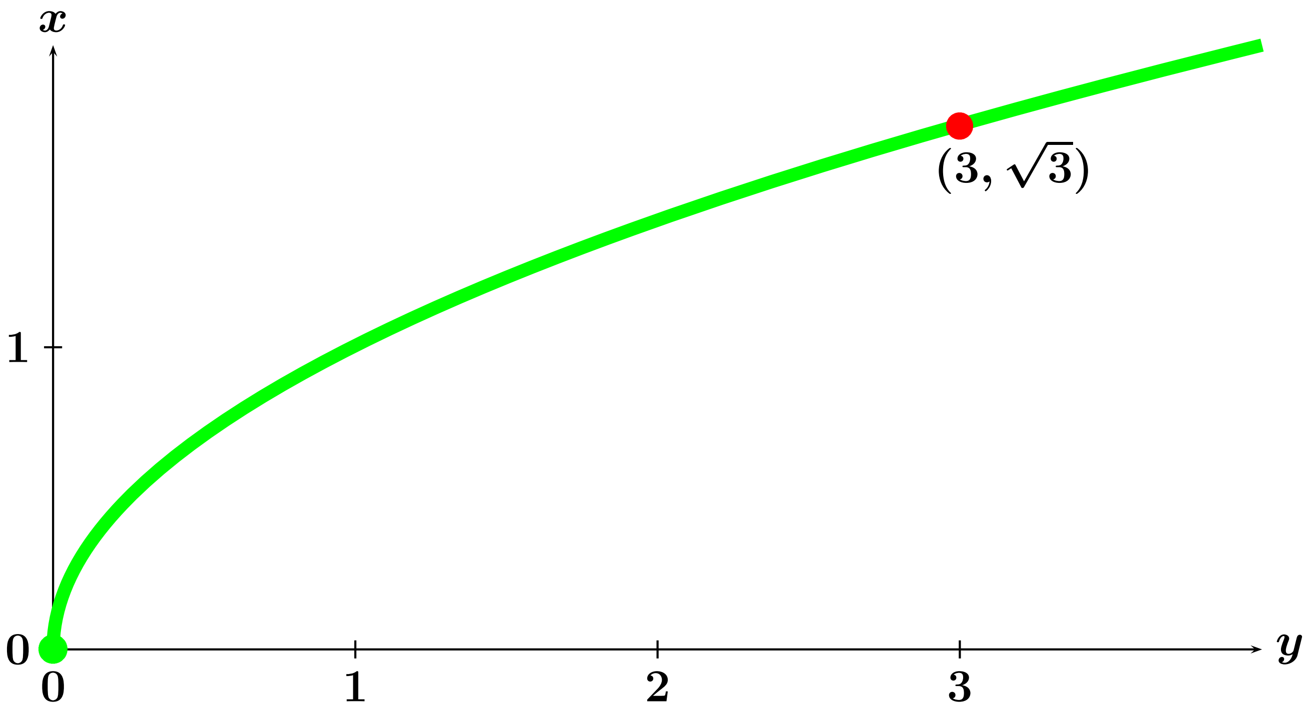
$$= \frac{1}{2\sqrt{y}}$$

$$\text{for } 0 < y < \infty.$$


y
3
2
1
0
0
1
x
$y = f(x) = x^2$


x
1
0
0
1
2
3
y
$x = f^{-1}(y) = \sqrt{y}$

$$x = f^{-1}(y) = \sqrt{y} \qquad \text{for } 0 \le y < \infty$$

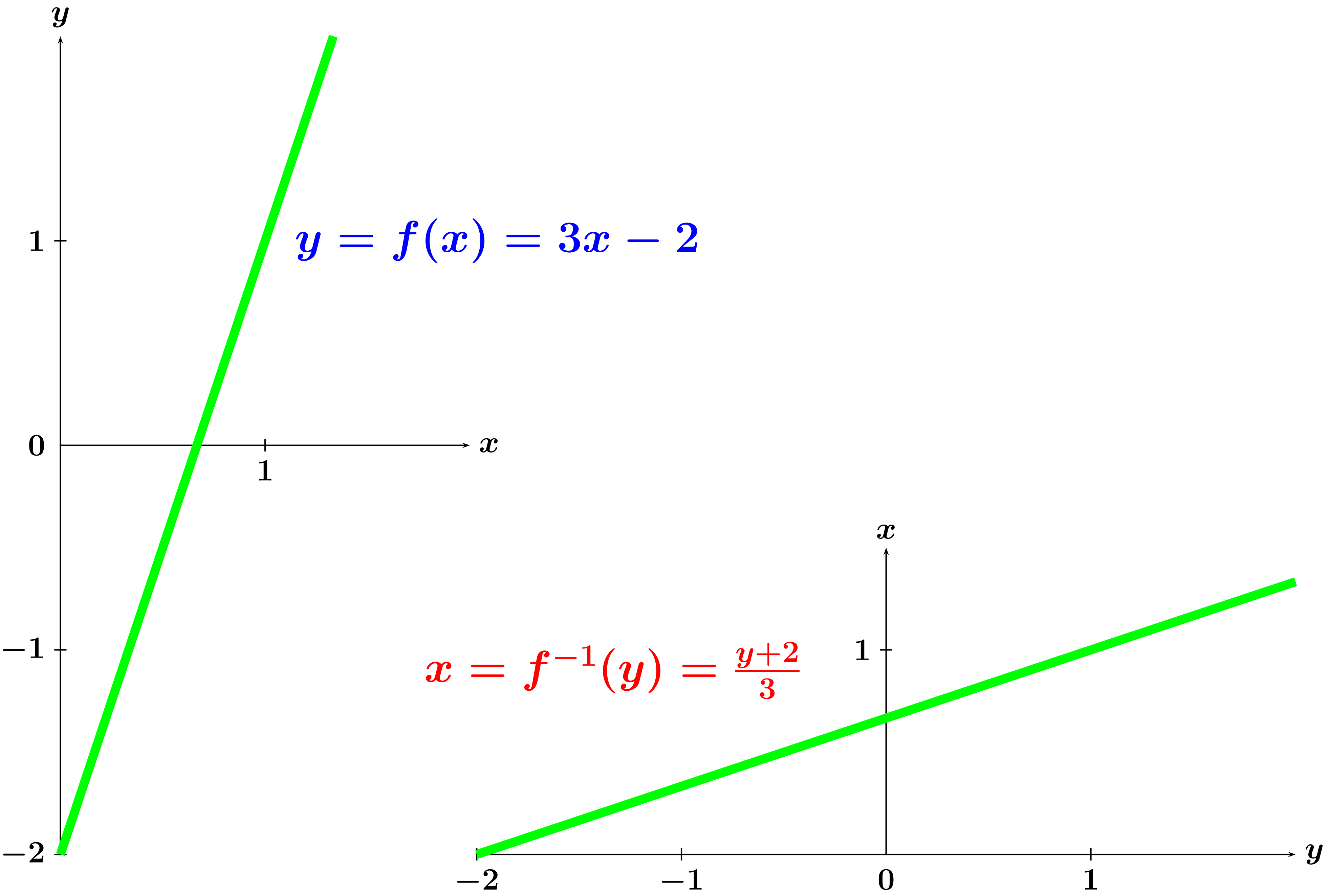


Example:

$$\begin{aligned} y &= f(x) = 3x - 2 \text{ for } -\infty < x < \infty \\ \frac{dy}{dx} &= f'(x) = 3 \text{ for } -\infty < x < \infty \end{aligned}$$

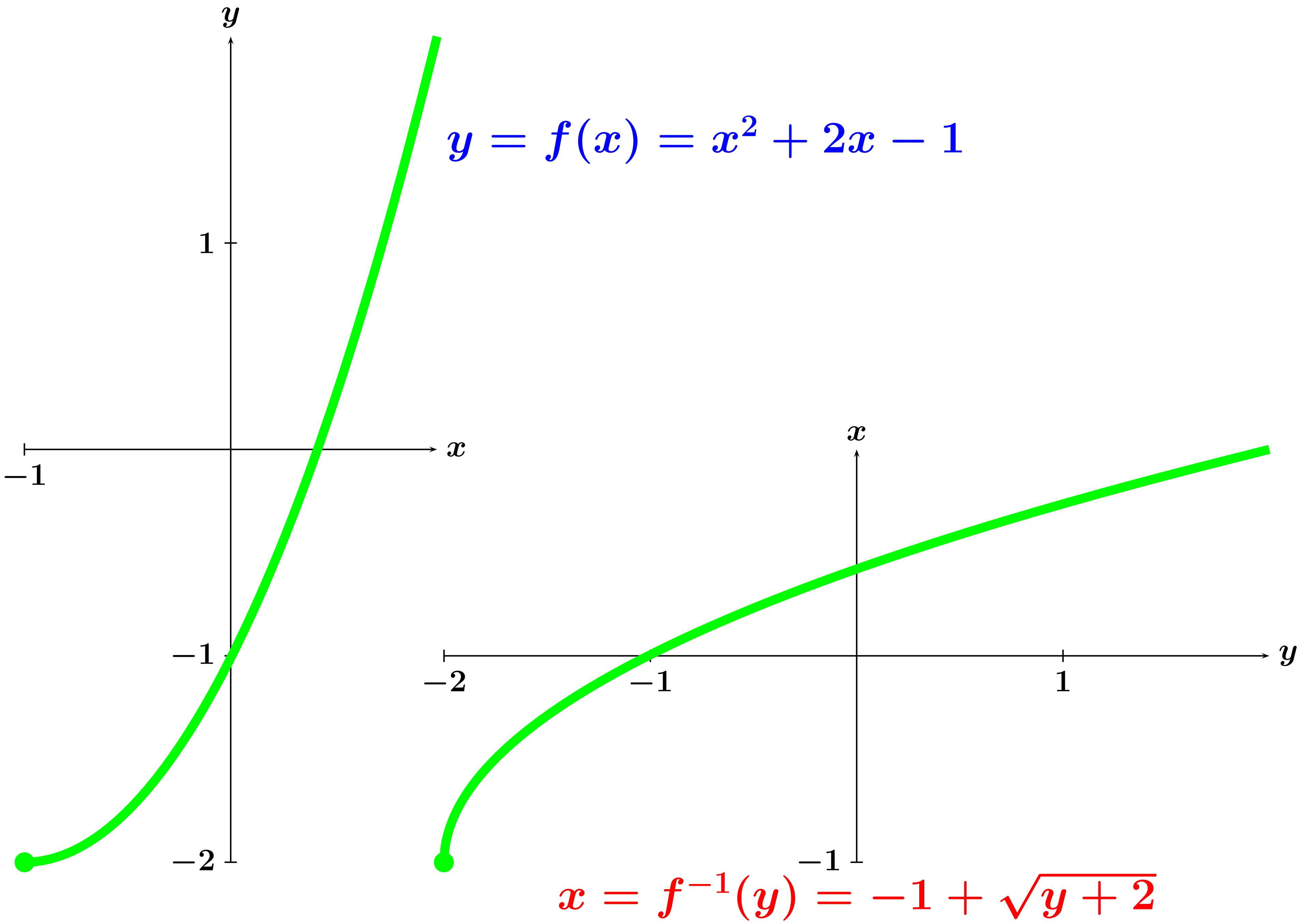
$$\begin{aligned} y &= f(x) = 3x - 2, \\ y + 2 &= 3x, \end{aligned}$$

$$\begin{aligned} x &= f^{-1}(y) = \frac{y+2}{3} \text{ for } -\infty < y < \infty \\ \frac{d}{dy} f^{-1}(y) &= \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3}. \end{aligned}$$


y
1
0
-1
-2
x
1
$y = f(x) = 3x - 2$
x
1
$x = f^{-1}(y) = \frac{y+2}{3}$
-2
-1
0
1
y

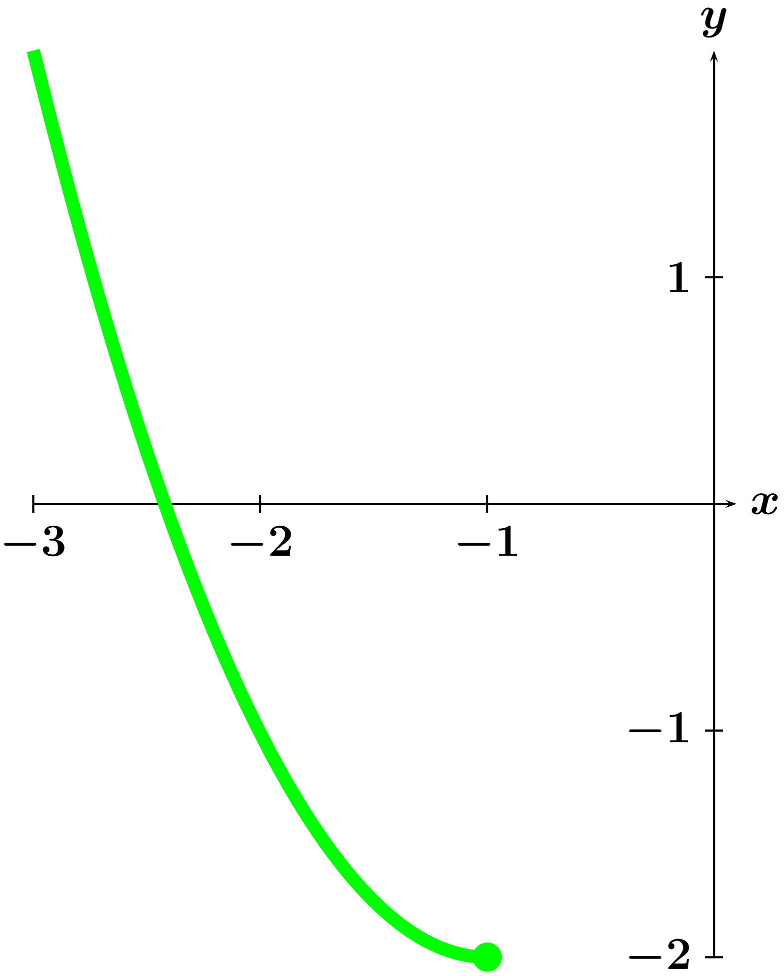
Example:

$$\begin{aligned}
y &= f(x) = x^2 + 2x - 1 \text{ for } -1 \leq x < \infty \\
\frac{dy}{dx} &= f'(x) = 2x + 2 \text{ for } -1 \leq x < \infty \\
x &= f^{-1}(y) = -1 + \sqrt{y+2} \text{ for } -2 \leq y < \infty \\
\frac{d}{dy} f^{-1}(y) &= \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x+2} \\
&= \frac{1}{2(-1+\sqrt{y+2})+2} = \frac{1}{-2+2\sqrt{y+2}+2} \\
&= \frac{1}{2\sqrt{y+2}} \quad \text{for } -2 < y < \infty.
\end{aligned}$$

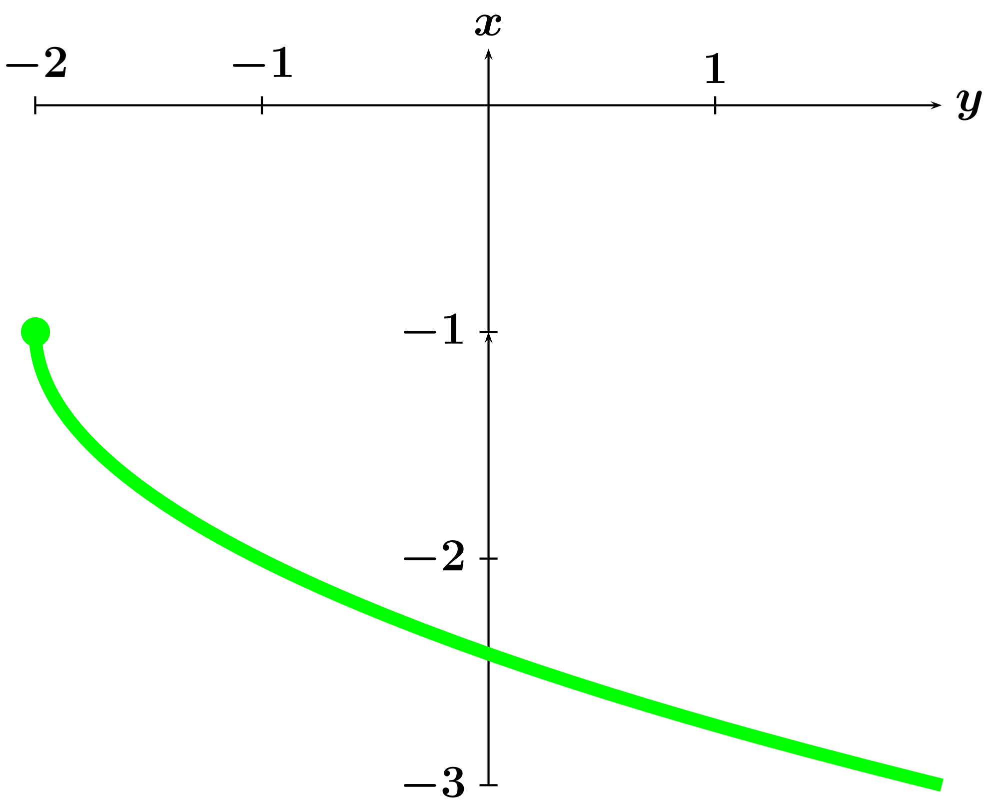

y
$y = f(x) = x^2 + 2x - 1$
1
x
-1
-1
-2
x
y
-2
-1
1
-1
$x = f^{-1}(y) = -1 + \sqrt{y+2}$

Example:

$$
\begin{aligned}
y &= f(x) = x^2 + 2x - 1 \text{ for } -\infty < x \leq -1 \\
\frac{dy}{dx} &= f'(x) = 2x + 2 \\
x &= f^{-1}(y) = -1 - \sqrt{y+2} \text{ for } -2 \leq y < \infty \\
\frac{d}{dy} f^{-1}(y) &= \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x+2} \\
&= \frac{1}{2(-1-\sqrt{y+2})+2} = \frac{1}{-2-2\sqrt{y+2}+2} \\
&= -\frac{1}{2\sqrt{y+2}} \quad \text{for } -2 < y < \infty.
\end{aligned}
$$



$$y = f(x) = x^2 + 2x - 1$$



$$x = f^{-1}(y) = -1 - \sqrt{y+2}$$