Handy Tests for Convergence of a Non-Negative Series $\sum a_n$ based on Comparisons with Geometric Series $\sum C^n$:

The Root Test, for Nonnegative Series:

$$\text{If } \lim_{n\to\infty} \sqrt[n]{a_n} = L < 1, \quad \text{then } \sum_{n=n_0}^{\infty} a_n \text{ converges.}$$

$$\text{If } \lim_{n\to\infty} \sqrt[n]{a_n} = L > 1, \quad \text{then } \sum_{n=n_0}^{\infty} a_n \text{ diverges.}$$

The Ratio Test, for Positive Series:

$$\text{If } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1, \quad \text{then } \sum_{n=n_0}^{\infty} a_n \text{ converges.}$$

$$\text{If } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1, \quad \text{then } \sum_{n=n_0}^{\infty} a_n \text{ diverges.}$$

$$\left(\text{Whenever } L \text{ equals } 1\text{, we compare with } \sum \frac{1}{x^p}\text{, etc., instead.}\right)$$

Example:

To test the convergence of $\sum \frac{3^k}{k4^k - 1}$**:**

using the ratio test,

$$\frac{a_{k+1}}{a_k} = \frac{\dfrac{3^{k+1}}{(k+1)4^{k+1} - 1}}{\dfrac{3^k}{k4^k - 1}} = \frac{\dfrac{3^{k+1}}{3^k}}{\dfrac{4 - \dfrac{1}{(k+1)4^k}}{\dfrac{k}{k+1} - \dfrac{1}{(k+1)4^k}}} \to \frac{3}{4} < 1,$$

or using the root test,

$$\sqrt[k]{\frac{3^k}{k4^k - 1}} = \frac{3}{4}\ \frac{1}{\sqrt[k]{k - \dfrac{1}{4^k}}} \to \frac{3}{4} < 1,$$

(if we apply L'Hospital's Rule to the logarithm)

For the Root Test, we shall often use:

Since $\lim_{n\to\infty} \frac{\ln n}{n} = 0, \quad \lim_{n\to\infty} \sqrt[n]{n} = 1, \qquad \boxed{\sqrt[n]{n} \sim 1}$

Since $\lim_{n\to\infty} \frac{\ln(\ln n)}{n} = 0, \quad \lim_{n\to\infty} \sqrt[n]{\ln n} = 1, \qquad \boxed{\sqrt[n]{\ln n} \sim 1}$

Since $\lim_{n\to\infty} \frac{\ln(\ln(\ln n))}{n} = 0, \qquad \boxed{\sqrt[n]{\ln(\ln n)} \sim 1}$

etc.

Root Test for Convergence of $\sum_{n=n_0}^{\infty} L^n \; n^p \; (\ln n)^q \; (\ln(\ln n))^r$**:**

$$\sqrt[n]{L^n \; n^p \; (\ln n)^q \; (\ln(\ln n))^r} \sim L \cdot 1^p \cdot 1^q \cdot 1^r = L,$$

which implies convergence for the series if $L < 1$

and divergence for the series if $L > 1$.

(The Root Test implies nothing if $L = 1$.**)**

For the Root Test, we can use Stirling's Approximation:

$$n! \sim \sqrt{2\pi n}\ n^n\ e^{-n}$$

$$(an)! \sim \sqrt{2\pi an}\ (an)^{an}\ e^{-an}$$

$$\sqrt[n]{(an)!} \sim \sqrt[2n]{2\pi an}\ (an)^a\ e^{-a} \sim \frac{a^a n^a}{e^a} \qquad \boxed{\sqrt[n]{(an)!} \sim \frac{a^a n^a}{e^a}}$$

Example:

Root Test verifying convergence of $\displaystyle\sum_{n=n_0}^{\infty} \frac{(an)!}{(bn)!}$, **if** $a < b$**:**

$$\sqrt[n]{\frac{(an)!}{(bn)!}} \sim \frac{\dfrac{a^a n^a}{e^a}}{\dfrac{b^b n^b}{e^b}} \sim \frac{a^a}{b^b e^{a-b}} n^{a-b},$$

which approaches 0, and is < 1**, as** n **approaches** ∞**.**

For the Ratio Test, we shall often use:

$$\boxed{\frac{n+1}{n} \sim 1} \quad \boxed{\frac{\ln(n+1)}{\ln n} \sim 1} \text{ and } \boxed{\frac{\ln(\ln(n+1))}{\ln(\ln n)} \sim 1}$$

Ratio Test for Convergence of $\displaystyle\sum_{n=n_0}^{\infty} L^n \; n^p \; (\ln n)^q \; (\ln(\ln n))^r$**:**

$$\frac{L^{n+1} \; (n+1)^p \; (\ln(n+1))^q \; (\ln(\ln(n+1)))^r}{L^n \; n^p \; (\ln n)^q \; (\ln(\ln n))^r}$$

$$\sim \frac{L^{n+1}}{L^n}\left(\frac{n+1}{n}\right)^p\left(\frac{\ln(n+1)}{\ln n}\right)^q\left(\frac{\ln(\ln(n+))}{\ln(\ln n)}\right)^r \sim L,$$

which implies convergence for the series if $L < 1$

and divergence for the series if $L > 1$**.**

(The Ratio Test implies nothing if $L = 1$**.)**

With factorials, most people know only about the Ratio Test,

with $\dfrac{(n+1)!}{n!} = n+1; \quad \dfrac{(n+2)!}{n!} = (n+2)(n+1);$ **etc.**

Example: Ratio Test for Convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ **:**

$$\frac{\dfrac{((n+1)!)^3}{(3(n+1))!}}{\dfrac{(n!)^3}{(3n)!}} = \left(\frac{(n+1)!}{n!}\right)^3 \frac{(3n)!}{(3n+3)!}$$

$$= \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \sim \frac{n^3}{(3n)^3} = \frac{1}{27} < 1,$$

Root Test:

$$\sqrt[n]{\frac{(n!)^3}{(3n)!}} = \frac{\sqrt[n]{(n!)^3}}{\sqrt[n]{(3n)!}} \sim \frac{\left(\dfrac{n}{e}\right)^3}{\dfrac{3^3 n^3}{e^3}} = \frac{1}{27} < 1.$$

More General Tests for Convergence of Non-Negative Series Which Depend on Inequalities, not Limits, for $\sqrt[n]{a_n}$ and $\dfrac{a_{n+1}}{a_n}$.

Another Root Test, for Nonnegative Series:

If $\sqrt[n]{a_n} \leq M < 1$, for $n \geq N$, then $\displaystyle\sum_{n=n_0}^{\infty} a_n$ converges.

If $\sqrt[n]{a_n} \geq 1$, for $n \geq N$, then $\displaystyle\sum_{n=n_0}^{\infty} a_n$ diverges.

Another Ratio Test, for Positive Series:

If $\dfrac{a_{n+1}}{a_n} \leq M < 1$, for $n \geq N$, then $\displaystyle\sum_{n=n_0}^{\infty} a_n$ converges.

If $\dfrac{a_{n+1}}{a_n} \geq 1$, for $n \geq N$, then $\displaystyle\sum_{n=n_0}^{\infty} a_n$ diverges.

How the Root Tests Assure Convergence of $\sum_{n=n_0}^{\infty} a_n$:

If $\sqrt[n]{a_n} \leq M < 1$, for $n \geq N$,

(which would be true if $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ and if $L < M < 1$)

then we would have $a_n \leq M^n$,

and $\sum_{n=n_0}^{\infty} a_n \leq \sum_{n=n_0}^{\infty} M^n$, which would converge.

How the Root Tests Assure Divergence of $\sum_{n=n_0}^{\infty} a_n$:

If $\sqrt[n]{a_n} \geq 1$, for $n \geq N$,

(which would be true if $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1$)

then we would have $a_n \geq 1$,

a_n would not converge to 0,

and $\sum_{n=n_0}^{\infty} a_n$ could not possibly converge.

How the Ratio Tests Assure Convergence of $\sum_{n=n_0}^{\infty} a_n$:

$$\text{If } \frac{a_{n+1}}{a_n} \le M < 1, \text{ for } n \ge N,$$

$$\text{(which would be true if } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \text{ and if } L < M < 1\text{)}$$

$$\text{then we would have } a_{n+1} \le a_n M, \text{ for } n \ge N,$$

$$a_{N+1} \le a_N M,$$

$$a_{N+2} \le a_{N+1} M \le a_N M^2,$$

$$a_{N+3} \le a_{N+2} M \le a_N M^3,$$

$$\cdots$$

$$a_{N+r} \le a_{N+r-1} M \le a_N M^r,$$

$$\sum_{n=N}^{\infty} a_n = \sum_{r=0}^{\infty} a_{N+r} \le \sum_{r=0}^{\infty} a_N M^r = a_N \sum_{r=0}^{\infty} M^r < \infty.$$

How the Ratio Tests Assure Divergence of $\sum_{n=n_0}^{\infty} a_n$:

$$\text{If } \frac{a_{n+1}}{a_n} \geq 1, \text{ for } n \geq N,$$

$$\text{(which would be true if } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1)$$

$$\text{then we would have } a_{n+1} \geq a_n, \text{ for } n \geq N,$$

The sequence of a_n would be positive and nondecreasing.

It would not converge to zero.

The series $\sum_{n=n_0}^{\infty} a_n$ would not converge.

How the Ratio Test Relates $\sum_{n=n_0}^{\infty} a_n$ to Geometric Series.

$$\text{If } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L, \text{ and if } \quad m < L < M$$

$$\text{then, for all } n > \text{some } n_0, \text{ we have } m < \frac{a_{n+1}}{a_n} < M,$$

$$m < \frac{a_{n_0+1}}{a_{n_0}} < M,$$

$$m < \frac{a_{n_0+2}}{a_{n_0+1}} < M,$$

$$\cdots$$

$$m < \frac{a_{n_0+r}}{a_{n_0+r-1}} < M.$$

$$\text{Multiply these to get } \; m^r < \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_{n_0+r}}{a_{n_0+r-1}} < M^r,$$

$$m^r < \frac{a_{n_0+r}}{a_{n_0}} < M^r,$$

How the Ratio Test Relates $\sum_{n=n_0}^{\infty} a_n$ to Geometric Series.

If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, and if $\quad m < \quad L \quad < M$

for some n_0, all $r \geq 0$, we have $\displaystyle m^r < \frac{a_{n_0+r}}{a_{n_0}} < M^r$,

$$a_{n_0} m^r < a_{n_0+r} < a_{n_0} M^r,$$

for all $n >$ this n_0, $\quad a_{n_0} m^{n-n_0} < a_n < a_{n_0} M^{n-n_0}$,

$$\frac{a_{n_0}}{m^{n_0}} m^n < a_n < \frac{a_{n_0}}{M^{n_0}} M^n,$$

$$\frac{a_{n_0}}{m^{n_0}} \sum_{n=n_0}^{\infty} m^n < \sum_{n=n_0}^{\infty} a_n < \frac{a_{n_0}}{M^{n_0}} \sum_{n=n_0}^{\infty} M^n.$$

$\sum a_n$ is smaller than any geometric series larger than $\sum L^n$.

$\sum a_n$ is larger than any geometric series smaller than $\sum L^n$.

How the Root Test Relates $\sum_{n=n_0}^{\infty} a_n$ to Geometric Series.

$$\text{If } \lim_{n\to\infty} \sqrt[n]{a_n} = L, \text{ and if } \quad m < L < M$$

$$\text{then, for all } n > \text{some } n_0, \text{ we have } m < \sqrt[n]{a_n} < M,$$

$$m^n < a_n < M^n,$$

$$\sum_{n=n_0}^{\infty} m^n < \sum_{n=n_0}^{\infty} a_n < \sum_{n=n_0}^{\infty} M^n,$$

$\sum a_n$ is smaller than any geometric series larger than $\sum L^n$.

$\sum a_n$ is larger than any geometric series smaller than $\sum L^n$.

$\sum a_n$ is closer to $\sum L^n$ than any other geometric series.

The limit $L = \lim_{n\to\infty} \sqrt[n]{a_n}$ of the Root Test
is equal to the least upper bound
of the numbers m which satisfy $\lim_{n\to\infty} \frac{a_n}{m^n} = \infty$.

The limit $L = \lim_{n\to\infty} \sqrt[n]{a_n}$ is also equal to the greatest lower bound of the numbers M which satisfy $\lim_{n\to\infty} \frac{a_n}{M^n} = 0$,

This makes the limit $L = \lim_{n\to\infty} \sqrt[n]{a_n}$ also equal to the limit
$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ of the Ratio Test,
whenever both tests can be performed on the same series $\sum a_n$.

What the Root and Ratio Tests accomplish
is to find the geometric series $\sum L^n$ closest to the series $\sum a_n$.

Warning!

If you are confusing the words "sequence" and "series", or if you are confusing any of the following sequences:

a_k, **the terms being added, which should** $\longrightarrow 0$,

$s_n = \sum_{k=c}^{n} a_k$, **the partial sums, which should** $\longrightarrow \sum_{k=c}^{\infty} a_k$,

$\dfrac{a_{k+1}}{a_k}$, **used in the ratio test, which approach a limit** L,

$\sqrt[n]{a_n}$, **used in the root test, which approach the same** L,

then you need to be devoting much more $\left\{\begin{array}{l}\textbf{time,}\\ \textbf{effort,}\\ \textbf{and thought}\end{array}\right.$

to this course right now, not later.

"Mene mene tekel upharsin!"