Elementary Partial Differential Equations*

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Introduction.

Partial differential equations (PDEs) is one of the oldest subjects in mathematical analysis. Its development extends back to Euler's work in the 1700s, together with Brooks Taylor and others.

Problems arising in the study of PDEs have motivated many of the principle developments in classical and modern analysis. For example, harmonic analysis (Fourier), complex analysis (Cauchy, Riemann), theory of integral equations (Fredholm, Hilbert), Hilbert and Banach space theory, fixed point theorems (Schauder), theory of distributions (L. Schwartz) and many others.

At present the theory of PDEs is one of the most active fields of research in modern mathematics. Each month *Mathematical Reviews* contains many pages of reviews of publications on PDE's. As another example Professor C. Miranda published a monograph on "PDEs of Elliptic Type" in 1954. It contained a bibliography of more than 600 research papers published between 1924-1953. In the revised edition published in 1968, Miranda estimated that to bring the bibliography up to date more than 1600 items would have to be added. The process has continued to accelerate.

At the present time, it is impossible to present in a single course a complete survey of what is known as PDEs and the properties of their solutions. Many advanced monographs exist and in many cases their contents scarcely overlap.

Plan for this course

A study of classical theories for some of the simplest PDEs. We shall use as a source, V. Smirnov, *A Course in Higher Mathematics*, vols. II and IV, and C. H. Wilcox, "Notes on PDEs."[1] The modern functional analytic theories of PDEs must wait for further courses.

Contents

[1]Used by permission.

Chapter 1. Heat Conduction in a Slab.

Some Classical PDEs[2]

$$\frac{\partial v}{\partial t} = k\Delta v = k(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \qquad \text{Heat or Diffusion Equation}$$

$$\frac{\partial^2 v}{\partial t^2} = c^2 \Delta v \qquad \text{Wave Equation}$$

$$\Delta v = \rho(x, y, z) \qquad \text{Poisson's Equation}$$

$$\Delta v = 0 \qquad \text{Laplace's Equation}$$

[2]See appendix I p. 117 for other examples.

Temperature $= v(x,t)$.

Equations:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad \text{for } \; 0 < x < l, t > 0$$

$$v(x,0) = g(x) \qquad \text{for } \; 0 \le x \le l \quad \text{(initial condition)}$$

$$v(0,t) = v_o \qquad \text{and } \; v(l,t) = v_1, \quad \text{for } \quad t \ge 0 \qquad \text{(boundary conditions)}$$

THE CLASSICAL QUESTIONS:

<u>Existence of a Solution?</u>

<u>Uniqueness of Solution?</u>

<u>Continuous Dependence on the Data?</u>

THE STEADY STATE LIMIT.

$v_s(x) = \lim_{t\to\infty} v(x,t)$ should satisfy

$$\frac{\partial^2 v_s}{\partial x^2} = 0, \qquad 0 < x < l, \quad \text{and} \quad v_s(0) = v_o, v_s(l) = v_1$$

It follows that

$$v_s(x) = v_o\left(\frac{l-x}{l}\right) + v_1\left(\frac{x}{l}\right)$$

THE REDUCED PROBLEM. $v(x,t) = v_s(x) + u(x,t)$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t},$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

Thus,

$$(1) \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad 0 < x < l, t > 0$$

$$(2) \qquad u(x,0) = g(x) - v_s(x) \equiv f(x) \qquad \text{for} \quad 0 \leq x \leq l$$

$$(3) \qquad u(0,t) = 0 \quad \text{and} \quad u(l,t) = 0 \quad \text{for} \quad t \geq 0$$

SEPARATION OF VARIABLES. Look for functions

$$u(x,t) = X(x)T(t) \not\equiv 0$$

which satisfy (1) and (3) (but not necessarily (2)).

$$XT' = kX''T$$

$$(1) \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \quad \text{const.} \quad = -\lambda$$

$$(3) \quad X(0)T(t) = 0 \;\; \text{and} \;\; X(l)T(t) = 0 \;\; \text{for} \;\; t \geq 0$$

These conditions are satisfied if

$$T'(t) + k\lambda T(t) = 0 \quad \text{for} \quad t > 0$$

and

$$X''(x) + \lambda X(x) = 0 \;\; \text{for} \;\; 0 < x < l$$

$$X(0) = 0 \;\; \text{and} \;\; X(l) = 0$$

These are problems involving constant coefficient linear ordinary differential equations and are therefore explicitly solvable:

$$X(x) = X_n(x) = \sin(\frac{\pi n x}{l}), \qquad \text{n} = 1,\, 2,\, 3,\, .\;.\;.$$

$$\lambda = \lambda_n = (\frac{\pi n}{l})^2$$

$$T(t) = T_n(t) = e^{-k(\frac{\pi n}{l})^2 t}$$

and

$$u(x,t) = u_n(x,t) = \sin(\frac{n\pi x}{l})e^{-k(\frac{n\pi}{l})^2 t}$$

Superposition. To solve (1), (2), (3) try

$$(4) \qquad u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l}) e^{-k(\frac{n\pi}{l})^2 t}$$

where $c_1, c_2, c_3...$ are to be determined.

Formal Solution.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l}) \qquad \text{(Fourier Sine Series)}$$

$$(5) \qquad c_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, \qquad \text{n} = 1,\ 2,\ 3,\ .\ .\ .$$

(5) follows from (4) by the orthogonality relations.[3] The formal solution is defined by (4), (5). Convergence theory for Fourier series may now be applied.

Convergence Theorem for Fourier Sine Series. Assume that

(a) $f(x) \in C[0,l]$

(b) $f(0) = f(l) = 0$

(c) $f'(x)$ is sectionally continuous on $[0,l]$.[4]

[3]I.e. $\int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{k\pi x}{l}) = 0$ when $n \neq k$. See page 130.

[4]A function is sectionally continuous if it is possible to divide its domain into a finite number of sections on which the function is continuous (or has removable discontinuities),

Then the Fourier sine series coefficients (5) satisfy

$$\sum_{n=1}^{\infty} |c_n| < \infty \tag{6}$$

moreover

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l}) \quad \text{for} \quad 0 \leq x \leq l$$

and the convergence is uniform and absolute on $0 \leq x \leq l$.[5]

NOTATION: $\Omega = \{(x,t) : 0 < x < l, t > 0\}$; $\overline{\Omega}$ = closure of Ω in $\mathbb{R}^2 = \{(x,t) : 0 \leq x \leq l, t \geq 0\}$.

CLASSICAL SOLUTION. A function u is a classical solution of the heat conduction problem in a slab if and only if $u \in C(\overline{\Omega})$, $\partial u/\partial t \in C(\Omega)$, $\partial^2 u/\partial x^2 \in C(\Omega)$ and (1), (2) and (3) all hold.

EXISTENCE THEOREM. If $f(x)$ satisfies (a), (b), (c) then the formal solution (4), (5) converges uniformly on $\overline{\Omega}$ and defines a classical solution.

Proof. (6) shows that (4) converges uniformly on $\overline{\Omega}$ and hence $u \in C(\overline{\Omega})$

bounded and has left and right-hand limits at each point in its domain. Naturally, all bounded continuous functions are also sectionally continuous, the converse of course, is not true.

[5] R. V. Churchill, *Fourier Series and Boundary Value Problems*, 2nd ed. McGraw-Hill 1963.

and (2) and (3) hold (for a detailed argument that this is so, see pages .

To prove that $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \in C(\Omega)$ and (1) holds note that

$$|c_n \sin \frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}| \leq |c_n| e^{-k(\frac{n\pi}{l})^2 t_o}$$

for all $x \in \mathbb{R}$ and $t \geq t_o$.

Hence the series

$$\sum_{n=1}^{\infty} n^2 c_n \sin \frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}$$

converges uniformly for all $x \in \mathbb{R}$ and $t \geq t_o$.[6] It follows that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in C(\Omega)$ can all be calculated by termwise differentiation and are continuous for all $x \in \mathbb{R}, t > 0$. Finally, (1) holds because each term in (4) is a solution of the heat equation. QED

Notation. $\mathbb{R}^2_+ = \{(x,t) : x \in \mathbb{R}, t > 0\}$ (Ω is as above).

$\Omega_T = \Omega \cap \{(x,t) : t < T\}$

$\overline{\Omega}_T = \text{closure of } \Omega_T = \overline{\Omega} \cap \{(x,t) : t \leq T\}$

$\partial\Omega = \text{boundary of } \Omega = \overline{\Omega} - \Omega$

[6]The Weierstrass M-Test may be used. See page 142-3.

$\Gamma_T = \partial\Omega \cap \overline{\Omega}_T$

Maximum Principle. Let $u \in C(\overline{\Omega})$ satisfy $\partial u/\partial t \in C(\Omega)$, $\partial^2 u/\partial x^2 \in C(\Omega)$ and the heat equation in Ω. Then for all $T > 0$,

$$\max_{\overline{\Omega}_T} u(x,t) = \max_{\Gamma_T} u(x,t)$$

Proof. (By contradiction.) Assume the conclusion is false, i.e., $\max_{\overline{\Omega}_T} u$ $(> \max_{\Gamma_T} u)$ occurs at $(x_o, t_o) \in \overline{\Omega}_T - \Gamma_T$. Define the function

$$v(x,t) = u(x,t) - \epsilon(t - t_o), \quad \epsilon > 0$$

Then for all such ϵ,

$$v(x_o, t_o) = u(x_o, t_o) = \max_{\overline{\Omega}_T} u > \max_{\Gamma_T} u$$

Let $\max v$ occur at $(x_1, t_1) \in \overline{\Omega}_T$. Then

$$v(x_o, t_o) = \max_{\overline{\Omega}_T} u > \max_{\Gamma_T} u + \epsilon t_o = \max_{\Gamma_T} v$$

provided ϵ is sufficiently small. Thus $\max v$ must occur at some $(x_1, t_1) \in \overline{\Omega}_T - \Gamma_T$. It follows that

$$\frac{\partial^2 v}{\partial x^2}(x_1, t_1) \le 0, \quad \frac{\partial v}{\partial t}(x_1, t_1) \ge 0$$

whence

$$\frac{\partial^2 u}{\partial x^2}(x_1, t_1) = \frac{\partial^2 v}{\partial x^2}(x_1, t_1) \leq 0$$

but

$$\frac{\partial u}{\partial t}(x_1, t_1) = \frac{\partial v}{\partial t}(x_1, t_1) + \epsilon > 0$$

which contradicts (1) (the heat equation). QED

Uniqueness Theorem. The BV problem (1), (2), (3) can have only one classical solution.

Proof. Let u_1 and u_2 be any two classical solutions with the same initial values $f(x)$. Then $u(x,t) = u_1(x,t) - u_2(x,t)$ is a classical solution with $f(x) \equiv 0$. Thus

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u = 0$$

i.e.[7]

$$u(x,t) \leq 0 \qquad \forall (x,t) \in \overline{\Omega}_T$$

Similarly, $-u(x,t)$ is a classical solution with $f(x) \equiv 0$. Following the same reasoning, $-u(x,t) \leq 0$. It follows that $u(x,t) \equiv 0$. QED

[7] $\forall$ is a logical symbol which simply means "for all."

The maximum principle implies that if $f(x)$ satisfies (a), (b), (c) and $u(x,t)$ is the corresponding classical solution then

$$\max_{\overline{\Omega}} |u(x,t)| \leq \max_{0 \leq x \leq l} |f(x)|$$

This implies

CONTINUOUS DEPENDENCE ON THE DATA. Let $\{f_n(x)\}$ be a sequence of functions satisfying (a), (b), (c). Let $\{u_n(x,t)\}$ be the corresponding solutions of (1), (2), (3). Suppose further that $f_n(x) \to 0$ uniformly as $n \to \infty$ on $0 \leq x \leq l$. Then $u_n(x,t) \to 0$ when $n \to \infty$, uniformly in $\overline{\Omega}$.

Chapter 2. Wave Propagation on a Taut String.

The small amplitude vibrations of a taut string, moving in a plane, are governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad\qquad c > 0$$

Interpretation

[See chalkboard illustration]

Note that the change of variable $\tau = ct$ reduces the wave equation to

$$(1) \qquad \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial x^2}$$

The integration[8] of (1) can be based on

Lemma 1 Let Ω be a domain in the (x, τ)-plane that is intersected by each line $x \pm \tau =$ const. in an interval (possibly empty) and let $(a_1, a_2), (b_1, b_2)$ be the smallest intervals such that

$$\Omega \subset \Omega_o = \{(x, \tau) : a_1 < x - \tau < a_2 \text{ and } b_1 < x + \tau < b_2\}$$

[See chalkboard illustration]

[8] By integration we mean finding the solution of the equation.

Assume that $u \in C^2(\Omega)$ and (1) holds for all $(x, \tau) \in \Omega$. Then there exist functions $f(\tau), g(\tau)$ such that

$$(a) \qquad f \in C^2(a_1, a_2), g \in C^2(b_1, b_2)$$

$$(b) \qquad u(x, \tau) = f(x - \tau) + g(x + \tau) \quad \text{in} \;\; \Omega$$

Remark 1 This is d'Alembert's solution of (1).

Remark 2 f and g are unique up to constant functions.[9]

Proof. Introduce new coordinates

$$\xi = x - \tau, \qquad \eta = x + \tau$$

and let

$$u(x, \tau) = v(\xi, \eta)$$

Then (chain rule for partial derivatives)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}, \qquad \frac{\partial u}{\partial \tau} = -\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^v}{\partial \eta^2}, \quad \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}$$

[9] That is, the pair (f, g) is equivalent to the pair $(f + C, g - C)$ where C is any constant.

Thus

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \tau^2} = 4\frac{\partial^2 v}{\partial \xi \partial \eta} = 0 \quad \text{in} \quad \Omega' = \{(\xi, \eta) : (x, \tau) \in \Omega\}$$

Also, $v \in C^2(\Omega')$. Thus for each $\eta_o \in (b_1, b_2)$, $\frac{\partial}{\partial \xi}(\frac{\partial v}{\partial \eta}) = 0$ on a non-empty ξ-interval (Ω' is connected) and hence

$$\frac{\partial v(\xi, \eta_o)}{\partial \eta} = G(\eta_o) \in C^1(b_1, b_2).$$

Repeating this with $\xi_o \in (a_1, a_2)$ gives $v(\xi, \eta) = f(\xi) + g(\eta)$ on Ω'.

where

$$g(\eta) = \int G(\eta) d\eta \in C^2(b_1, b_2)$$

and hence $f = v - g \in C^2(a_1, a_2)$. QED

Corollary. Under the hypotheses of Lemma 1 u has an extension $u' \in C^2(\Omega_o)$ which satisfies (1) in Ω_o.

Wave Propagation on a Long String. In the (x, τ) - plane consider the domains

$$\Omega = \{(x, \tau) : \tau > 0, a < x < b\}, \Delta = \{(x, \tau) : \tau > 0, \tau + a < x < b - \tau\}$$

[See chalkboard illustration]

Let $u \in C^2(\Omega)$ describe a motion of the string. By the lemma the values of u in $\Delta \subset \Delta_o$ are independent of what happens at the ends of the string and $\exists f, g \in C^2(a,b)$[10] such that

$$u(x,\tau) = f(x-\tau) + g(x+\tau) \quad \text{in} \quad \Delta$$

Moreover, f and g can be determined by the initial values

$$u(x,0) = u_o(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial \tau} = u_1(x), \qquad a < x < b$$

Indeed,

$$u(x,0) = f(x) + g(x) = u_o(x), f'(x) + g'(x) = u_o'(x)$$
$$\frac{\partial u(x,0)}{\partial \tau} = -f'(x) + g'(x) = u_1(x)$$

Thus

$$2f'(x) = u_o'(x) - u_1(x), 2g'(x) = u_o'(x) + u_1(x)$$

$$2f(x) = u_o(x) - \int_a^x u_1(\xi)d\xi + C, 2g(x) = u_o(x) + \int_a^x u_1(\xi)d\xi + C'$$

[10] $\exists$ is a logical symbol meaning "there exists."

Hence

$$2(f(x)+g(x)) = 2u_o(x) + C + C' = 2u_o(x)$$

whence

$$C + C' = 0 \quad \text{or} \quad C' = -C$$

Thus

$$u(x,\tau) = \frac{1}{2}u_o(x-\tau) - \frac{1}{2}\int_a^{x-\tau} u_1(\xi)d\xi + C$$
$$+\frac{1}{2}u_o(x+\tau) + \frac{1}{2}\int_a^{x+\tau} u_1(\xi)d\xi - C$$

or

$$(2) \qquad u(x,\tau) = \frac{1}{2}\{u_o(x-\tau) + u_o(x+\tau)\} + \int_{x-\tau}^{x+\tau} u_1(\xi)d\xi, \qquad (x,\tau) \in \Delta.$$

This leads us to the

Initial Value Problem for the Wave Equation.

In the idealized case of an infinitely long string, $a = -\infty, b = \infty$, (2) gives the solution of the boundary value problem

$$(3) \qquad \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} \qquad -\infty < x < \infty, \quad \tau > 0$$

$$(4) \qquad u(x,0) = u_o(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial \tau} = u_1(x) \quad \text{for} \quad -\infty < x < \infty$$

Notation. $\mathbb{R}^2_+ = \{(x,\tau) : -\infty < x < \infty, \quad \tau > 0\}$.

Classical Solution. A function u is a classical solution of the Initial Value Problem (3), (4) if and only if $u \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ and (3) and (4) hold.

Existence. If $u_o \in C^2(\mathbb{R}), u_1 \in C^1(\mathbb{R})$ then (2) defines a classical solution.

Uniqueness. The argument leading to (2) shows that any classical solution must be given by (2). Hence classical solutions are unique.

Continuous Dependence on the Data. If is clear for (2) that if $u^{(n)}(x,\tau)$ corresponds to data $u_o^{(n)}(x), u_1^{(n)}(x)$ then if $(u_o^{(n)}(x), u_1^{(n)}(x)) \to 0$ uniformly on bounded intervals then $u^{(n)}(x,\tau) \to 0$ uniformly on bounded subsets of $\mathbb{R}^2_+$ (as $n \to \infty$).

Wave Propagation on a String of Finite Length.

The displacement $u(x,\tau)$ is a solution of the boundary value (BV) problem

$$(5) \qquad \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < l, \tau > 0$$

$$(6) \qquad u(0,\tau) = 0 \quad \text{and} \quad u(l,\tau) = 0 \quad \text{for} \quad \tau \geq 0$$

$$(7) \qquad u(x,0) = u_o(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial \tau} = u_1(x) \quad \text{for} \quad 0 \leq x \leq l$$

The solution may be constructed by separation of variables as with the heat equation.

Separation of Variables. Look for functions

$$u(x,\tau) = X(x)T(\tau)$$

which satisfy (5), (6). Then

$$XT'' = X''T$$

or

$$\frac{T''}{T} = \frac{X''}{X} = \quad \text{const.} \quad = -\lambda$$

consequently

$$(8) \qquad T'' + \lambda T = 0, \quad \tau > 0$$

$$(9) \qquad X'' + \lambda X = 0 \quad \text{for} \quad 0 < x < l$$

$$X(0) = 0, \quad \text{and} \quad X(x) = 0 \quad \text{for} \quad 0 < x < l$$

Observe that (9) is the same BV problem which occurred in the reduction of the heat equation. Hence

$$\lambda = \lambda_n = (\frac{n\pi}{l})^2, \qquad X(x) = X_n(x) = \sin(\frac{n\pi x}{l}), \qquad \text{n} \ = \ 1, 2, 3, 4, . \ . \ .$$

The corresponding $T(\tau)$ factors are

$$T(\tau) = T_n(\tau) = A_n \cos(\frac{n\pi\tau}{l}) + B_n \sin(\frac{n\pi\tau}{l})$$

Thus (5) (6) have the solutions

$$u_n(x,\tau) = \{A_n \cos(\frac{n\pi\tau}{l}) + B_n \sin(\frac{n\pi\tau}{l})\} \sin(\frac{n\pi x}{l}), \qquad \text{n} \;=\; 1,\,2,\,3,\,4,\,.\;\;.\;\;.$$

Principle of Superposition. To solve (5), (6), (7) try

$$(10) \qquad u(x,\tau) = \sum_{n=1}^{\infty} \{A_n \cos(\frac{n\pi\tau}{l}) + B_n \sin(\frac{n\pi\tau}{l})\} \sin(\frac{n\pi x}{l})$$

$$\frac{\partial u(x,\tau)}{\partial \tau} = \sum_{n=1}^{\infty} \frac{n\pi}{l} \{-A_n \sin(\frac{n\pi\tau}{l}) + B_n \cos(\frac{n\pi\tau}{l})\} \sin(\frac{n\pi x}{l})$$

Formal Solution.

$$u(x,0) = u_o(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$$

$$\frac{\partial u(x,0)}{\partial \tau} = u_1(x) = \sum_{n=1}^{\infty} \frac{n\pi}{l} B_n \sin(\frac{n\pi x}{l})$$

Thus

$$(11) \qquad A_n = \frac{2}{l} \int_0^l u_o(x) \sin \frac{n\pi x}{l} dx$$

$$\frac{n\pi}{l} B_n = \frac{2}{l} \int_0^l u_1(x) \sin \frac{n\pi x}{l} dx \qquad \text{n} \;=\; 1,\,2,\,3,\,4,\,.\;\;.\;\;.$$

The formal solution is defined by (10), (11). We may now proceed to define the

Classical Solution. u is a classical solution of the vibrating string problem (5), (6), (7) if and only if $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and (5), (6), (7) hold. Here $\Omega = \{(x,\tau) : \quad 0 < x < l, \quad \tau > 0\}$, $\overline{\Omega} = \{(x,\tau) : 0 \le x \le l, \quad \tau \ge 0\}$.

The existence of a classical solution can be proved by using convergence theory for the Fourier sine series, as in the case of the heat equation, but this does not give the best results (the weakest possible smoothness conditions on the initial values). Instead Fourier convergence theory will be used in a different way to <u>sum</u> the formal series (10), (11). For simplicity, only the special case where

$$\frac{\partial u(x,0)}{\partial \tau} = u_1(x) \equiv 0$$

will be discussed. In this case, the formal solution is

$$(10)' \qquad u(x,\tau) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi\tau}{l} \sin \frac{n\pi x}{l}$$

$$(11)' \qquad A_n = \frac{2}{l} \int_0^l u_o(x) \sin \frac{n\pi x}{l} dx$$

To sum $(10)'$, $(11)'$ note that

$$(12) \qquad 2\cos\frac{n\pi\tau}{l}\sin\frac{n\pi x}{l} = \sin\frac{n\pi(x-\tau)}{l} + \sin\frac{n\pi(x+\tau)}{l}$$

and hence $(10)'$ can be written as

$$(13) \qquad u(x,\tau) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi(x-\tau)}{l} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi(x+\tau)}{l}$$

The two series in this expression are Fourier sine series for $u_o(x)$ evaluated at $x-\tau$ and $x+\tau$ respectively. Now if $u_o(x)$ has a convergent sine series then $\forall \xi \in \mathbb{R}$

$$U_o(\xi) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi\xi}{l} = \begin{cases} u_o(\xi) & \text{if } 0 \le \xi \le l; \\ \\ \text{Odd periodic extension of } u_o(\xi) & \\ \text{with period } 2l & \\ \text{for all other values of } \xi. & \end{cases}$$

Thus, formally,

$$(14) \qquad u(x,\tau) = \frac{1}{2}[U_o(x-\tau) + U_o(x+\tau)]$$

This will define a classical solution if and only if $U_o \in C^2(\mathbb{R})$. This is true if and only if

$$(15) \qquad u_o \in C^2[0,l]$$

and

$$(16) \quad u_o(0) = 0, u_o(l) = 0, u_o''(0) = 0, u_o''(l) = 0^{11}$$

This proves the

EXISTENCE THEOREM. If u_o satisfies (15), (16) then (13), (14) defines a classical solution of (5), (6), (7) with $u_1(x) \equiv 0$.

The case where $u_o(x) \equiv 0$ and $u_1(x) \neq 0$ can be treated similarly.

ANOTHER DERIVATION OF THE SOLUTION BASED ON LEMMA 1

Let u be a classical solution of the finite string problem, i.e.

$$(1) \qquad u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \quad \text{and} \quad \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad \Omega$$

$$(2) \qquad u(x,0) = u_o(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial \tau} = u_1(x) \quad \text{for} \quad 0 \leq x \leq l$$

$$(3a) \qquad u(0,t) = 0 \quad \text{for} \quad t \geq 0$$

$$(3b) \qquad u(l,t) = 0 \quad \text{for} \quad t \geq 0^{12}$$

[11] These conditions are necessary to allow a smooth $2l$ periodic extension of u_o.
[12] The conditons (3a), (3b) state that the string is immobilized at the ends.

It will be shown that Lemma 1 leads to a unique construction of $u(x,\tau)$ in $\overline{\Omega}$.

Step 1. (1) $\implies$ $u(x,\tau) = f(x-\tau) + g(x+\tau)$ in Ω_o.

where

$$\Omega_o = \{(x,\tau) : -\tau < x < l + \tau\}$$

$$f \in C^2(-\infty, l) \quad \text{and} \quad g \in C^2(0,\infty)^{13}$$

It will be convenient to transform this into the (ξ,η) - plane where

$$\xi = x - \tau, \qquad \eta = x + \tau$$

$$\Omega \to \Omega' = \{(\xi,\eta) : 0 < \eta + \xi < 2l \;\text{ and }\; \eta - \xi > 0\}$$

$$\Omega_o \to \Omega_o' = \{(\xi,\eta) : -\infty < \xi < l \;\text{ and }\; \eta > 0\}$$

[See chalkboard illustration.]

Step 2.

(1), (3) $\implies$ (see page 11)

[13] See pages 10, 11.

(4) $$f(\xi) = \frac{1}{2}u_o(\xi) - \frac{1}{2}\int_0^\xi u_1(x)dx \quad \forall \;\; 0 \leq \xi \leq l$$

(5) $$g(\eta) = \frac{1}{2}u_o(\eta) - \frac{1}{2}\int_0^\eta u_1(x)dx \quad \forall \;\; 0 \leq \eta \leq l$$

<u>Step 3</u>.

(1), (3a) $\implies u(0,\tau) = f(-\tau) + g(\tau) = 0, \forall \tau \geq 0$

or

(6) $$f(-\tau) = -g(\tau), \forall \tau \geq 0$$

<u>Step 4</u>.

(1), (3b) $\implies u(l,\tau) = f(l-\tau) + g(l+\tau) = 0, \quad \forall \tau \geq 0$

or

(7) $$f(l-\tau) = -g(l+\tau), \quad \forall \tau \geq 0$$

<u>Lemma</u> (1), (3a), (3b) $\implies$

(8) $$g(\tau + 2l) = g(\tau), \forall \tau \geq 0$$

(9) $$f(\tau + 2l) = f(\tau), \forall \tau \leq -l$$[14]

Proof.

(7) $\implies f(-\tau) = -g(2l + \tau), \forall \tau \geq -l$

(6) $\implies f(-\tau) = -g(\tau), \forall \tau \geq 0$

These two together give (8).

(7) $\implies f(-\tau - l) = -g(\tau + 3l), \forall \tau + 2l \geq 0$

(7) and (8) $\implies f(l - \tau) = f(-l - \tau), \forall \tau \geq 0$

$\implies f(\tau + l) = f(\tau - l), \forall \tau \leq 0$ which $\implies$ (9). QED.

Construction of f and g.

- (4), (5) define f, g on $[0, l]$

[14]In other words, g is $2l$- periodic and f nearly so.

- (5), (6) extend f to $[-l, 0]$

- (4), (7) extend g to $[l, 2l]$

- (8) extends g to $[2l, \infty)$

- (9) extends f to $(-\infty, -l]$

[See chalkboard illustration.]

Specifically,

$$f(\xi) = \begin{cases} \frac{1}{2}u_o(\xi) - \frac{1}{2}\int_0^{\xi} u_1(x)dx, & 0 \leq \xi \leq l \\ \\ -\frac{1}{2}u_o(-\xi) - \frac{1}{2}\int_0^{-\xi} u_1(x)dx, & -l \leq \xi \leq 0 \end{cases}$$

$$f'(\xi) = \begin{cases} \frac{1}{2}u_o'(\xi) - \frac{1}{2}u_1(\xi), & 0 \leq \xi \leq l \\ \\ \frac{1}{2}u_o'(-\xi) + \frac{1}{2}u_1(-\xi), & -l \leq \xi \leq 0 \end{cases}$$

$$
g(\eta) = \begin{cases} \frac{1}{2}u_o(\eta) + \frac{1}{2}\int_0^{\eta} u_1(x)dx, & 0 \le \eta \le l \\ \\ -\frac{1}{2}u_o(2l-\eta) + \frac{1}{2}\int_0^{2l-\xi} u_1(x)dx, & l \le \xi \le 2l \end{cases}
$$

$$
g'(\eta) = \begin{cases} \frac{1}{2}u_o'(\eta) + \frac{1}{2}u_1(\eta), & 0 \le \eta \le l \\ \\ \frac{1}{2}u_o'(2l-\eta) - \frac{1}{2}u_1(2l-\eta), & l \le \eta \le 2l \end{cases}
$$

If f and g have sufficient smoothness, they will verify (16) [see page 16]:

Recall that $v(\xi,\eta) = f(\xi) + g(\eta)$. This holds in particular for $\xi \le l, \eta \ge 0$. Along the line segment $\xi = 0, 0 \le \eta \le l$ we have by smoothness of v,

$$
v(0^+,\eta) = v(0^-,\eta) \Leftrightarrow f(0^+) = f(0^-) \Leftrightarrow u_o(0^+) = -u_o(0^+) \quad \text{or} \quad \underline{u_o(0^+) = 0}
$$

$$\frac{\partial v(0^+,\eta)}{\partial \xi} = \frac{\partial v(0^-,\eta)}{\partial \xi} \Leftrightarrow f'(0^+) = f'(0^-) \Leftrightarrow u_1(0^+) = -u_1(0^+) \text{ or } \underline{u_1(0^+) = 0}$$

$$\frac{\partial^2 v(0^+,\eta)}{\partial \xi^2} = \frac{\partial^2 v(0^-,\eta)}{\partial \xi^2} \Leftrightarrow f''(0^+) = f''(0^-) \Leftrightarrow u_o''(0^+) = -u_o''(0^+) \text{ or } \underline{u_o''(0^+) = 0}$$

Similarly, considering the segment $0 \le \xi \le l, \eta = l$ gives

$u_o(l^-) = 0, u_1(l^-) = 0, u_o(l^-) = 0$

Uniqueness Theorem. $\exists$ at most one solution to (1), (2), (3a), (3b).

Proof. It must be given by $u(x,\tau) = f(x-\tau) + g(x+\tau)$ where f and g are defined as above.

Corollary For a solution of (1), (2), (3a), (3b) to exist it is necessary that

(10) $u_o \in C^2[0,l]$, $u_1 \in C^1[0,l]$

(11) $u_o(0^+) = 0, u_1(0^+) = 0, u_o(0^+) = 0$

(12) $u_o(l^-) = 0, u_1(l^-) = 0, u_o(l^-) = 0$

Existence Theorem If u_o, u_1 satisfy (10), (11), (12) then $\exists$ a solution of (1), (2), (3a), (3b). The proof is the construction of f and g above.

Continuous Dependence on the Data. Let $\{u_o^n(x)\}$, $\{u_1^n(x)\}$ be a sequence of initial data with corresponding solutions $\{u^n(x,\tau)\}$. Then

$$\{u_o^n(x)\} \to 0, \{u_1^n(x)\} \to 0 \quad \text{uniformly on} \quad 0 \le x \le l \quad \Longrightarrow$$

$$u^n(x,\tau) \to 0 \quad \text{uniformly on} \quad \overline{\Omega}$$

<u>Proof</u>. This is evident from (4) and (5) and their extensions.

Periodicity. Note that (8), (9) $\Longrightarrow$ $u(x,\tau+2l) = f(x-\tau-2l) + g(x+\tau+2l) = u(x,\tau)$, i.e. every motion of the string has period $2l$. This is also evident from the Fourier series for $u(x,\tau)$.

Chapter 3. Steady Temperature in a Circular Cylinder.

Consider a long solid cylindrical rod of radius a. The temperature $T = u(x, y, z, t)$ in such a rod satisfies the heat equation:

$$\frac{\partial u}{\partial t} = K(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

Assume that

$$u(x, y, z, t) = g(x, y), \quad \text{for} \quad x^2 + y^2 = a^2, t \geq 0^{15}$$

Then we expect that

$$\lim_{t \to \infty} u(x, y, z, t) = u_s(x, y), \qquad \forall \quad x^2 + y^2 \leq a^2$$

The corresponding steady temperature should be a solution of the BV problem Dirichlet's Problem for Laplace's Equation:[16]

$$\begin{cases} \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = 0 \quad \text{for} \quad x^2 + y^2 < a^2 \\ \\ u_s(x, y) = g(x, y) \quad \text{for} \quad x^2 + y^2 = a^2 \end{cases}$$

We shall study this problem. It is natural to introduce

[15]That is, the temperature at the surface of the cylinder depends on neither z nor t.
[16]Dirichlet problems are those in which the unknown function is required to have specified behavior at the boundary of its domain.

Polar Coordinates. Put $T = u_s(x,y) = v(r,\theta)$ where

$$x = r\cos\theta, y = r\sin\theta$$

Then

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

Differentiation gives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{r^2} = \frac{-\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Applying the chain rule for partial derivatives:

$$\frac{\partial u_s}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \equiv F_1(r,\theta)$$

$$\frac{\partial u_s}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \equiv F_2(r,\theta)$$

Differentiating both sides of these equations yields

$$\frac{\partial^2 u_s}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial F_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos^2\theta \frac{\partial^2 v}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r}$$

$$\frac{\partial^2 u_s}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} = \frac{\partial F_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin^2\theta \frac{\partial^2 v}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial r}$$

Thus

$$\frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

BV PROBLEM FOR $v(r,\theta)$. If we write $f(\theta)$ for $g(a\cos\theta, a\sin\theta)$ then

$$(1) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \quad \text{for} \quad 0 < r < a, \quad \text{and all} \quad \theta$$

$$(2) \qquad v(a,\theta) = f(\theta) \quad \text{for all} \quad \theta$$

$$(3) \quad v(r,\theta + 2\pi) = v(r,\theta) \quad \text{for all} \quad 0 \le r \le a \quad \text{and all} \quad \theta$$

$$(4) \qquad v(0,\theta) = v_o = \text{a finite constant for all} \quad \theta$$

The condition (3) is an obvious geometric fact. Rotation around the vertical axis of the cylinder by 2π should not effect the temperature. Condition (4) is certainly true for the steady-state temperature along the vertical axis of the cylinder. But it is a necessary condition for the solution of the problem, since (1) has a singularity at $r = 0$. (4) excludes non-physical solutions.

Separation of Variables. Look for functions

$$v(r, \theta) = R(r)\Theta(\theta)$$

which satisfy (1), (3), (4). Then (1) implies

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''$$

Multiply this by $r^2/R\Theta$ to get

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \mu$$

μ is called the separation constant.[17] This and (3) gives

$$\Theta''(\theta) + \mu\Theta(\theta) = 0$$

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

[17]See page 3 for the slab problem. - λ was the separation constant.

The solutions of this *eigenvalue* problem are $\mu = \mu_n = n^2, n = 0, 1, 2, 3, ...,$

$$\Theta(\theta) = \Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

The equation for $R(r)$, with $\mu = n^2$, is

$$r^2 R'' + rR'(r) - n^2 R(r) = 0, 0 \le r \le a$$

This is an Euler equation with solutions of the form r^α or $\ln r$. We try $R(r) = r^\alpha$. Substitution into the equation gives

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = [\alpha^2 - n^2]r^\alpha = 0$$

whence

$$\alpha^2 - n^2 = 0, \quad \text{or} \quad \alpha = \pm n, n = 1, 2, 3, ...$$

For $n = 0$, $rR'' + R' = 0 \implies R' = \frac{D_o}{r}, R_o = C_o + D_o \ln r$

(4) requires that $D_o = 0$. Thus the separated solutions are

$$v_n(r, \theta) = r^n(A_n \cos n\theta + B_n \sin n\theta), n = 0, 1, 2, 3, ...$$

Superposition. Try to satisfy (1)-(4) by a linear combination

$$(5) \qquad v(r, \theta) = A_o + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

Formal Solution

$$(6) \qquad v(a,\theta) = f(\theta) = A_o + \sum_{n=1}^{\infty}(a^n A_n \cos n\theta + a^n B_n \sin n\theta)$$

Suppose that the Fourier series of $f(\theta)$ is

$$f(\theta) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

$$(7) \qquad \begin{cases} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)d\phi, n = 0, 1, 2, ... \\ \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)d\phi, n = 1, 2, ... \end{cases}$$

This agrees with (6) if

$$A_o = \frac{1}{2}a_o, \quad a^n A_n = a_n, a^n B_n = b_n, n = 1, 2, ...$$

Thus the formal solution can be written:

$$(8) \qquad v(r,\theta) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}(\frac{r}{a})^n(a_n \cos n\theta + b_n \sin n\theta)$$

The existence of solutions of the Dirichlet problem for the disk can be discussed by applying convergence theorems for Fourier series to the formal solution (8). However, we shall take a different approach to show that the series in (8) can be summed.

The Poisson Integral. Still proceeding formally we have, from (7) and (8),

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi + \sum_{n=1}^{\infty}(\frac{r}{a})^n \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\}d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi + \sum_{n=1}^{\infty}(\frac{r}{a})^n \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n(\theta-\phi)d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\{1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta-\phi)\}d\phi$$

Remark. The last step is easy to justify if $r < a$ and $f \in L_1(-\pi,\pi)$.

Now[18]

$$(\frac{r}{a})^n \cos n\psi = Re\{(\frac{r}{a})^n e^{in\psi}\} = Re\{(\frac{re^{i\psi}}{a})^n\}$$

and

$$\sum_{n=1}^{\infty}(\frac{re^{i\psi}}{a})^n = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + ... = \frac{z}{1-z}$$

$$= \frac{\frac{r}{a}e^{i\psi}}{1-\frac{r}{a}e^{i\psi}} = \frac{\frac{r}{a}e^{i\psi}(1-\frac{r}{a}e^{-i\psi})}{|1-\frac{r}{a}e^{i\psi}|^2}$$

[18]Re stands for "real part." Thus for any complex number $z = c + ib$, $Re(z) = c$.

$$= \frac{\frac{r}{a}e^{i\psi} - (\frac{r}{a})^2}{(1 - \frac{r}{a}cos\psi)^2 + \frac{r}{a}\sin^2\psi}$$

Thus

$$\sum_{n=1}^{\infty} (\frac{r}{a})^n \cos n\psi = \frac{\frac{r}{a}\cos\psi - (\frac{r}{a})^2}{1 - 1\frac{r}{a}\cos\psi + (\frac{r}{a})^2}$$

Hence

$$1 + 2\sum_{n=1}^{\infty} (\frac{r}{a})^n \cos n\psi = \frac{1 - 2\frac{r}{a}\cos\psi + (\frac{r}{a})^2 + 2\frac{r}{a}\cos\psi - 2(\frac{r}{a})^2}{1 - 2\frac{r}{a}\cos\psi + (\frac{r}{a})^2}$$

Simplifying,

$$1 + 2\sum_{n=1}^{\infty} (\frac{r}{a})^n \cos n\psi = \frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos\psi}$$

The latter expression is called the "Poisson kernel." Again formally,

$$(9) \qquad v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\theta - \phi)}d\phi, \quad \text{for} \quad 0 \leq r < a$$

The corresponding formula in rectangular coordinates is

$$(10) \quad u_s(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - x^2 - y^2}{a^2 + x^2 + y^2 - 2a(x\cos\phi + y\sin\phi)}g(a\cos\phi, a\sin\phi)d\phi$$

This function is called Poisson's integral. It may be used to prove the existence of solutions.

<u>Classical Solutions</u>. $u_s(x,y)$ is a classical solution of the Dirichlet problem for the Laplace equation in the disk $\Omega = \{(x,y) : x^2+y^2 < a^2\} \Leftrightarrow$ $u_s \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfies

$$(11) \qquad \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = 0 \text{ in } \Omega$$

$$(12) \qquad u_s(x,y) = g(x,y) \quad \text{on} \quad \partial\Omega = \{(x,y) : x^2+y^2 = a^2\}$$

A necessary condition for the existence of a classical solution is $g \in C(\partial\Omega)$. It will be shown, using Poisson's integral, that this is also a sufficient condition!

<u>Existence Theorem</u>. For all $g(x,y) \in C(\partial\Omega)$ the function

$$u_s(x,y) = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2-x^2-y^2}{a^2+x^2+y^2-2a(x\cos\phi+y\sin\phi)} g(a\cos\phi, a\sin\phi)\,d\phi, & r < a \\ \\ g(x,y), \qquad r = a \end{cases}$$

is a classical solution of the Dirichlet problem (11), (12).

<u>Proof</u>. The main difficulty is establishing that the two values of u_s match up at the boundary. In other words, we must eventually show that $\lim_{r\to a} u_s(x,y) = g(x,y)$ with some uniformity in θ (i.e., direction). We

attack the easier parts first.

Step 1. Show $u_s \in C^2(\Omega)$ and $\frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = 0$ in Ω.

To show this, let $0 < \delta < a$ and define

$$\Omega_\delta = \{(x, y) : r = \sqrt{x^2 + y^2} < a - \delta\}$$

Then if $\psi = \theta - \phi$

$$a^2 - 2ra\cos\psi + r^2 = a^2 - 2ra + r^2 + 2ra(1 - \cos\psi)$$

$$= (a - r)^2 + 2ra(1 - \cos\psi) \geq (a - r)^2 > \delta^2$$

Hence

$$\frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\theta - \phi)} = 1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta - \phi), r < a \tag{13}$$

defines a function which has partial derivatives of all orders and satisfies Laplace's equation on Ω_δ for each δ such that $0 < \delta < a$. Hence (10) defines a function $u_s \in C^2(\Omega)$ (indeed $C^\infty(\Omega)$) such that (11) holds.

Step 2. $u_s \in C(\overline{\Omega})$. It suffices to prove continuity at the boundary of Ω. Note that

- $$\frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\theta - \phi)} \to 0 \quad \text{when} \quad r \to a \;\text{ if }\; \phi \neq \theta.$$

- $\dfrac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)} = \dfrac{a^2-r^2}{(a-r)^2} = \dfrac{a+r}{a-r} \to +\infty \;\text{ if }\; \phi=\theta.$
- $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \frac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)}d\phi \equiv 1, \quad \forall r<a, -\pi\le\theta\le\pi \quad \text{(by 13)}$

[See chalkboard illustration.]

Thus, for $\delta>0$ sufficiently small

$$v(r,\theta)-f(\theta_o) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)}\{f(\phi)-f(\theta_o)\}d\phi^{19}$$

$$= \frac{1}{2\pi}\int_{\theta_o-\delta}^{\theta_o+\delta} \frac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)}\{f(\phi)-f(\theta_o)\}d\phi$$

$$+\frac{1}{2\pi}\int_{|\theta_o-\phi|\ge\delta} \frac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)}\{f(\phi)-f(\theta_o)\}d\phi$$

$$= I_1(r,\theta)+I_2(r,\theta)$$

Choose $\delta=\delta_1(\epsilon)>0$ such that[20]

$$|f(\phi)-f(\theta_o)| < \frac{\epsilon}{2} \quad \forall\;\phi \;\text{ with }\; |\phi-\theta_o|\le\delta_1(\epsilon)$$

[19] Recall $a^2+r^2-2ra\cos(\theta-\phi)>0$.

[20] Observe by definition (see (6)) that $g(x,y)\in C(\partial\Omega) \Leftrightarrow f(\theta)\in C(\mathbb{R})$ and $f(\theta+2\pi)=f(\theta), \quad \forall\,\theta\in\mathbb{R}$.

Then

$$|I_1(r,\theta)| \leq \frac{1}{2\pi}\int_{\theta_o-\delta}^{\theta_o+\delta} \frac{a^2-r^2}{a^2+r^2-2ra\cos(\theta-\phi)}|f(\phi)-f(\theta_o)|d\phi \tag{15}$$

$$\leq \frac{\epsilon}{2}, \forall\theta \in \mathbb{R}, r < a.$$

Next observe that

$$a^2+r^2-2ra\cos(\theta-\phi) \geq 2ra(1-\cos(\theta-\phi)) = 4ra\sin^2\frac{\theta-\phi}{2}$$

Take

$$|\theta-\theta_o| < \frac{\delta_1(\epsilon)}{2}, |\theta_o-\phi| \geq \delta_1(\epsilon)$$

Then

$$|\theta-\phi| \geq |\theta_o-\phi| - |\theta-\theta_o| > \frac{\delta_1(\epsilon)}{2}$$

and hence

$$a^2+r^2-2ra\cos(\theta-\phi) \geq 2ra(1-\cos(\theta-\phi) > 4ra\sin^2\frac{\delta_1(\epsilon)}{4} > 0$$

Thus if

$$M = \max_{0 \le \theta \le 2\pi} |f(\theta)|$$

$$|I_2(r,\theta)| \le \frac{1}{2\pi} \int_{|\theta_o - \phi| \ge \delta_1(\epsilon)} \frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\theta - \phi)} \{|f(\phi)| + |f(\theta_o)|\} d\phi$$

(16)

$$\le \frac{2M}{2\pi} \int_{|\theta_o - \phi| \ge \delta_1(\epsilon)} \frac{a^2 - r^2}{4ra\sin^2 \frac{\delta_1(\epsilon)}{4}} d\phi \le 2M \frac{a^2 - r^2}{4ra\sin^2 \frac{\delta_1(\epsilon)}{4}} < \frac{\epsilon}{2}$$

provided $r > a - \delta_2(\epsilon)$ for $\delta_2(\epsilon)$ sufficiently small, and $|\theta - \theta_o| < \delta_1(\epsilon)/2$.

Combining (14), (15), (16) gives

$$|v(r,\theta) - f(\theta_o)| \le |I_1(r,\theta)| + |I_2(r,\theta)| < \epsilon$$

$$\forall \;\; (x,y) = (r\cos\theta, r\sin\theta) \quad \text{with} \quad a - \delta_2(\epsilon) < r < a, |\theta - \theta_o| < \frac{\delta_1(\epsilon)}{2}.$$

This shows that $u_s(x,y) - g(x_o,y_o) \to 0$ uniformly as $(x,y) \to (x_o,y_o)$ and this completes the proof. QED.

The Mean Value Theorem for Laplace's Equation. Let

$$(1) \qquad \Omega \quad \text{be a domain (any open set) in the } \ (x,y) \ \text{ plane.}$$

$$(2) \qquad u \in C^2(\Omega) \quad \text{and} \quad \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = 0 \ \text{ in } \quad \Omega.$$

$$(3) \qquad D(x_o, y_o, R) = \{(x,y) : (x - x_o)^2 + (y - y_o)^2 < R^2\} \subset \Omega.$$

Then

$$(3) \qquad u(x_o, y_o) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_o + r\cos\theta, y_o + r\sin\theta) d\theta \quad \text{for} \ \ 0 \le r \le R.$$

In other words, at any point in the domain, u is equal to its mean value along any circle surrounding it in Ω.

Proof. Define

$$v(r,\theta) = u(x_o + r\cos\theta, y_o + r\sin\theta)$$

Then

$$(5) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \quad \text{for} \quad 0 < r < R, \theta \in \mathbb{R}.$$

[The reader should check (5) by comparing with pp. 33-34.]

$$(6) \qquad v(r, \theta + 2\pi) = v(r, \theta)$$

Now let

$$(7) \qquad A(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_o + r\cos\theta, y_o + r\sin\theta)d\theta$$

Then

$$A''(r) + \frac{1}{r}A'(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})v(r,\theta)d\theta$$

$$= -\frac{1}{2\pi r^2}\int_{-\pi}^{\pi}\frac{\partial^2 v}{\partial \theta^2}d\theta = 0 \quad \text{by (5), (6)}$$

Thus $A(r) = c_o + c_1 \ln r$, $0 < r < R$. But $A(0) = \lim_{r\to 0^+} A(r) = u(x_o, y_o)$ by (7). Hence

$c_1 = 0$ and therefore $A(r) = c_o = u(x_o, y_o)$. QED

Corollary (The Maximum Principle). If (1), (2) hold and u is not constant in Ω then it can have no local maximum or minimum in Ω.

Proof. (By contradiction.) Assume u has a local max at $(x_o, y_o) \in \Omega$ then for R sufficiently small, $u(x,y) < u(x_o, y_o)$ for all $(x,y) \in D(x_o, y_o, R) - (x_o, y_o)$. This violates (3).

COROLLARY. If $\overline{\Omega}$ is compact,

$$u \in C(\overline{\Omega}) \cap C^2(\Omega)$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \Omega$$

then the maximum and minimum of u occur on $\partial\Omega$.

UNIQUENESS THEOREM. If $\overline{\Omega}$ is compact then the Dirichlet problem for Laplace's equation in Ω has at most one classical solution. Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be any two solutions with the same values on $\partial\Omega$. Then $u(x,y) = u_1(x,y) - u_2(x,y)$ satisfies the conditions of the preceding corollary and vanishes on $\partial\Omega$. Hence $u(x,y) \equiv 0$ on $\overline{\Omega}$ or

$$u_1(x,y) \equiv u_2(x,y)$$

QED.

CHAPTER 4. BASIC CONCEPTS IN THE THEORY OF HEAT CONDUCTION.

Temperature. The notion of the temperature of a body (at a point) is an intuitive concept. A more precise definition can be based on thermodynamics. In any case it is measurable by thermometers, thermocouples and many other devices, The temperature relative to a fixed scale is measured by a real number. Scales include

- Centigrade or Celsius Scale $T_C = 0^\circ \leftrightarrow$ Freezing water.

- Fahrenheit Scale $T_F = 32^\circ + 9/5 T_C$

- Kelvin (or Absolute) Scale $T_K = T_C + 273.16^\circ$

The Kelvin scale is derived from thermodynamic principles. It has the property that every body has a temperature

$$T_K \geq 0$$

Quantity of Heat. Heat is a form of energy. The basic unit of heat energy is the calory (also spelled calorie), defined by the property that 1 calory = Quantity of heat needed to heat 1 gram of water from 14.5°C to 15.5°C.

Specific Heat of a Solid. The lower case c will be used to represent this quantity. For each substance, the amount of heat required to raise the temperature of 1 gram by 1°C is called the specific heat of the substance. Actually, the specific heat varies slightly with the temperature, but we shall treat it as a constant. Some approximate values for familiar substances are

Substance	Specific Heat
Water	1.0000 (at 15°C)
Glass	.20
Cork	.48
Copper	.0914
Silver	.0556

Heat Transfer Mechanisms. Three different modes of heat transfer are distinguished:

Conduction (or Diffusion) - This means the direct transfer by contact of adjacent particles.

Convection - This means the transfer of heat in a fluid due to motion of the fluid.

Radiation - This means the transfer of heat by conversion to electromagnetic waves and their propagation through space.

In solid bodies heat transfer in the interior is assumed to take place by pure conduction. It may be necessary to consider convection and/or radiation at the surface of a solid.

The Physical Principles Governing Heat Conduction.

There are two principles

I. The Conservation of Heat Energy. This is usually formulated as the statement that if V is any volume in a solid then

Net Change in quantity of heat in V during any time interval =

Net Flux of heat through the surface of V during the same time interval.

This statement will be quantified below in several cases.

II. Fourier's Law of Heat Conduction. This states that the rate of flow of heat at any point in a solid is a function of the temperature gradient at that point.

This will also be quantified in several different cases below. To begin, the case of 1-dimensional heat flow in a plate (slab) is discussed.

Heat Flow in a Plate. Imagine a large uniform plate (or wall) with

$$\text{Area} = A, \qquad \text{Thickness} = l.$$

Assume that the two faces are kept at fixed temperature T_o and $T_1 \neq T_o$, $T(x)$ being the temperature (independent of time) on a plane parallel to the faces of the plate at depth x[21] and let

$$Q = \text{Quantity of heat in the plate from}$$

$$\text{side 0 to side 1 in } t \text{ units of time.}^{22}$$

In this case Fourier's law states that

$$Q \propto \frac{(T_1 - T_o)}{A} l$$

Thus one may write

$$(1) \qquad Q = -K\frac{(T_1 - T_o)At}{l}$$

where the constant of proportionality $K =$ the thermal conductivty of the

[21] Hence $T_o = T(0), T_1 = T(l)$.

[22] Therefore the quantity of heat in the plate between $x = 0$ and some interior depth $x = x_o$ would be $Q(x_o, t) = -\frac{(T(x_o) - T_o)At}{l}$.

plate. The value of K is characteristic of the material of which the plate is made (it may vary somewhat with the thermal state of the material, but we shall assume it is constant).

Remarks

a. Heat flows from high temperature regions to low temperature regions. Hence, with the definition of Q given above, $K > 0$.

b. The insulating value of a layer of insulation is proportional to its thickness.

c. Our confidence in Fourier's law is based on both direct experiment and the accuracy of many predictions based on the law.

Heat Flux. The quantity

$$q = \frac{Q}{At} = -K\frac{(T_1 - T_o)}{l} \qquad \frac{\text{cal.}}{m^2\text{sec.}}$$

is called the heat flux through the plate. Some representative values of the thermal conductivity are

Substance	Thermal Conductivity
Water	.00144
Glass	.0028
Cork	.0001
Copper	.93
Silver	1.00

Steady Temperature Profile in a Plate. Introduce a coordinate x normal to the surface of the plate, the left side corresponding to $x = 0$ and x_o corresponding to some interior point. Between any two interior points, $x_o, x_o + \Delta x$ the net increase in heat content is determined as *specific heat times volume times net temperature increase over time* is approximately (assuming Δx is small) $= c\rho A \Delta x[T(x_o, t + \Delta t) - T(x_o, t)]$ where ρ is mass density. Meanwhile, the heat energy entering $[x_o, x_o + \Delta x]$ over the time interval $[t, t + \Delta t]$ is $(q(x_o, t) - q(x_o + \Delta x, t))A\Delta t$. Applying the principle of conservation of heat energy tells us that these two quantities should be equal and using the fact that T (and therefore q) is not a function of time, we have that $q(x_o) = \text{ const. } = q$.

Thus

$$q = -K\frac{T(x_o) - T_o}{x_o}$$

the heat flux through the layer $0 \leq x \leq x_o$ of the plate is independent of x_o. This implies that for any x between 0 and l,

$$T(x) = T_o - \frac{q}{K}x = T_o + \frac{T_1 - T_o}{l}x$$

Non-Steady Temperatures in a Plate. Now suppose that T_o and T_1 are functions of t:

$$T_o = T_o(t), \quad T_1 = T_1(t)$$

In this case the temperature in the plate will be a function of t and x, the coordinate introduced above:

$$T = T(x,t), \qquad 0 \leq x \leq l$$

Consider the very thin slab parallel to the plate

Fourier's law for steady temperatures suggests that the instantaneous flux of heat through this slab, at time t, is approximately

$$q(x_o,t) \simeq -K\frac{T(x_o+\Delta x,t) - T(x_o,t)}{\Delta x} \simeq -K\frac{\partial T(x_o,t)}{\partial x}$$

Fourier assumed this law was exact in the limit as $\Delta x \to 0$.

<u>Fourier's Law of Transient Heat Flow (1 Dimension)</u>

$$q(x_o,t) = -K\frac{\partial T(x_o,t)}{\partial x}$$

As in the case of steady heat flow, our confidence in Fourier's law is based on the accuracy of many predictions based on it.

<u>The Heat Equation for Non-Steady Temperatures in a Plate</u>. Apply the

conservation of energy principle to a portion $(x, x+\Delta x)$ of the plate and a time interval $(t, t+\Delta t)$:

The net increase of heat content of $(x, x+\Delta x)$ during $(t, t+\Delta t)$

$$= c\,\overbrace{\rho\,\underbrace{A\Delta x}_{\text{volume}}}^{\text{mass}}\underbrace{[T(x,t+\Delta t)-T(x,t)]}_{\text{net temperature increase}}$$

The heat energy entering $(x, x+\Delta x)$ during $(t, t+\Delta t)$ is

$$\underbrace{\int_t^{t+\Delta t} Aq(x,t)dt}_{\text{heat entering at } x} - \underbrace{\int_t^{t+\Delta t} Aq(x+\Delta x,t)dt}_{\text{heat leaving at } x+\Delta x} = -A\int_t^{t+\Delta t}\int_x^{x+\Delta x}\frac{\partial q}{\partial x}dt$$

By conservation of energy the two quantities are equal and thus,

$$-A\int_t^{t+\Delta t}\int_x^{x+\Delta x}\frac{\partial q}{\partial x}dt = c\rho A\Delta x[T(x,t+\Delta t)-T(x,t)]$$

or

$$-\frac{1}{\Delta t}\int_t^{t+\Delta t}\frac{1}{\Delta x}\int_x^{x+\Delta x}\frac{\partial q}{\partial x}dt = c\rho\frac{T(x,t+\Delta t)-T(x,t)}{\Delta t}$$

Making $\Delta t, \Delta x \to 0$ gives

$$-\frac{\partial q(x,t)}{\partial x} = c\rho\frac{\partial T(x,t)}{\partial t}$$

Applying Fourier's law,

$$K\frac{\partial^2 T}{\partial x^2} = c\rho\frac{\partial T}{\partial t}$$

or

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

where

$k = \frac{K}{c\rho} =$ "thermal diffusivity" or the "diffusion coefficient."

k measures the speed with which heat is conducted through a substance. The following table gives approximate values of k for some familiar substances

Substance	Thermal Diffusivity
water	.00144
glass	.0058
cork (ground)	.0014
copper	1.14
silver	1.71

Thus cork is roughly 1000 times better as an insulator than copper.

To determine the temperature in a plate as a solution of the heat equation we must know T at some initial time (say $t = 0$). This gives

Initial Condition.

$$T(x,0) = f(x) \quad \text{(a given function)} \;\; \text{for} \;\; 0 \leq x \leq l$$

Boundary Conditions. In addition the temperatures at the two faces of the plate must be controlled in some way. Several possibilities will be considered.

Surface Temperatures Given:

$$T(0,t) = T_o(t) \quad \text{(a given function)} \;\; \text{for} t \geq 0$$

$$T(l,t) = T_1(t) \quad \text{(a given function)} \quad \text{for} \;\; t \geq 0$$

Surface Heat Flux Given. Instead of specifying the surface temperatures one may specify

$$q(0,t) = -K\frac{\partial T(0,t)}{\partial x} = q_o(t) \quad \text{(a given function)} \;\; t \geq 0$$

or

$$\frac{\partial T(0,t)}{\partial x} = g_o(t) \quad (= \frac{q_o(t)}{-K}) \quad \text{for} \quad t \geq 0$$

Similarly

$$\frac{\partial T(l,t)}{\partial x} = g_1(t) \quad \text{for} \quad t \geq 0$$

may be given.

Convection Boundary Condition. If the plate face at $x = 0$ is cooled by convection into a fluid at temperature T_e then, to a good approximation,

$$-q(0,t) = K\frac{\partial T(0,t)}{\partial x} = H(T(0,t) - T_e)$$

where $H =$ "outer conductivity" $=$ const.

Thus

$$\frac{\partial T(0,t)}{\partial x} - hT(0,t) = -hT_e, \qquad t \geq 0$$

where $h = H/K$. Note that since heat flows from hot to cold, $H \geq 0, h \geq 0$.

Mixed Boundary Conditions. We can have one of the above conditions at $x = 0$ and a different one at $x = l$.

Linear Diffusion of Heat in a Slender Non-Uniform Rod.

In this context "slender" means that the temperature in the rod can be described by a function

$$T = T(x,t)$$

where x measures distance along the rod.

[See blackboard illustration.]

The physical characteristics of the rod are described by the real-valued <u>positive</u> functions

$A = A(x) =$ cross-sectional area of rod

$P = P(x) =$ perimeter of rod

$\rho = \rho(x) =$ linear density of rod

$c = c(x) =$ specific heat of rod

$K = K(x) =$ thermal conductivity of rod

$H = H(x) =$ outer conductivity of rod

If the rod is cooling through its surface into an environment with temperature $T_e(x)$ then (cf. p. 59)

$$\text{Net heat energy entering } \ (x, x+\Delta x) \ \text{ during } \ (t, t+\Delta t)$$

$$= -\int_t^{t+\Delta t} [A(x+\Delta x)q(x+\Delta x,t) - A(x)q(x,t)]dt$$

$$-\int_x^{x+\Delta x}\int_t^{t+\Delta t} H(x)[T(x,t)-T_e(x)]P(x)dtdx$$

and

Net increase in heat content of $(x, x{+}\Delta x)$ during $(t, t{+}\Delta t)$ (assuming Δx is small)

$$= c(x)\rho(x)A(x)\Delta x[T(x,t+\Delta t)-T(x,t)]$$

Equating these and dividing by $\Delta x\Delta t$ gives

$$-\frac{1}{\Delta t}\int_t^{t+\Delta t}[\frac{A(x+\Delta x)q(x+\Delta x,t)-A(x)q(x,t)]}{\Delta x}dt$$

$$-\frac{1}{\Delta x}\int_x^{x+\Delta x}\frac{1}{\Delta t}\int_t^{t+\Delta t} H(x)[T(x,t)-T_e(x)]P(x)dtdx$$

$$= c(x)\rho(x)A(x)[\frac{T(x,t+\Delta t)-T(x,t)}{\Delta t}]$$

Making $\Delta x, \Delta t \to 0$ gives

$$-\frac{\partial}{\partial x}(A(x)q(x,t)) - H(x)P(x)[T(x,t)-T_e(x)] = c(x)\rho(x)\frac{\partial T(x,t)}{\partial t}$$

Combining this and Fourier's law for a non-uniform rod:

$$q(x,t) = -K(x)\frac{\partial T(x,t)}{\partial x}$$

gives the heat diffusion equation for a slender non-uniform rod:

$$c(x)\rho(x)A(x)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(A(x)K(x)\frac{\partial T}{\partial x}) - H(x)P(x)[T(x,t) - T_e(x)]$$

This has the form

$$\frac{\partial T}{\partial t} - LT = F(x)$$

where $F(x)$ is a known function and

$$Lu = p_o(x)\frac{\partial^2 u}{\partial x^2} + p_1(x)\frac{\partial u}{\partial x} + p_2(x)u$$

with

$$p_o(x) = \frac{K(x)}{c(x)\rho(x)} > 0$$

$$p_1(x) = \frac{1}{\rho(x)c(x)A(x)}\frac{d}{dx}(A(x)K(x))$$

$$p_2(x) = -\frac{H(x)P(x)}{\rho(x)c(x)A(x)} < 0$$

It is interesting that the most general linear second order operator[23] L can arise in this way; i.e., by suitable choice of $A(x), P(x)$, etc. The only restrictions are that $p_o(x) > 0, p_2(x) < 0$.

The initial and boundary conditions given on pp. 58-59 are appropriate for the non-uniform rod. In addition we will consider the

<u>Fourier Ring Problem</u>. Imagine bending a slender rod of length l into a ring and joining the ends. Then the physical identity of the ends $x = 0$ and $x = l$ gives the

<u>Periodic Boundary Condition</u>.

$$T(0,t) = T(l,0) \quad \text{and} \quad K(0)\frac{\partial T(0,t)}{\partial x} = K(l)\frac{\partial T(l,t)}{\partial x}, \quad t \geq 0$$

[23] See appendix I.

Diffusion of Heat in 3 Space Dimensions. Consider a heat conducting solid body occupying a domain $\Omega \subset \mathbb{R}^3$. The thermal state of the body is characterized by a temperature field

$$T = T(x,t), \quad x = (x_1, x_2, x_3) \in \Omega$$

(x is a vector quantity).

Let $c = c(x) =$ the specific heat of the body at $x \in \Omega$.
$\rho = \rho(x) =$ density of the body at $x \in \Omega$.

Thus if $V \subset \Omega$ is any volume in the body

$$Q(t) = \int_V c(x)\rho(x)T(x,t)dx = \quad \text{Total quantity of heat in } V \text{ at time } \ t$$

where $dx = dx_1 dx_2 dx_3$.[24] In the important case where the body is homogeneous c and ρ are constants and $Q(t) = c\rho \int_V T(x,t)dx$.

The flow of heat in the body is described by a vector field

$$\vec{q}(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t)) = \quad \underline{\text{Heat Flux Field}}$$

[24]The single integral sign is customary in modern mathematics. In elementary calculus one often sees multiple integral signs corresponding to the space dimension.

To interpret $\vec{q}$ let dS be a (small) surface element in Ω with unit normal vector $\vec{\nu}$.

[See blackboard illustration.]

Then

$$\vec{q}(x,t) \bullet \vec{\nu}(x)dS = \text{ Quantity of heat crossing } dS \text{ per unit time at time } t$$

In particular, if V is any volume inside the body with boundary ∂V having exterior unit normal $\vec{\nu}(x)$ then the surface integral

$$\int_{\partial V} \vec{q}(x,t) \bullet \vec{\nu}(x)dS = \text{ Quantity of heat leaving } V \text{ per unit time at time } t.$$

<u>Conservation of Heat Energy</u>. The conservation of heat energy law becomes, in this context the statement

$$\int_{\partial V} \vec{q}(x,t) \bullet \vec{\nu}(x)dS = -\frac{dQ}{dt}$$

To obtain a differential equation we can use the divergence theorem:

$$\int_V \nabla \bullet \vec{A} dx = \int_{\partial V} \vec{\nu} \bullet \vec{A} dS$$

or

$$\int_V \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = \int_{\partial V} (\nu_1 A_1 + \nu_2 A_2 + \nu_3 A_3) dS$$

Applying this to $\vec{q}$ gives

$$\int_V \nabla \bullet \vec{q} dx = \int_{\partial V} \vec{\nu} \bullet \vec{q} dS$$

Hence the conservation of heat energy principle can be written

$$\int_V \nabla \bullet \vec{q} dx = -\int_V c(x)\rho(x) \frac{\partial T(x,t)}{\partial t} dx$$

or

$$\int_V \nabla \bullet \vec{q} dx + c(x)\rho(x) \frac{\partial T(x,t)}{\partial t} dx = 0$$

If $T(x,t), \vec{q}(x,t)$ are C^1 functions (in $\Omega \times \mathbb{R}$) and $c(x), \rho(x) \in C^1(\Omega)$ then the integrand in the last integral is continuous in Ω (for any fixed t). Since the identity holds for <u>all</u> volumes $V \subset \Omega$ it follows that

$$(1) \qquad \nabla \bullet \vec{q} + c(x)\rho(x) \frac{\partial T(x,t)}{\partial t} = 0, \; x \in \Omega, t \in \mathbb{R}$$

Indeed, if

$$\nabla \bullet \vec{q}(x_o, t) + c(x_o)\rho(x_o)\frac{\partial T(x_o, t)}{\partial t} > 0$$

then the same inequality holds in a neighborhood $V = V(x_o)$ of x_o, by continuity, and we have a contradiction. Equation (1) is the conservation of heat energy principle in differential form.

Fourier's Law of Heat Conduction for a 3-Dimensional Isotropic Body

This may be formulated as the statement

$$(2) \qquad \vec{q}(x, t) = -K(x)\nabla T(x, t)$$

where $K = K(x) =$ the thermal conductivity at x.

Note that $K > 0$ (heat flows from hot to cold). Moreover, K is constant in a homogeneous body.

Heat Equation for an Inhomogeneous Isotropic Body.

Combining (1) and (2) by eliminating $\vec{q}$ gives

$$(3) \qquad \frac{\partial T(x, t)}{\partial t} - \frac{1}{c(x)\rho(x)}\nabla \bullet (K(x)\nabla T(x, t)) = 0$$

In the anisotropic case, the thermal conductivity causes variation in direction away from the temperature gradient. This can be expressed mathematically by allowing K to be a matrix quantity. In the isotropic, homogeneous case (3) becomes

$$(4) \qquad \frac{\partial T(x,t)}{\partial t} - k\left(\frac{\partial^2 T(x,t)}{\partial x_1^2} + \frac{\partial^2 T(x,t)}{\partial x_2^2} + \frac{\partial^2 T(x,t)}{\partial x_3^2}\right) = 0$$

where

(5) $k = K/c\rho =$ Thermal diffusivity of the body.

Initial Condition. To determine $T(x,t)$ for a given body one must construct a solution of (3) with a given initial temperature distribution

$$(6) \qquad T(x,0) = f(x), x \in \Omega$$

In addition, information on temperature and heat flux on the boundary $\partial\Omega$ must be given. Several possibilities will be considered.

Surface Temperature Given:

$$(7) \qquad T(x,t) = \phi(x,t) \quad \text{(a given function) for } x \in \partial\Omega, t \geq 0$$

Surface Heat Flux Given:

$$(8) \qquad \vec{q}(x,t) \bullet \vec{\nu}(x) = -K(x)\nabla T(x,t) \bullet \vec{\nu}(x) = \phi(x,t)$$

$$\text{(a given function) for } \; x \in \partial\Omega, t \geq 0$$

where $\vec{\nu}(x)$ is the unit normal vector to $\partial\Omega$ at x directed <u>out</u> of Ω.

<u>Convection Boundary Condition</u>.

$$(9) \quad \vec{q}(x,t) \bullet \vec{\nu}(x) = -K(x)\nabla T(x,t) \bullet \vec{\nu}(x) = H(x)(T(x,t) - T_o(x))$$

$$\text{for } \; x \in \partial\Omega, t \geq 0$$

where

$H = H(x) =$ "outer conductivity" of $\partial\Omega$ at x $(H \geq 0)$

and

$T_o(x) =$ exterior temperature at $x \in \partial\Omega$. This can be written (if $\partial T/\partial\vec{\nu} = \nabla T \bullet \vec{\nu}, \vec{\nu}$ <u>out</u> of Ω)

$$(10) \qquad \frac{\partial T}{\partial \vec{\nu}} + hT = hT_o, x \in \partial\Omega, t \geq 0$$

where $h = h(x) = H(x)/K(x) \geq 0$. (10) is sometimes called the <u>Robin</u>[25] boundary condition.

[25] After Victor Gustave Robin, 19th century physicist.

Mixed Boundary Conditions are also possible where $\partial\Omega = S_1 \cup S_2 \cup \ldots S_k$ and one of the above BCs holds on each S_j.

The above considerations lead us to formulate the following BV Problems for the heat equation (4) (only the case of homogeneous bodies will be considered).

BV Problem 1 (Dirichlet Condition on $\partial\Omega$). Find a function $u(x,t), x \in \overline{\Omega} \subset \mathbb{R}^3, t \geq 0$ such that

$$\frac{\partial u}{\partial t} - k(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}) = 0, x \in \Omega, t > 0 \tag{11}$$

$$u(x,0) = f(x), x \in \overline{\Omega} \tag{12}$$

$$u(x,t) = \phi(x,t), x \in \partial\Omega, t \geq 0 \tag{13}$$

where $f(x)$ and $\phi(x,t)$ are prescribed functions on $\overline{\Omega}$ and $\partial\Omega \times [0,\infty]$, respectively.

Definition. A classical solution of BV Problem 1 is a real-valued function $u \in C(\overline{\Omega} \times [0,\infty)) \cap C^2(\Omega \times (0,\infty))$ which satisfies (11), (12), (13).

BV Problem 2 (Neumann Condition on $\partial\Omega$).[26] Find a function $u(x,t), x \in \overline{\Omega}, t \geq 0$ such that

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = 0, x \in \Omega, t > 0 \tag{14}$$

$$u(x,0) = f(x), x \in \overline{\Omega} \tag{15}$$

$$\frac{\partial u(x,t)}{\partial \nu} \equiv \nabla u(x,t) \bullet \nu(x) = \phi(x,t), x \in \partial\Omega, t \geq 0 \tag{16}$$

where $f(x)$ and $\phi(x,t)$ are prescribed functions.

Definition A classical solution of BV Problem 2 is a real-valued function $u \in C^1(\overline{\Omega} \times [0,\infty)) \cap C^2(\Omega \times (0,\infty))$ which satisfies (14), (15), (16).

BV Problem 3 (Robin condition on $\partial\Omega$). Find a function $u(x,t), x \in \overline{\Omega}, t \geq 0$ such that

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = 0, x \in \Omega, t > 0 \tag{17}$$

[26] Named for German mathematician Carl Gottfried Neumann, cofounder of the mathematical research journal *Mathematische Annalen.*

$$u(x,0) = f(x), x \in \overline{\Omega} \tag{18}$$

$$\frac{\partial u(x,t)}{\partial \nu} + hu = \phi(x,t), x \in \partial\Omega, t \geq 0 \tag{19}$$

where $f(x)$ and $\phi(x,t)$ are prescribed functions and $h > 0$ is a prescribed constant.

Diffusion of Heat in 2 Space Dimensions. If

$$\Omega = \{(x_1, x_2, x_3) : (x_1, x_2) \in \Omega', -\infty < x_3 < \infty\}$$

and if the prescribed functions in BV Problems 1, 2 and 3 are all independent of x_3 then $u = u(x_1, x_2, t)$ (also independent of x_3). In this case the BV problems reduce to problems with one less space variable.

The Maximum Principle. The maximum theorem for the heat equation in one space dimension formulated and proved on pages 10-11, can be generalized to higher dimensions. It may be formulated as follows.

Theorem. Let Ω be a bounded domain in $\mathbb{R}^3$. Let $u \in C(\overline{\Omega} \times [0, \infty))$, $\partial u/\partial t \in C(\Omega \times (0, \infty))$, $\partial^2 u/\partial x_i^2 \in C(\Omega \times (0, \infty))$, $(i = 1, 2, 3)$ and

$$\frac{\partial u}{\partial t} - k(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}) = 0, x \in \Omega, t > 0$$

Let

$$\Gamma_T = (\overline{\Omega} \times \{0\}) \cup \{(x, t) : x \in \partial\Omega, 0 \leq t \leq T\}$$

Then $\forall\, T > 0$

$$\max_{x \in \overline{\Omega}, 0 \leq t \leq T} u(x, t) = \max_{\Gamma_T} u(x, t)$$

The proof is essentially the same as the one on pages 6-7. In fact, the proof works for any number $n \geq 1$ of space variables.

Uniqueness Theorem for BV Problem 1 in Bounded Domains. The theorem states that BV Problem 1 for a bounded domain Ω has at most one classical solution. The result is an immediate consequence of the maximum principle. The proof is the same as the one for one space dimension (pages 7-8 above).

Uniqueness Theorems for BV Problems 2 and 3 in Bounded Domains. In the case of BV Problems 2 and 3 the uniqueness of classical solutions does not follow immediately from the maximum principle as it does for BV Problem 1. Another method of proving uniqueness will now be given that works for these two problems. The following notation will be used in the proof.

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3})$$

$$\triangle u = \nabla \bullet \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

Note also the identity

$$\nabla \bullet (u \nabla v) = \nabla u \bullet \nabla v + u \triangle v \qquad (20)$$

Theorem If $\Omega \in \mathbb{R}^3$ is a bounded domain for which the divergence theorem holds (this is a type of restriction on $\partial\Omega$) then BV problems 2 and 3 have at most one classical solution.

Proof. Note that BV problem 3 with h = 0 is the same as BV problem 2. Hence it will be enough to discuss BV problem 3 with condition $h \geq 0$.

To prove that at most one classical solution exists we suppose that $u_1(x,t)$ and $u_2(x,t)$ are any two classical solutions with the same "data" $f(x), \phi(x,t)$ and consider the difference

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

By linearity, $u(x,t)$ is a classical solution of BV problem 3 with data $f(x) \equiv 0$ in Ω, $\phi(x,t) \equiv 0$ on $\partial\Omega \times [0,\infty)$. Now consider the function

$$J(t) = \int_\Omega u(x,t)^2 dx, \quad t \geq 0$$

$J(t)$ is finite $\forall\ t \geq 0$ because Ω is bounded and $u(.,t) \in C(\overline{\Omega}), \forall\ t \geq 0$. Moreover, it is easy to verify that

$$J(t) \in C[0,\infty) \cap C^1(0,\infty)$$

and

$$J'(t) = 2\int_\Omega u(x,t)\frac{\partial u(x,t)}{\partial t} dx, \quad t \geq 0$$

Using the heat equation for u gives

$$J'(t) = 2\int_\Omega u(x,t) \bigtriangleup u(x,t) dx$$

Combining this with (20), with $v = u$, gives

$$J'(t) = 2\int_\Omega \nabla \bullet (u\nabla u) dx - 2\int_\Omega |\nabla u(x,t)|^2 dx$$

Next, using the divergence theorem gives

$$J'(t) = 2\int_{\partial\Omega} u\nabla u \bullet \nu(x)dS - 2\int_{\Omega} |\nabla u|^2 dx$$
$$= 2\int_{\partial\Omega} u\frac{\partial u}{\partial \nu}dS - 2\int_{\Omega} |\nabla u|^2 dx$$

Finally, using (19) with $\phi(x,t) \equiv 0$ gives

$$J'(t) = -2h\int_{\partial\Omega} u(x,t)^2 dS - 2\int_{\Omega} |\nabla u(x,t)|^2 dx \leq 0, \quad \forall t > 0$$

because $h \geq 0$. Moreover, the initial condition (18) with $f(x) \equiv 0$ implies $J(0) = 0$. Thus we deduce that

$$J(t) = J(0) + \int_0^t J'(\tau)d\tau = \int_0^t J'(\tau)d\tau \leq 0, \quad \forall\ t \geq 0 \tag{21}$$

But, since $u(.,t) \in C(\overline{\Omega}), \quad \forall\ t \geq 0$, (21) implies[27] that

$$u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0, \quad \forall \ \ x \in \overline{\Omega}, t \geq 0$$

i.e.,

$$u_1(x,t) = u_2(x,t)$$

QED.

Remark 1. The same method also works for BV problem 1 if the divergence theorem holds for the domain Ω.

[27] by the definition of J

Remark 2. The proof works for any number $n \geq 1$ of space dimensions.[28]

[28]The divergence theorem for $n = 1$ dimensions is just the integration by parts formula.

CHAPTER 5.

<u>THE CAUCHY PROBLEM FOR THE HEAT EQUATION IN 1 SPACE DIMENSION</u>.

So far, our results for the heat equation have been restricted to bounded domains. The Cauchy problem asks for a function $u(x, t)$ such that

$$(1) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \quad t > 0$$

$$(2) \qquad u(x, 0) = u_o(x) \quad \text{for} \quad -\infty < x < \infty$$

where $u_o(x)$ is a prescribed function.

In this chapter the existence, uniqueness and construction of a solution of (1), (2) is studied. It will be seen that uniqueness holds only if restrictions are placed on the initial value function $u_o(x)$ <u>and</u> the class of solutions $u(x, t)$ admitted.

It will be helpful to think of the solution of (1), (2) as describing the temperature distribution in an infinite uniform rod whose surface is insulated. Thus temperature will satisfy (1) if

$$(3) \qquad k = \frac{K}{c\rho} = 1$$

which also can be achieved by the proper choice of time unit. It will also be assumed that

$$(4) \qquad c\rho = 1 \quad \text{(choice of unit of heat)}$$

and

$$(5) \qquad A = 1 \quad \text{(choice of unit of length)}$$

Diffusion of Heat From a Point Source - The Fundamental Solution. Suppose that the infinite uniform rod is at temperature 0 for $t < 0$ and imagine that at time $t = 0$ 1 unit of heat is introduced in a section $x_o \le x \le x_o + \Delta x$ of the rod. If the heat is distributed uniformly in $[x_o, x_o + \Delta x]$ the result will be an increase in temperature of the section from 0 to a uniform temperature $T_o =$ const. in $[x_o, x_o + \Delta x]$. By the definition of specific heat and the assumptions (3), (4), (5)

$$\text{Net increase in heat content of } [x_o, x_o+\Delta x] = 1 = c \times \underbrace{\rho A \Delta x}_{\text{mass of section}} \times (T_o - 0) = T_o \Delta x$$

Thus the introduction of unit heat, uniformly distributed in $[x_o, x_o+\Delta x]$, produces an initial temperature distribution

$$(6) \qquad u_o(x) = u_{\Delta x}(x, x_o) = \begin{cases} \frac{1}{\Delta x}, & x_o \le x \le x_o + \Delta x \\ 0 & \text{elsewhere} \end{cases}$$

Let $u_{\Delta x}(x, x_o, t)$ denote the subsequent temperature distribution, i.e., the solution of (1), (2) with $u_o(x)$ defined by (6) (it is assumed for the moment to exist).

[See blackboard illustration.]

The total quantity of heat in the infinite rod for any time t, is

$$(7) \qquad Q_{\Delta x} = \int_{-\infty}^{\infty} c\rho A u_{\Delta x}(x, x_o, t)dx = \int_{-\infty}^{\infty} u_{\Delta x}(x, x_o, t)dx$$

The "conservation of energy" principle implies that this should be independent of t. Indeed, this is verified by the following calculation:

$$Q'_{\Delta x}(t) = \int_{-\infty}^{\infty} \frac{\partial u_{\Delta x}(x, x_o, t)}{\partial t} dx = \int_{-\infty}^{\infty} \frac{\partial^2 u_{\Delta x}(x, x_o, t)}{\partial x^2} dx = 0$$

provided $\partial u_{\Delta x}(x, x_o, t)/\partial x \to 0$ when $x \to \pm\infty$. This is clearly the case. Thus

$$(8) \qquad Q_{\Delta x}(t) = Q_{\Delta x}(o^+) = 1, \quad \forall \ t > 0$$

<u>The Limiting Case of a Point Source</u>. Now make $\Delta x \to 0$ and assume, tentatively, that the limit

$$(9) \qquad \phi(x, x_o, t) = \lim_{\Delta x \to 0} u_{\Delta x}(x, x_o, t)$$

exists. This will be verified below. The limiting function $\phi(x, x_o, t)$ should describe the temperature distribution at time $t > 0$ due to a unit amount of heat released at the point x_o at time $t = 0$.

Intuitively, this corresponds to the Cauchy problem (1), (2) with the initial distribution

$$u_o(x) = \delta(x - x_o) \qquad \text{(Dirac } \delta\text{-function)}$$

but use of distribution theory will be avoided here. The limit function (9) may be expected to have the following properties

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \qquad \forall \;\; x, x_o \in \mathbb{R} \;\text{ and }\; t > 0 \tag{10}$$

$$\lim_{t \to 0^+} \phi(x, x_o, t) = 0 \quad \forall \;\; x \neq x_o \tag{11}$$

$$\int_{-\infty}^{\infty} \phi(x, x_o, t) dx = 1 \quad \forall \;\; x_o, t > 0 \tag{12}$$

$$\phi(x, x_o, t) \geq 0, \quad \forall x, x_o \in \mathbb{R}, t > 0 \tag{13}$$

A function $\phi(x, x_o, t)$ having these properties will now be constructed. As a working hypothesis, it will be assumed that there is exactly one function having properties (10)-(13). This assumption will be shown to lead to a construction of $\phi(x, x_o, t)$. The latter is part of folklore in PDEs, summarized by phrase, "uniqueness implies existence."

Step 1. Let $\phi(x, x_o, t)$ be the function having properties (10)-(13) and consider the function $\phi'(x, x_o, t)$ defined by

$$\phi'(x, x_o, t) = \phi(x - \xi, x_o - \xi, t), \qquad \xi \in \mathbb{R} \tag{14}$$

It is clear that for each fixed $\xi \in \mathbb{R}$ ϕ' also satisfies (10)-(13). Hence, by the assumed uniqueness of ϕ

$$\phi(x, x_o, t) = \phi(x - \xi, x_o - \xi, t) \quad \forall \;\; x, x_o, \xi \in \mathbb{R}, t > 0 \tag{15}$$

Taking $x_o = \xi$ in (14) gives

$$\phi(x, x_o, t) = \phi(x - x_o, t) \quad \forall \;\; x, x_o \in \mathbb{R}, t > 0 \tag{16}$$

where $\phi(x, t)$ is the function defined by

$$\phi(x, t) = \phi(x, 0, t) \tag{17}$$

Thus $\phi(x, t)$ satisfies

$$\frac{\partial \phi(x, t)}{\partial t} = \frac{\partial^2 \phi(x, t)}{\partial x^2} \qquad \forall \;\; x \in \mathbb{R} \;\; \text{and} \;\; t > 0 \tag{18}$$

$$(19) \qquad \lim_{t\to 0^+} \phi(x,t) = 0 \quad \forall \;\; x \neq 0$$

$$(20) \qquad \int_{-\infty}^{\infty} \phi(x,t)dx = 1 \quad \forall \;\; t > 0$$

$$(21) \qquad \phi(x,t) \geq 0, \quad \forall x \in \mathbb{R}, t > 0$$

Step 2. Consider the function

$$(22) \qquad \phi'(x,t) = \alpha^n \phi(\alpha x, \alpha^2 t), \quad \alpha > 0$$

For any fixed $\alpha > 0$ it satisfies (18), (19) and (21). Moreover,

$$\int_{-\infty}^{\infty} \phi'(x,t)dx = \alpha^n \int_{-\infty}^{\infty} \phi(\alpha x, \alpha^2 t)dx = \alpha^{n-1} \int_{-\infty}^{\infty} \phi(y, \alpha^2 y)dy = 1$$

if and only if $\alpha^{n-1} = 1$; i.e., $\alpha = 1$ or $n = 1$. Thus $\alpha\phi(\alpha(x - x_o), \alpha^2 t)$ has the properties (10)-(13) for any $\alpha > 0$ and hence by the assumed uniqueness

$$(23) \qquad \phi(x,t) = \phi(x,0,t) = \alpha\phi(\alpha x, \alpha^2 t) \quad \forall \;\; \alpha > 0$$

Taking $\alpha = t^{-\frac{1}{2}}$ gives

$$(24) \qquad \phi(x,t) = t^{-\frac{1}{2}}\phi(t^{-\frac{1}{2}}x)$$

where $\phi(x)$ is the function of $x \in \mathbb{R}$ defined by

$$\phi(\xi) = \phi(\xi, 1), \qquad \xi \in \mathbb{R} \tag{25}$$

To find the properties of $\phi(\xi)$ corresponding to (18)-(21) note that differentiating (24) gives

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2}t^{-\frac{3}{2}}\phi(\frac{x}{t^{\frac{1}{2}}}) - \frac{1}{2}t^{-\frac{1}{2}}t^{-\frac{3}{2}}x\phi'(\frac{x}{t^{\frac{1}{2}}}) = -\frac{1}{2}t^{\frac{3}{2}}(\phi(\xi) + \xi\phi'(\xi))$$

where

$$\xi = \frac{x}{t^{\frac{1}{2}}}$$

Similarly,

$$\frac{\partial \phi(x,t)}{\partial x} = \frac{1}{t}\phi'(\frac{x}{t^{\frac{1}{2}}}), \qquad \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{t^{\frac{3}{2}}}\phi''(\xi)$$

Whence,

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{\partial \phi(x,t)}{\partial t} = \frac{1}{t^{\frac{3}{2}}}\{\phi''(\xi) + \frac{\xi}{2}\phi'(\xi) + \frac{1}{2}\phi(\xi)\} = 0$$

Also,

$$\int_{-\infty}^{\infty} \phi(x,t)dx = t^{\frac{1}{2}}\int_{-\infty}^{\infty} \phi(t^{-\frac{1}{2}}x)dx = \int_{-\infty}^{\infty} \phi(\xi)d\xi = 1$$

and

$$\lim_{t\to 0^+} \phi(x,t) = \lim_{t\to 0^+} \frac{1}{x}\frac{x}{t^{\frac{1}{2}}}\phi(\frac{x}{t^{\frac{1}{2}}}) = \frac{1}{x}\lim_{\xi\to\pm\infty} \xi\phi(\xi) = 0$$

for all $x \neq 0$ ($x > 0 \Leftrightarrow \xi \to \infty$, $x < 0 \Leftrightarrow \xi \to -\infty$.)

Thus (24) has properties (18) - (21) if and only if $\phi(\xi)$ satisfies the conditions

$$(26) \quad \frac{d^2\phi}{d\xi^2} + \frac{\xi}{2}\frac{d\phi}{d\xi} + \frac{1}{2}\phi = 0 \qquad \forall\ \xi \in \mathbb{R}$$

$$(27) \qquad \lim_{t\to\pm\infty} \xi\phi(\xi) = 0$$

$$(28) \qquad \int_{-\infty}^{\infty} \phi(\xi)d\xi = 1$$

$$(29) \qquad \phi(\xi) \geq 0 \qquad \forall\ \xi \in \mathbb{R}$$

It will be shown that (26)-(28) determine $\phi(\xi)$ uniquely.

<u>Step 3</u>. To integrate (26) note that

$$\frac{d}{d\xi}(\frac{d\phi}{d\xi} + \frac{\xi}{2}\phi) \equiv \frac{d^2\phi}{d\xi^2} + \frac{\xi}{2}\frac{d\phi}{d\xi} + \frac{1}{2}\phi$$

Thus (26) is equivalent to the 1st order equation

$$(30) \quad \frac{d\phi}{d\xi} + \frac{\xi}{2}\phi = K = \text{ const.} \quad \xi \in \mathbb{R}$$

If $K = 0$ then separation of variables gives

$$\phi = ce^{\frac{-\xi^2}{4}}$$

where c is any constant. The method of variation of constants[29] then gives the general solution

$$(31) \qquad ce^{-\xi^2/4} + c'e^{-\xi^2/4}\int_0^{\xi} e^{\tau^2/4}d\tau$$

where c and c' are arbitrary constants. To determine them, conditions (27)-(29) will be used. Note that, by repeated use of l'Hôpital's rule

$$\lim_{\xi\to\pm\infty} \xi e^{-\xi^2/4}\int_0^{\xi} e^{\tau^2/4}d\tau = \lim_{\xi\to\pm\infty} \frac{\xi \int_0^{\xi} e^{\tau^2/4}d\tau}{e^{\xi^2/4}}$$

$$= \lim_{\xi\to\pm\infty} \frac{\int_0^{\xi} e^{\tau^2/4}d\tau + \xi e^{\xi^2/4}}{\frac{1}{2}\xi e^{\xi^2/4}}$$

$$= 2 + \lim_{\xi\to\pm\infty} \frac{e^{\xi^2/4}}{\frac{1}{2}\xi e^{\xi^2/4} + \frac{\xi^2}{4}e^{\xi^2/4}} = 2$$

Thus (31) will satisfy (27) if and only if $c' = 0$. Thus

$$(32) \qquad \phi(\xi) = ce^{-\xi^2/4}$$

To find c we have, by (28)

[29]Also called variation of parameters.

$$\int_{-\infty}^{\infty} \phi(\xi)d\xi = c\int_{-\infty}^{\infty} e^{\xi^2/4}d\xi = c(4\pi)^{1/2} = 1$$

i.e. $c = (4\pi)^{-1/2}$ and hence

$$(33) \qquad \phi(\xi) = (4\pi)^{-1/2}e^{\xi^2/4}$$

Note that (29) is also satisfied. Finally, combining (33), (24) and (16) gives

$$(34) \qquad \phi(x, x_o, t) = (4\pi t)^{-1/2}e^{-(x-x_o)^2/4t}$$

and we have proved the

Lemma. If $\exists$ a unique function $\phi(x, x_o, t)$ having properties (10)-(13) then that function is given by (34).

Corollary. The function (34) does have the properties (10)-(13).

The corollary follows from the derivation and can also be verified directly.

Remark. The function defined by (34) is called the fundamental solution of the heat equation. Physically, it gives the temperature in an infinite uniform rod due to a unit of heat energy placed at x_o at time $t = 0$.

Formal Solution of the Cauchy Problem. Returning to the problem (1), (2) (p. 78), let us assume that $u_o \in C(\mathbb{R})$ and approximate it by a step function:

$$u_o(x) \approx \sum_{i=-\infty}^{\infty} u_o(x_i')\chi_i(x)$$

where $x_i < x_i' < x_{i+1}$ and

$$\chi_i(x) = \begin{cases} 1 & x_i \leq x < x_{i+1} \\ \\ 0 & \text{elsewhere} \end{cases}$$

The temperature distribution due to a single term

$$u_o(x_i')\chi_i(x)$$

is (see p. 79)

$$u(x,t) = (u_o(x_i')\Delta x)u_{\Delta x}(x, x_i, t) \approx \phi(x, x_i, t)u_o(x_i')\Delta x$$

Summing over i gives the approximation

$$u(x,t) \approx \sum_{-\infty}^{\infty} \phi(x, x_i, t)u_o(x_i')\Delta x$$

with an approximation that improves as $\Delta x \to 0$. This suggests that the Cauchy problem has the solution

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x,\xi,t) u_o(\xi) d\xi$$

or, more explicitly

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} u_o(\xi) d\xi \tag{35}$$

Remark. (35) is usually called Poisson's solution, although it appears to have been discovered first by Laplace (1809).

Form of the Fundamental Solution. Note that for small t, $\phi(x, x_o, t)$ is concentrated near $x = x_o$. One see this by doing various plots with a computer algebra system.

This suggests that if $u(x,t)$ is defined by (35) and $u_o \in C(\mathbb{R})$ then for t small

$$\begin{aligned} u(x_o,t) &\approx \frac{1}{(4\pi t)^{1/2}} \int_{x_o-\delta}^{x_o+\delta} e^{-(x_o-\xi)^2/4t} u_o(\xi) d\xi \\ &\approx \frac{u_o(x_o)}{(4\pi t)^{1/2}} \int_{x_o-\delta}^{x_o+\delta} e^{-(x_o-\xi)^2/4t} d\xi) \\ &\approx u_o(x_o) \end{aligned}$$

This argument can be made more precise.[30]

[30] See D. V. Widder, *The Heat Equation.* New York: Academic Press, 1975; also Widder, "Postive Temperatures on an Infinite Rod." *Transactions of the American Mathematical Society* 55 (1944): 85-95.

Notation. Let $\Omega \subset \mathbb{R}^2$ be a domain (a connected open set). Then

$$(36) \qquad H(\Omega) = \{u(x,t) : \frac{\partial u}{\partial t} \in C(\Omega), \frac{\partial^2 u}{\partial x^2} \in C(\Omega), \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad \Omega)\}$$

THEOREM 1. Assume that $\exists$ a constant $a > 0$ such that

$$(37) \qquad u_o(x)e^{-ax^2} \in L_1(\mathbb{R})$$

Then[31] the integral in (35) converges for all (x,t) in the domain $\Omega_{1/4a}$ where

$$(38) \qquad \Omega_T = \mathbb{R} \times (0,T)$$

and defines a function $u \in H(\Omega_{1/4a})$.

Proof. Formal differentiation of (35) gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} = \frac{1}{4t^2}\int_{-\infty}^{\infty}(x-\xi)^2\phi(x,\xi,t)u_o(\xi)d\xi - \frac{1}{2t}\int_{-\infty}^{\infty}\phi(x,\xi,t)u_o(\xi)d\xi$$

The theorem will be established if it is shown that the last two integrals converge uniformly in some neighborhood of each point $(x_o,t_o) \in \Omega_{1/4a}$.

[31](37) is a growth limiting condition on u_o. It may grow fast but not too fast!

Choose $\delta > 0$ such that

$$N(x_o, t_o) = \{(x,t) : |x - x_o| < \delta, |t - t_o| < \delta\} \subset \Omega_{1/4a}$$

It will be shown that the integrals are dominated by convergent integrals whose integrands are independent of $(x,t) \in N(x_o, t_o)$. Consider first the two integrals restricted over the partial range $[R, \infty)$ where $R > x_o + \delta$. We have for $x_o - \delta < x < x_o + \delta$, $\xi \geq R, 0 < t_o - \delta < t < t_o + \delta < t/4a$

$$\phi(x,\xi,t) = \frac{1}{(4\pi t)^{1/2}} e^{-(x-\xi)^2/4t} \leq \frac{1}{(4\pi t)^{1/2}} e^{-(\xi - x_o - \delta)^2/4t}$$

$$\leq \frac{1}{(4\pi t)^{1/2}} e^{-(x-\xi)^2/4(t_o+\delta)} \leq \frac{1}{4\pi(t_o - \delta))^{1/2}} e^{-(\xi - x_o - \delta)^2/4(t_o+\delta)}$$

$$= C\phi(x_o + \delta, \xi, t_o + \delta)$$

where

$$C = \sqrt{\frac{t_o + \delta}{t_o - \delta}}$$

Similarly, $\forall\ (x,t) \in N(x_o, t_o)$ and $R \leq \xi < \infty$

$$(x - \xi)^2 \phi(x, \xi, t) \leq C(\xi - x_o + \delta)^2 \phi(x_o + \delta, \xi, t_o + \delta)$$

Since $t_o + \delta < 1/4a$ it follows that

$$\phi(x_o + \delta, \xi, t_o + \delta) e^{a\xi^2} \leq K_1 \qquad \forall\ \xi \in [R, \infty)$$

$$(\xi - x_o + \delta)^2 \phi(x_o + \delta, \xi, t_o + \delta) e^{a\xi^2} \leq K_2 \qquad \forall \;\; \xi \in [R, \infty)$$

Then

$$|\phi(x, \xi, t) u_o(\xi)| \in C K_1 |u_o(\xi) e^{-a\xi^2}| \in L_1([R, \infty))$$

for all $(x, t) \in N(x_o, t_o)$. The remaining integrals can be treated in the same way. QED.

<u>Theorem 2</u>. Assume that u_o satisfies (37) and that the limits $u_o(x_o+)$ and $u_o(x_o-)$ exist. Then the function $u \in H(\Omega_{1/4a})$ defined by (35) satisfies[32]

$$\text{(39)} \qquad \limsup_{x \to x_o, t \to 0^+} |u(x, t)| \leq \max(|u_o(x_o+)|, |u_o(x_o-)|)$$

If in addition, $u_o(x_+) = u_o(x_o-)$ then

$$\text{(40)} \qquad \lim_{x \to x_o, t \to 0^+} u(x, t) = u_o(x_o+)$$

<u>Remark</u>. The notation $x \to x_o, t \to 0$ means that $(x, t) \to (x_o, 0)$ through points of the half plane $t > 0$. The approach is otherwise unrestricted.

<u>Proof of Theorem 2</u>. Let $M = \max(|u_o(x_o+)|, |u_o(x_o-)|)$. Then $\forall \;\; \epsilon > 0 \;\; \exists \;\; \delta > 0$ such that

$$\text{(41)} \qquad |u_o(\xi)| < M + \epsilon \quad \text{ for } \quad |\xi - x_o| \leq \delta$$

[32]See Appendix IV.

Thus if

$$u(x,t) = \int_{-\infty}^{x_o-\delta} \phi(x,\xi,t)u_o(\xi)d\xi + \int_{x_o-\delta}^{x_o+\delta} \phi(x,\xi,t)u_o(\xi)d\xi + \int_{x_o+\delta}^{\infty} \phi(x,\xi,t)u_o(\xi)d\xi$$

$$= I_1 + I_2 + I_3$$

then

$$|I_2| \leq (M+\epsilon)\int_{-\infty}^{\infty} \phi(x,\xi,t)u_o(\xi)d\xi = M+\epsilon, \quad \forall \;\; x \in \mathbb{R}, t > 0.$$

Moreover, if $|x - x_o| < \rho < \delta, \xi \geq x_o + \delta$ then $-x_o - \rho < x < x_o + \rho, \;\; \xi - x \geq \xi - x_o - \rho$ and hence

$$\phi(x,\xi,t) \leq \frac{1}{(4\pi t)^{1/2}} e^{-(\xi - x_o - \rho)/4t} = \phi(x_o+\rho,\xi,t)$$

Thus if $|x - x_o| < \rho$ then

$$|I_3| \leq \int_{x_o+\delta}^{\infty} \phi(x_o+\rho,\xi,t)|u_o(\xi)|d\xi$$

Now

$$\phi(x_o+\rho,\xi,t)e^{a\xi^2}$$

is a <u>decreasing</u> <u>function</u> of ξ for $\xi \geq x_o + \delta$ and all t sufficiently small. (In fact it has its maximum at $\xi = (x_o+\rho)/(1-4at)$.) Thus

$$\phi(x_o+\rho,\xi,t)e^{a\xi^2} \leq \phi(x_o+\rho,x_o+\delta,t)e^{a(x_o+\delta)^2}, \quad \xi \geq x_o+\delta$$

or

$$\phi(x_o+\rho,\xi,t) \leq [\phi(\rho,\delta,t)e^{a(x_o+\delta)^2}]e^{-a\xi^2}, \quad \xi \geq x_o+\delta$$

and

$$|I_3| \leq [\phi(\rho,\delta,t)e^{a(x_o+\delta)^2}]\int_{x_o+\delta}^{\infty} e^{-a\xi^2}|u_o(\xi)d\xi$$

$\forall$ (x,t) with $|x-x_o|<\rho, t>0$. Making $t\to 0^+$ gives

$$\limsup_{x\to x_o, t\to 0^+} |I_3(x,t)| = 0$$

The integral I_1 has the same property. Thus by (42)

$$\limsup_{x\to x_o, t\to 0^+} |u(x,t)| \leq M+\epsilon, \quad \forall \ \epsilon>0$$

which implies (39). To prove (40), write

$$u(x,t)-u_o(x_o+) = \int_{-\infty}^{\infty}\phi(x,\xi,t)[u_o(\xi)-u_o(x_o+)]d\xi = I_1'+I_2'+I_3'$$

as above. Following the same reasoning as above,

$$|I_2'(x,t)| \le \sup_{|x-x_o|\le\delta} |u_o(\xi) - u_o(x_o+)| + \epsilon$$

and the argument used above gives

$$\limsup_{x\to x_o, t\to 0^+} |u(x,t) - u_o(x_o+)| \le \sup_{|\xi-x_o|\le\delta} |u_o(\xi) - u_o(x_o+)|, \quad \forall \ \ \delta > 0$$

If $u_o(x_o-) = u_o(x_o+)$ the last expression tends to zero with δ which proves (40). QED.

Theorems 1 and 2 can be combined to give an existence theorem for the Cauchy problem. The notation Ω_T as above and

$$\dot{\Omega}_T = \mathbb{R} \times [0,T)$$

will be used.

Definition. A classical solution of the Cauchy problem (1), (2) (p. 78) in a domain Ω_T is a function

$$u \in H(\Omega_T) \cap C(\dot{\Omega}_T)$$

such that $u(x,0) = u_o(x)$ for all $x \in \mathbb{R}$.

Observe that $u_o \in C(\mathbb{R})$ is a necessary condition for the existence of a classical solution. Theorems 1 and 2 imply that continuity and condition (37) are sufficient. More precisely, the following theorem holds.

Existence Theorem. Assume that

$$\text{(a)} \qquad\qquad u_o \in C(\mathbb{R})$$

$$\text{(b)} \quad \exists \;\; a > 0 \;\; \text{such that} \;\; u_o(x)e^{-ax^2} \in L_1(\mathbb{R})$$

Then the integral (35) defines a classical solution of the Cauchy problem in the domain $\Omega_{1/4a}$.

Proof. Theorems 1 and 2 imply that $u \in H(\Omega_{1/4a})$ and that

$$\lim_{x \to x_o, t \to 0^+} u(x,t) = u_o(x_o)$$

for all x_o. This property and the continuity of u_o imply that $u \in C(\dot{\Omega}_{1/4a})$ and $u(x,0) = u_o(x), \;\; \forall \;\; x \in \mathbb{R}$. QED.

Corollary. Assume that

$$\text{(a)} \qquad u_o \in C(\mathbb{R})$$

$$\text{(b)} \qquad \exists M \;\; \text{such that} \;\; |u_o(x)| \leq M, \;\; \forall \;\; x \in \mathbb{R}$$

Then (35) defines a classical solution of the Cauchy problem in $\Omega_\infty = \mathbb{R}^2_+$. Moreover

$$|u(x,t)| \leq M \qquad \forall \; (x,t) \in \mathbb{R}^2_+$$

.

Proof. The boundedness condition (b) implies that condition (b) of the

Existence Theorem holds for all $a > 0$. (35) implies the second part of the corollary. QED.

The heat equation is strongly linked to the notion of absolute zero. In fact it is possible to show that:

Widder's Representation Theorem. A real-valued function $u(x,t)$ has the properties

$$(1) \qquad u \in H(\Omega_T)$$

$$(2) \qquad u(x,t) \geq 0, \quad \forall \ (x,t) \in \Omega_T$$

if and only if there exists a real non-decreasing function $\alpha(x)$ defined on $\mathbb{R}$ such that

$$(3) \qquad u(x,t) = \int_{-\infty}^{\infty} \phi(x,\xi,t) d\alpha(\xi) \quad \forall \ \ (x,t) \in \Omega_T$$

The class of solutions defined by (3) is more general than that defined by Poisson's integral with locally integrable $u_o(x)$. However we will not prove this theorem since it requires some elementary measure theory.

Chapter 6. Steady Temperature in a Finite Cylinder.

In Chapter 3 (p. 32) we considered the case of a long cylinder. We now consider the case of a finite cylinder. Let Ω be a cylinder of radius a and length l. Let the x_3-axis be the axis of the cylinder with $x_3 = 0$ corresponding to the base of the cylinder and $x_3 = l$ the top. The steady-state temperature u_s will satisfy

$$\frac{\partial^2 u_s}{\partial x_1^2} + \frac{\partial^2 u_s}{\partial x_2^2} + \frac{\partial^2 u_s}{\partial x_3^2} = \Delta u_s = 0, \quad x = (x_1, x_2, x_3) \in \Omega \tag{1}$$

$$u_s(x_1, x_2, 0) = f(x_1, x_2), \quad x_1^2 + x_2^2 \leq a^2 \tag{2}$$

$$u_s(x_1, x_2, l) = g(x_1, x_2), \quad x_1^2 + x_2^2 \leq a^2 \tag{3}$$

$$u_s(x_1, x_2, x_3) = h(x_1, x_2, x_3), \quad x_1^2 + x_2^2 = a^2, 0 \leq x_3 \leq l \tag{4}$$

(2)-(4) respectively correspond to the prescribed temperature at the bottom, top and side of Ω. Observe that conditions (2)-(3) suggest the consistency conditions $f(x_1, x_2) = h(x_1, x_2, 0), g(x_1, x_2) = h(x_1, x_2, l), \quad x_1^2 + x_2^2 = a^2$. The circular nature of the problem suggests a change of coordinates, $(x_1, x_2, x_3) \to (r, \theta, x_3)$ which is the familiar cylindrical coordinate transformation.

Put $u_s(x_1, x_2, x_3) = v(r, \theta, x_3)$. The calculations on pp. 33-34 show that v satisfies

$$(5) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial x_3^2} = 0, \quad 0 < r < a, \ \text{ all } \theta, \ 0 < x_3 < l$$

$$(6) \qquad v(r,\theta,0) = f(r,\theta), \quad x_1^2 + x_2^2 \le a^2$$

$$(7) \qquad v(r,\theta,l) = g(r,\theta), \quad x_1^2 + x_2^2 \le a^2$$

$$(8) \qquad v(a,\theta,x_3) = h(a,\theta,x_3), \quad 0 \le x_3 \le l$$

$$(9) \qquad v(0,\theta,x_3) \ \text{ is finite.}$$

where $f(r,\theta)$ is shorthand for $f(r\cos\theta, r\sin\theta)$, etc.

To simplify our computations, we shall make the assumption that the boundary temperatures are independent of θ (in other words, $f(r,\theta) = f(r)$, etc). Hence the steady-state temperature will be independent of θ and satisfies

$$(10) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x_3^2} = 0, \quad 0 < r < a, \ \text{ all } \theta, \ 0 < x_3 < l$$

$$(11) \qquad v(r,0) = f(r), \quad x_1^2 + x_2^2 \le a^2$$

$$(12) \qquad v(r,l) = g(r), \quad x_1^2 + x_2^2 \le a^2$$

$$(13) \qquad v(a,x_3) = h(x_3), \quad 0 \le x_3 \le l$$

$$(14) \qquad v(0, x_3) \text{ is finite.}$$

As with the vibrating string problem, it is convenient to break this problem into simpler problems.

BV Problem 1:

$$(15) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x_3^2} = 0, \quad 0 < r < a, \ \text{ all } \theta, \ 0 < x_3 < l$$

$$(16) \qquad v(r, 0) = f(r), \quad x_1^2 + x_2^2 \le a^2$$

$$(17) \qquad v(r, l) = 0, \quad x_1^2 + x_2^2 \le a^2$$

$$(18) \qquad v(a, x_3) = 0, \quad 0 \le x_3 \le l$$

$$(19) \qquad v(0, x_3) \text{ is finite.}$$

BV Problem 2:

$$(20) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x_3^2} = 0, \quad 0 < r < a, \ \text{ all } \theta, \ 0 < x_3 < l$$

$$(21) \qquad v(r, 0) = f(r), \quad x_1^2 + x_2^2 \le a^2$$

$$(22) \qquad v(r, l) = g(r), \quad x_1^2 + x_2^2 \le a^2$$

$$(23) \qquad v(a, x_3) = h(x_3), \quad 0 \le x_3 \le l$$

$$(24) \qquad v(0, x_3) \text{ is finite.}$$

BV Problem 3:

$$(25) \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x_3^2} = 0, \quad 0 < r < a, \text{ all } \theta, \ 0 < x_3 < l$$

$$(26) \qquad v(r, 0) = 0, \quad x_1^2 + x_2^2 \leq a^2$$

$$(27) \qquad v(r, l) = 0, \quad x_1^2 + x_2^2 \leq a^2$$

$$(28) \qquad v(a, x_3) = h(x_3), \quad 0 \leq x_3 \leq l$$

$$(29) \qquad v(0, x_3) \text{ is finite.}$$

We shall only consider BV Problem 1, as the others are similar.

We proceed by means of separation of variables. Assume that $v(r, x_3) = R(r)X(x_3)$.

(15) leads to

$$\frac{1}{r}(rR')'X + X''R = 0, \quad 0 < r < a, \ 0 < x_3 < l$$

and hence

$$\frac{\frac{1}{r}(rR')'}{R} = -\frac{X''}{X} = -\lambda^2$$

and this gives

$$(30) \qquad \frac{1}{r}(rR')' + \lambda^2 R = 0$$

$$(31) \qquad R(0) \text{ finite}$$

$$(32) \qquad R(a) = 0$$

$$(33) \qquad X'' - \lambda^2 X = 0$$

$$(34) \qquad X(l) = 0$$

(31), (32) and (34) are required by (19), (18) and (17), respectively. (30) is a form of Bessel's equation.[33]

Fortunately, the solutions to (33) are easily found. Indeed, $X(x_3) = A\cosh\lambda x_3 + B\sinh\lambda x_3$. (34) will be used later.

Solutions of Bessel's Equation. It is helpful to return to the heat equation: $\partial u/\partial t = \Delta u$.

Suppose we have a solution of the heat equation of the form $u(x_1, x_2, x_3, t) = w(x_1, x_2)e^{-\lambda^2 t}$. Substitution of this u into the heat equation gives

$$\Delta w + \lambda^2 w = 0$$

or in polar coordinates ($x_1 = r\cos\theta, x_2 = r\sin\theta$):

$$(35) \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \lambda^2 v = 0$$

[33]Named for German astronomer and mathematician Friedrich Bessel. The solutions of (30)-(32) are known as Bessel functions, or (for obvious reasons) cylinder functions. Bessel established basic properties of the solutions of (30)-(32) in 1824.

$$(36) \qquad v(r, \theta + 2\pi) = v(r, \theta)$$

$$(37) \qquad v(0, \theta) \text{ finite}$$

Now observe that (30) is a special case of (35) in case v is independent of θ. Write (35) as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \lambda^2 v = 0$$

and integrate with respect to θ:

$$\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial r^2} d\theta + \frac{1}{r}\int_{-\pi}^{\pi} \frac{\partial v}{\partial r} d\theta + \frac{1}{r^2}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2} d\theta + \lambda^2 \int_{-\pi}^{\pi} v d\theta = 0$$

The third integral may be computed and based on (36) is zero. Thus we have

$$\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial r^2} d\theta + \frac{1}{r}\int_{-\pi}^{\pi} \frac{\partial v}{\partial r} d\theta + \lambda^2 \int_{-\pi}^{\pi} v d\theta = 0$$

or

$$\frac{\partial^2}{\partial r^2}\int_{-\pi}^{\pi} v d\theta + \frac{1}{r}\frac{\partial}{\partial r}\int_{-\pi}^{\pi} v d\theta + \lambda^2 \int_{-\pi}^{\pi} v d\theta = 0$$

Thus we may write

$$(38) \qquad R(r) = \int_{-\pi}^{\pi} v d\theta$$

Thus every solution of $\Delta w + \lambda^2 w = 0$ generates a solution of (30). We may exploit this to obtain information about solutions of (30) which has no solutions in terms of standard elementary functions. Suppose that $w = e^{ax_1+bx_2}$. Substituting this into $\Delta w + \lambda^2 w = 0$ yields $a^2 + b^2 + \lambda^2 = 0$. We may choose a and b in any way, so select $a = 0, b = i\lambda$. This gives $w = e^{ix_2\lambda}$. Introducing polor coordinates again gives

$$(39) \qquad v(r,\theta) = e^{ir\lambda\sin\theta}$$

Using (38) we have

$$(40) \qquad R(r) = \int_{-\pi}^{\pi} e^{ir\lambda\sin\theta} d\theta$$

It is convenient to introduce a normalizing factor into (40):

$$(41) \qquad R(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ir\lambda\sin\theta} d\theta$$

This specifies that $R(0) = 1$. Using Euler's formula and that sin is an odd function,

$$(42) \qquad R(r) = \frac{1}{\pi}\int_{0}^{\pi} \cos(ir\lambda\sin\theta) d\theta$$

Now substitution of the Maclaurin series for cosine into (42) gives

$$R(r) = \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j(\lambda r)^{2j}}{(2j)!}\int_{0}^{\pi} \sin^{2j}\theta d\theta$$

Reduction formulae can now be used to write

$$\int_0^\pi \sin^{2j}\theta d\theta = \frac{\pi(2j)!}{2^{2j}(j!)^2}$$

It follows that $R(r)$ has the power series representation

$$(43) \qquad R(r) = \sum_{j=0}^{\infty} \frac{(-1)^j(\lambda r)^{2j}}{j!^2 2^{2j}}$$

(43) may be used to obtain approximate values of $R(r)$.

Other forms of $R(r)$ exist and provide other kinds of information. For example, make the substitution $\omega = \theta - \pi/2$ into (42) which results in

$$(44) \qquad R(r) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(\lambda r \cos\omega)d\omega$$

Make the substitution $t = \cos\omega$ into (44) to get

$$(45) \qquad R(r) = \frac{2}{\pi}\int_0^1 \frac{\cos(\lambda r t)}{\sqrt{1-t^2}}dt$$

The standard notation for $R(r)$ is $J_o(r\lambda)$. From (42) we observe that

$$(46) \qquad |J_o(r\lambda)| \leq 1$$

For ease of computation, write $\lambda = 1$ in (32). (We can always recover R by substituting λr for r.) Then

$$J_o(r) = \frac{2}{\pi}\int_0^1 \frac{\cos rt}{\sqrt{1-t^2}}dt \tag{47}$$

Now change variables again with $rt = r - \tau$. Then (47) becomes

$$J_o(r) = \frac{2}{\pi}\int_o^r \frac{\cos(r-\tau)}{\sqrt{2r\tau}\sqrt{1-\frac{\tau}{2r}}}d\tau \tag{48}$$

Applying the cosine addition formula we get

$$J_o(r) = \frac{2}{\pi\sqrt{r}}\int_o^r \frac{\cos(\tau)}{\sqrt{2\tau}\sqrt{1-\frac{\tau}{2r}}}d\tau\cos r + \frac{2}{\pi\sqrt{r}}\int_o^r \frac{\sin(\tau)}{\sqrt{2\tau}\sqrt{1-\frac{\tau}{2r}}}d\tau\sin r \tag{49}$$

As $r \to \infty$

$$\int_o^r \frac{\cos(\tau)}{\sqrt{2\tau}\sqrt{1-\frac{\tau}{2r}}}d\tau \to \int_o^\infty \frac{\cos(\tau)}{\sqrt{2\tau}}d\tau$$

with a similar result for the sine integral. Both limits can be computed explicitly, i.e.,

$$\int_o^\infty \frac{\sin(\tau)}{\sqrt{2\tau}}d\tau = \int_o^\infty \frac{\cos(\tau)}{\sqrt{2\tau}}d\tau = \sqrt{\pi}/2$$

Consequently,

$$J_o(r) = \frac{\sqrt{2}}{\sqrt{\pi r}}[\cos(r - \frac{\pi}{4}) + o(r)] \quad \text{as} \quad r \to \infty$$

Thus $J_o(\lambda r)$ behaves like $\cos(\lambda r - \pi/4)$ for large r. Consequently when r is large, J will have zeros in each of the intervals $[\frac{m\pi}{\lambda} + \frac{\pi}{4}, \frac{(m+1)\pi}{\lambda} + \frac{\pi}{4}]$. Arranging the zeros of $J_o(r)$ as

$$0 < \beta_1 < \beta_2 < \beta_3 ...$$

we see that $J_o(r)$ has properties similar to sine and cosine, although it is not periodic.

Since (30) is a second order equation, it has two linearly independent solutions. Of course there are many such pairs. Selecting another solution in addition to J is partly a matter of usefulness. None of these solutions are important for our heat conduction problem however: they are all unbounded at $r = 0$ which violates (31). However, a common second solution, known as Weber's function is given by

$$Y_o(\lambda r) = \frac{2}{\pi} J_o(\lambda r) \ln r + \sum c_k r^{2k}$$

Such solutions become important if we change the geometry of the problem, for example instead of a finite cylinder, imagine a cylinder with a hole bored down the center. Then r cannot be zero.

Finishing the Solution to BV Problem 1. The condition $R(a) = 0$ implies that $\lambda_n = \beta_n / a$. Hence we have

$$R_n(r) = J_o(\lambda_n r), X_n(x_3) = A_n \cosh \lambda_n x_3 + B_n \sinh \lambda_n x_3$$

Recall that $X(l) = 0$. Hence, $A_n \cosh \lambda_n l + B_n \sinh \lambda_n l = 0$. This implies that

$$A_n = -B_n \tanh \lambda_n l$$

and we may define

$$v_n(r, x_3) = B_n J_o(\lambda_n r)(-\tanh \lambda_n l \cosh \lambda_n x_3 + \sinh \lambda_n x_3)$$

<u>Formal Solution</u>. We define

$$v(r, x_3) = \sum_{n=1}^{\infty} v_n(r, x_3), \quad v(r, 0) = f(r) \tag{50}$$

Then the question of convergence for the series (50) is decided in part by Fourier-Bessel convergence theory.

<u>Theorem</u>. Let $f'(r)$ be sectionally continuous on $[0, a]$. Then for each r, $0 < r < a$, the series,

$$\sum_{n=1}^{\infty} A_n J_o(\lambda_n r) = \frac{f(r+) + f(r-)}{2}$$

converges and the coefficients A_n are given by

$$A_n = \frac{2}{a^2 J_o'(\beta_n)} \int_0^a f(r) J_o(\lambda_n r) r dr$$

We could define the notion of a classical solution and consider an existence theorem in much the same was as we have already done. However, we shall instead consider a similar problem.

Wave Motion in Two Dimensions.

Le Ω be a circular membrane (e.g., drumhead) of radius a, occupying the region $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 \leq a^2\}$.

If the membrane or drumhead is under tension, it will vibrate when struck. For a low-mass drumhead, the transverse displacement from equilibrium u satisfies the equation

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 (\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}), \quad x_1^2 + x_2^2 < a^2, \quad t > 0$$

$$(2) \qquad u(x_1, x_2, 0) = u_o(x_1, x_2), \quad x_1^2 + x_2^2 \leq a^2$$

$$(3) \qquad \frac{\partial u(x_1, x_2, 0)}{\partial t} = u_1(x_1, x_2), \quad x_1^2 + x_2^2 \leq a^2$$

$$(4) \qquad u(x_1, x_2, t) = 0, \quad x_1^2 + x_2^2 = a^2, \quad t \geq 0$$

Conditions (2) and (3) specify the initial shape and velocity of the drumhead, while condition (4) says that the drum skin is secured at the edge. As with the vibrating string problem, we make the simplifying assumption that $u_1 \equiv 0$ (see p. 22). To further simplify our computations, let us assume that a change of variable has been done to allow us to substitute 1 for c (see p. 13).

Changing to polar coordinates ($u(x_1, x_2, t) = v(r, \theta, t)$) gives us the problem

$$(5) \qquad \frac{\partial^2 v}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, \quad 0 < r < a, \quad t > 0$$

$$(6) \qquad v(r,\theta,0) = u_o(r\cos\theta, r\sin\theta) = f(r,\theta), \quad \forall \;\; 0 \le r \le a, \theta \in \mathbb{R}$$

$$(7) \qquad \frac{\partial v(r,\theta,0)}{\partial t} = 0, \quad \forall \; 0 \le r \le a, \theta \in \mathbb{R}$$

$$(8) \qquad v(r,\theta,t) = 0, \quad r = a, \quad \forall \; t \ge 0$$

$$(9) \qquad v(r,\theta + 2\pi, t) = v(r,\theta,t), \quad \forall \;\; \theta \in \mathbb{R}, \quad t \ge 0$$

$$(10) \qquad v(r,\theta,t) \;\; \text{finite}, \quad r \to 0, \quad \forall \;\; \theta \in \mathbb{R}, t \ge 0$$

$$(11) \qquad v(a,\theta,0) = 0, \forall \;\; \theta \in \mathbb{R}$$

<u>Separation of Variables</u>.

We perform this in two stages:

<u>Stage 1</u>. Assume $v(r,\theta,t) = w(r,\theta)T(t)$. Substitution into (5) gives

$$T''w = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r})T + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}T$$

or

$$\frac{T''}{T} = \frac{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}}{w} = -\lambda^2$$

Thus

$$T'' + \lambda^2 T = 0, \quad T'(0) = 0 \;\; \text{(by (7)}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda^2 w = 0$$

The last equation will be familiar from the previous steady-temperature problem.

Stage 2. Assume that $w(r,\theta) = R(r)\Theta(\theta)$. Substitution into the last equation gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r})\Theta + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2}R + \lambda^2 R\Theta = 0$$

It follows that

$$\frac{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r})}{R}r^2 + \lambda^2 r^2 = -\frac{\frac{\partial^2\Theta}{\partial\theta^2}}{\Theta} = \mu^2$$

This implies

$$(12) \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + (\lambda^2 - \frac{\mu^2}{r^2})R = 0$$

$$R(a) = 0, \quad R(0) \ \text{ bounded}$$

$$(13) \qquad \Theta'' + \mu^2\Theta = 0,$$

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

Solving (13) leads us to

$$(14) \qquad \mu = \pm n, \quad n = 0, 1, 2, 3, 4, ... \quad \Theta(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

(12) becomes

$$(15) \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + (\lambda^2 - \frac{n^2}{r^2})R = 0 \quad n = 1, 2, 3, ...$$

(12) is known as Bessel's equation of order μ, while (15) is Bessel's equation of integer order. Equation (30), p. 102 is properly Bessel's equation of order 0. Its treatment on pp. 103-106 above suggests a similar course to understand (15). We multiply both sides of (see top of page 112)

$$(16) \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda^2 w = 0$$

by $e^{-in\theta}$ and then integrate:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\int_{-\pi}^{\pi} we^{-in\theta}d\theta) + \frac{1}{r^2}\int_{-\pi}^{\pi}\frac{\partial^2 w}{\partial \theta^2}e^{-in\theta}d\theta + \lambda^2\int_{-\pi}^{\pi} we^{-in\theta}d\theta = 0$$

Integration by parts (twice) on the second integral yields

$$(17) \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\int_{-\pi}^{\pi} we^{-in\theta}d\theta) - \frac{n^2}{r^2}\int_{-\pi}^{\pi} we^{-in\theta}d\theta + \lambda^2\int_{-\pi}^{\pi} we^{-in\theta}d\theta = 0$$

This implies that $R(r) = \int_{-\pi}^{\pi} we^{-in\theta}d\theta$ is a solution to (15). Equation (39), p. 105 gives a possible solution to (16) which leads to

$$(18) \qquad R(r) = \int_{-\pi}^{\pi} e^{ir\lambda \sin\theta} e^{-in\theta} d\theta$$

As before, use of Euler's formula shows that we may simplify (18) to

$$(19) \qquad R(r) = J_n(\lambda r) = \frac{1}{\pi} \int_0^{\pi} \cos(r\lambda \sin\theta - n\theta) d\theta$$

where J_n is the Bessel function of order n (the π appears in front of the integral for standardization purposes as before). We observe that $J_n(0) = 0$ for $n \neq 0$ however.

The following <u>Fourier-Bessel Expansion Theorem</u> is useful.

<u>Theorem</u>. Suppose that $f'(r)$ is sectionally continuous on $[0, a]$. For each $r, 0 < r < a$,

$$\frac{f(r+) + f(r-)}{2} = \sum_{m=1}^{\infty} C_{nm} J_n(\lambda_{nm} r)$$

Moreover, the function $J_n(x)$ has zeros $0 < \beta_{n1} < \beta_{n2} < \beta_{n3} < .... \to \infty$ and $\lambda_{nm} = \beta_{nm}/a$. Further the coefficients C_{nm} are found from

$$(20) \qquad C_{nm} = \frac{2}{a^2 J_{n+1}^2(\lambda_{nm} a)} \int_0^a f(r) J_n(\lambda_{nm} r) r dr$$

By the last equation on page 111, we have $T(t) = A \cos \lambda t$. This suggests the solution of (5) is

$$(21) \quad v(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \underbrace{C_{nm}J_n(\lambda_{nm}r)}_{R}\underbrace{(A_n\cos n\theta + B_n \sin n\theta)}_{\Theta}\underbrace{D_n\cos\lambda_{nm}t}_{T}$$

By construction, (21) is a formal solution to (5). It also satisfies (7), (8), (9), (10), (11) by construction. (6) requires that

$$(22) \quad f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}J_n(\lambda_{nm}r)(A_n\cos n\theta + B_n \sin n\theta)$$

(11) also suggests $f(a,\theta) = 0$. (22) is a double series, a combination of Fourier series and a Fourier-Bessel series. Fortunately, it can be shown that this series and its first and second partial derivatives converge uniformly when f is twice continuously differentiable and has smoothness at the boundary of Ω. Moreover the coefficients $C_{nm}A_n, C_{nm}B_n$ are computed as:

$$C_{nm}A_n = \frac{2}{\pi a^2 J_{n+1}^2(\lambda_{nm}a)}\int_0^a\int_{-\pi}^{\pi} J_n(\lambda_{nm}r)\cos(n\theta)f(r,\theta)rd\theta dr$$

and similarly for $C_{nm}B_n$.

Definition. Let $v \in C^2((0,a)\times[-\pi,\pi]\times(0,\infty)) \cap C^1([0,a]\times[-\pi,\pi]\times[0,\infty))$ and $v(r,\theta+2\pi,t) = v(r,\theta,t), \forall\, 0 \le r \le a, \theta, t \ge 0$. Moreover, suppose v satisfies (5)-(11) for some $f(r,\theta)$. Then v is called a classical solution of the vibrating membrane (drumhead) problem.

Existence Theorem. If $f(r,\theta) \in C^2(\Omega) \cap C^1(\overline{\Omega})$, for all r, $f(r,\theta+2\pi) = f(r,\theta)$ and for all θ, $f(a,\theta) = 0$ then there exists a classical solution v of

(5)-(11) given by (21).

The functions

$$v_{nm}(r,\theta,t) = \underbrace{C_{nm}J_n(\lambda_{nm}r)}_{R}\underbrace{(A_n\cos n\theta + B_n\sin n\theta)}_{\Theta}\underbrace{D_n\cos\lambda_{nm}t}_{T}$$

are called modes of vibration or harmonics for the drumhead. The vibration of a drumhead has been the source of many interesting questions. For example, if the shape is not circular, can the shape be determined by merely listening to it? The answer is yes if the drumhead is convex and has a reasonably smooth edge.[34] Otherwise, the answer is no in general.

[34]Zelditch, "Spectral Determination of Analytic Bi-axisymmetric Plane Domains," *Geometric and Functional Analysis* 10/3 (2000): 628-677.

Chapter 7. The Propagation of Electromagnetic Waves and The Fourier Transform.

Appendix I: Classification of PDEs[35]

PDEs can be classified in different ways. Some of these are (A) geometric form, (B) historical significance, (C) associated persons, (D) "algebraic" form.

As an example of (A) consider an equation of the type

$$\alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial x \partial y} + \gamma \frac{\partial^2 u}{\partial y^2} + \delta \frac{\partial u}{\partial x} + \epsilon \frac{\partial u}{\partial y} = F(x, y) \tag{1}$$

This equation is classified as "2nd order" because the highest order derivative is 2. The values of the coefficients may be used to give a further useful classification. This is based on obvious correspondence to the equation

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y = F \tag{2}$$

Geometrically, this equation describes one of the conic sections: ellipse, hyperbola, parabola, line. Which one depends on the relative value of the coefficients. The equation (1) is classified according as equation (2) is: elliptic, hyperbolic, parabolic. For example,

$$y - x^2 = 0 \tag{3}$$

[35] The perusal of any university library will reveal many monographs and research articles on equations of various types. The survey of types of equations below is by no means exhaustive.

is a parabola. Hence

$$\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

is classified as parabolic. We see that the heat equation is parabolic, while the wave equation (vibrating string) is hyperbolic.

It is possible for an equation to change its geometric classification if its coefficients are variable. A simple example is the Tricomi equation:

$$\frac{\partial^2 u}{\partial x^2} = x \frac{\partial^2 u}{\partial y^2}$$

This equation is elliptic where $x < 0$ and hyperbolic when $x > 0$.

Another important classification is delivered by considering the algebraic form (D) of the equation rather than the geometric form. For example,

$$u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial y^3} = x \tag{5}$$

The first term, $u \partial u / \partial x$, is identified as "nonlinear."[36] The reason for this is that the form does not distribute over sums and constant multiples.

$$4 \frac{\partial^3 u}{\partial z^3} \tag{6}$$

is linear since

[36]Nonlinear equations often present unique challenges and require customized techiques to study their solutions and properties.

$$4\frac{\partial^3(u_1+u_2)}{\partial z^3} = 4\frac{\partial^3 u_1}{\partial z^3} + 4\frac{\partial^3 u_2}{\partial z^3} \tag{7}$$

while

$$(u_1+u_2)\frac{\partial(u_1+u_2)}{\partial x} \neq u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_2}{\partial x} \tag{8}$$

The linear equations are precisely those of the form

$$\sum_{n_1,n_2,\ldots,n_k} \alpha_{n_1 n_2 n_3 \ldots n_k}(x)\frac{\partial^{|\beta|}u}{\partial x^{\beta}} = F(x) \tag{9}$$

where $\beta = \{n_1, n_2, ..., n_k\}$, $x = (x_1, x_2, ..., x_k)$, $\partial x^{\beta} = \partial x_1^{n_1}\partial x_2^{n_2}...\partial x_k^{n_k}$ and $|\beta| = n_1 + n_2 + n_3 + ... + n_k$.

No terms involving products or other nonlinear functions of u and/or its derivatives are linear.

Classification (C) by name of person or thing. The heat and wave equations are examples. Examples of (C) and (D) classification include a nonlinear variant of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin u(x,t) \tag{10}$$

the sine-Gordon equation.

Another nonlinear wave equation is the Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{11}$$

KdV describes shallow water waves, internal waves in density stratified media, some kinds of plasma waves, etc.

Sine-Gordon is a nonlinear hyperbolic equation whose name arose as a parody of the Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = m^2 u(x,t) \tag{12}$$

a well-known linear equation from particle physics.

An interesting parabolic equation whose solutions behave in ways like solutions of the wave equation is the Schrödinger equation:

$$i\frac{\partial u(x,t)}{\partial t} = -\frac{1}{2m}\Delta u + V(x)u \tag{13}$$

The Schrödinger equation describes how the quantum state of a system evolves. The solutions give probability densities which assign the probable "location" of the particle or system.

PDEs are not in any way restricted to scalar-valued functions. For example, Maxwell's equations (in a vacuum) may be written:

$$\frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ -\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 & 0 & 0 & 0 \end{pmatrix} \vec{u} \tag{14}$$

where

$$\vec{u}(x_1, x_2, x_3, t) = \begin{pmatrix} e_1(x_1, x_2, x_3, t) \\ e_2(x_1, x_2, x_3, t) \\ e_3(x_1, x_2, x_3, t) \\ h_1(x_1, x_2, x_3, t) \\ h_2(x_1, x_2, x_3, t) \\ h_3(x_1, x_2, x_3, t) \end{pmatrix} \tag{15}$$

In the anisotropic case, we introduce another 6×6 matrix, $E(x_1, x_2, x_3)$ as

$$E\frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ -\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 & 0 & 0 & 0 \end{pmatrix} \vec{u} \tag{16}$$

or

$$\frac{\partial \vec{u}}{\partial t} = E^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ -\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 & 0 & 0 & 0 \end{pmatrix} \vec{u} \tag{17}$$

The geometric classification can be extended by means of matrix theory to higher dimensional 2nd order equations. Suppose we have an equation with n variables:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + cu = 0 \tag{18}$$

Then (18) is classified as elliptic when the matix (a_{ij}) has all positive eigenvalues, or all negative; hyperbolic when only one eigenvalue is negative and the rest are positive or vice versa; parabolic when one eigenvalue is zero and the others have the same sign. Obviously this does not cover all possible cases - there could be two positive and two negative eigenvalues for example. The geometric classification may be extended to systems of equations like (17).

Appendix II: Bessel Functions and Sturm-Lioville Problems

Sturm-Lioville problems often arise in separation of variables in PDEs.[37] The Bessel equations are examples of *singular* Sturm-Lioville problems. Let B stand for the operator

$$B = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) \tag{19}$$

so that

$$Bu = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) \tag{20}$$

using the product rule,

$$Bu = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \tag{21}$$

Now suppose that

$$\lim_{r\to 0^+} u(r) < \infty, \lim_{r\to 0^+} ru'(r) = 0, \lim_{r\to 0^+} v(r) < \infty, \lim_{r\to 0^+} rv'(r) = 0 \tag{22}$$

Observe that

$$\int_c^a [vBu - uBv]rdr = \int_c^a [v\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - u\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r})]rdr$$

[37] Phillip Hartman, *Ordinary Differential Equations.* Cambrige: Cambridge Univ. Press, 2002.

$$= \int_c^a v \frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})dr - \int_c^a u \frac{\partial}{\partial r}(r\frac{\partial v}{\partial r})dr$$

Integration by parts on both integrals gives

$$-[rv(r)u'(r) - u(r)v'(r)]|_c^a$$

By (22)

$$\lim_{c \to 0^+} \int_c^a [vBu - uBv]rdr = \int_0^a [vBu - uBv]rdr \tag{23}$$

$$= \lim_{c \to 0^+} -[rv(r)u'(r) - ru(r)v'(r)]|_c^a = 0 \tag{24}$$

Thus

$$\int_0^a [vBu - uBv]rdr = 0 \tag{25}$$

Equation (25) is called Green's formula for B.

Now consider the singular Sturm-Lioville problem

$$Bu = \lambda^2 u, \quad u(r) \text{ is bounded as } \ r \to 0^+, \quad u(a) = 0 \tag{26}$$

Suppose u_1, u_2 are two solutions to (26) corresponding to λ_1, λ_2.

Observe that

$$u_2 B u_1 = \lambda_1^2 u_1 u_2, \quad u_1 B u_2 = \lambda_2^2 u_2 u_1 \tag{27}$$

Subtracting these two equations and integrating we have

$$\int_0^a [u_2 B u_1 - u_1 B u_2] r dr = (\lambda_1^2 - \lambda_2^2) \int_o^a u_1 u_2 r dr \tag{28}$$

By (23)-(24), (26),

$$\int_o^a u_1 u_2 r dr = 0 \tag{29}$$

Now observe that (26) is Bessel's equation and $J_o(\lambda r)$ satisfies (26) and (22) (by (19) p. 114). Hence

$$\int_o^a J_o(\lambda_n r) J_o(\lambda_m r) r dr = 0 \tag{30}$$

when $n \neq m$. Equation (30) is called an orthogonality relation. Now suppose that

$$f(r) = \sum_{m=1}^{\infty} A_m J_o(\lambda_m r) \tag{31}$$

Multiply both sides of (31) by $J_o(\lambda_n r) r$ and integrate. This gives

$$\int_0^a f(r) J_o(\lambda_n r) r dr = \int_0^a \sum_{m=1}^{\infty} A_m J_o(\lambda_m r) J_o(\lambda_n r) r dr \tag{32}$$

$$= \sum_{m=1}^{\infty} \int_0^a A_m J_o(\lambda_m r) J_o(\lambda_n r) r dr = \int_0^a A_n J_o(\lambda_n r) J_o(\lambda_n r) r dr \tag{33}$$

This shows that A_n is given by

$$A_n = \frac{\int_0^a f(r) J_o(\lambda_n r) r dr}{\int_0^a J_o^2(\lambda_n r) r dr} \tag{34}$$

Now recall

$$\int_c^a [vBu - uBv]rdr = -[rv(r)u'(r) - ru(r)v'(r)]|_c^a \tag{35}$$

Using (26) with $u = J_o(\mu r), v = J_o(\lambda_m r)$ we have

$$\int_0^a J_o(\mu r)J_o(\lambda_m r)rdr = \frac{a\lambda_m J_o(\mu a)J_o'(a\lambda_m)}{\mu - \lambda_m} \tag{36}$$

Since $J_o(a\lambda_m) = 0$, take the limit of both sides of (36) (l'hôpital's rule can be used on the right hand side) as $\mu \to \lambda_m$

$$\int_0^a J_o^2(\lambda_m r)rdr = \frac{a^2}{2}[J_o'(\lambda_m a)]^2 \tag{37}$$

The Bessel functions $J_n(\lambda r)$ of order n also satisfy orthogonality relations. Indeed, there are numerous identities satisfied by these functions. Setting $B_1 u = Bu + \frac{n^2}{r^2}u$ we see Bessel's equation ($\mu = n$) can be written as

$$B_1 u = \lambda^2 u \tag{38}$$

Now

$$\int_c^a (vB_1 u - uB_1 v)rdr = \int_c^a (vBu - uBv + v\frac{n^2}{r^2}u - u\frac{n^2}{r^2}v)rdr \tag{39}$$

$$= \int_c^a (vBu - uBv)rdr = -[rv(r)u'(r) - ru(r)v'(r)]|_c^a \tag{40}$$

The same argument used above now shows that

$$\int_o^a J_n(\lambda_{nm_2} r) J_n(\lambda_{nm_1} r) r dr = 0 \tag{41}$$

when $m_1 \neq m_2$.

The coefficients in (20), p. 114 are computed following the same method as (32)-(34) above.

Appendix III: Fourier Series

We have seen that Fourier series arise in the solution of a number of boundary value problems involving PDEs. To examine some elementary facts about such series, consider a typical Fourier coefficient, say for a sine series:

$$c_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi x}{l}) f(x)dx \tag{42}$$

The important question is how fast c_n approaches 0 as n gets large. Observe that if $f'(x)$ is sectionally continuous, integration by parts leads to

$$c_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi x}{l}) f(x)dx = -\frac{2}{n\pi}\cos(\frac{n\pi x}{l})f(x)|_0^l + \frac{2}{n\pi}\int_0^l \cos(\frac{n\pi x}{l})f'(x)dx \tag{43}$$

Note that if $f(0) = f(l) = 0$ then the first term on the right side of (43) vanishes. Next, suppose that $f''(x)$ is sectionally continuous. Then we may repeat the integration by parts:

$$\begin{aligned} c_n &= \frac{2}{l}\int_0^l \sin(\frac{n\pi x}{l}) f(x)dx \\ &= -\frac{2l}{n^2\pi^2}\sin(\frac{n\pi x}{l})f'(x)|_0^l + \frac{2l}{n^2\pi^2}\int_0^l \sin(\frac{n\pi x}{l})f''(x)dx \end{aligned} \tag{44}$$

Since $f''(x)$ is sectionally continuous, we can estimate c_n by $|c_n| \leq \frac{K}{n^2}$. The comparison test for infinite series shows that $\sum c_n$ converges absolutely ((6)

on page 8) and that $\sum c_n \sin(\frac{n\pi x}{l})$ also converges uniformly and absolutely. The theorem on page 7 is rather better than this, since it requires nothing of $f''(x)$.

One can conclude from the theorem on page 7 that if $f'(0) = f'(l)$ and $f''(x)$ is sectionally continuous on $[0, l]$ that it is possible to expand $f'(x)$ as a Fourier series as well, etc.

Continuity is not necessary in the consideration of Fourier series. Indeed, the most natural requirement turns out to be a condition like[38]

$$\int_0^l |f(x)|^2 dx < \infty \tag{45}$$

however, in applications, continuity is often physically obvious.

Fourier sine series appeared in our study of heat in a slab. Fourier cosine series have much the same properties, and will arise in the slab problem when Neumann boundary conditions appear. We have also seen general Fourier series appear in our study of heat in a long cylinder ((7) p. 37).

Fourier series arise, just as Bessel series, from the solution of a Sturm-Lioville problem.[39] In particular, a problem such as

$$X''(x) + \lambda^2 X(x) = 0, \quad X(0) = X(l) = 0 \tag{46}$$

[38] The reason for this stems from the orthogonality properties of the sine and cosine functions. One can see almost immediately that (45) implies that $\sum |c_n|^2 < \infty$.

[39] Fourier himself discovered Fourier series when proposing a solution to the problem of heat in a slab: *Therie Analytique de la Chaleur*, 1822.

Moreover, Sturm-Lioville problems always lead to solutions with properties which are Fourier-like. Indeed, such series are often called generalized Fourier series. Thus Bessel series are generalized Fourier series.

The question of whether a given function can be expanded as a Fourier series is a complicated one and has led to very complex mathematics. Functions, even highly discontinuous ones, which satisfy (45) always have convergent Fourier series, at least in a certain general sense:

$$\lim_{n\to\infty}\int_{-l}^{l}|\sum_{k=0}^{n}a_k\cos(\frac{k\pi x}{l})+b_k\sin(\frac{k\pi x}{l})-f(x)|^2dx=0 \tag{47}$$

A very complex result, due to Lennart Carleson shows that functions satisfying (45) may be expanded as a Fourier series in the usual sense, except at certain exceptional points which together form a set of zero length.[40] On the other hand, it has been shown that if one only requires that

$$\int_0^l |f(x)|dx<\infty \tag{48}$$

then the Fourier series for $f(x)$ may not converge at any point at all. Functions satisfying (45) also satisfy (48), but not necessarily the other way around. Hence such extreme examples must satisfy (48) but not (45).

[40] "On convergence and growth of partial sums of Fourier series." *Acta Mathematica* 116 (1966): 135-157. Carleson's proof is so complex that it took considerable time for other mathematicians to verify it. No simplified proof has been found. Condition (45) was generalized to $\int_{-l}^{l}|f(x)|^p dx<\infty$ for any fixed real value $p>1$ by Richard Hunt in 1967.

Orthogonality Relations.

The trigonometric functions satisfy orthogonality relations (see footnote 3 p. 7) which may be established directly or by means similar to those used in appendix II above. Such relations include

$$\int_0^l \sin(\frac{n\pi x}{l})\sin(\frac{k\pi x}{l})dx = 0 \quad \text{when} \quad n \neq k \tag{49}$$

$$\int_{-l}^l \sin(\frac{n\pi x}{l})\sin(\frac{k\pi x}{l})dx = 0 \quad \text{when} \quad n \neq k \tag{50}$$

$$\int_{-l}^l \sin(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l})dx = 0 \quad \text{when} \quad n \neq k \tag{51}$$

$$\int_{-l}^l \cos(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l})dx = 0 \quad \text{when} \quad n \neq k \tag{52}$$

$$\int_0^l \cos(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l})dx = 0 \quad \text{when} \quad n \neq k \tag{53}$$

As an example, let us prove (53). We use the identity

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Thus

$$\cos(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l}) = \frac{1}{2}(\cos(\frac{n\pi x}{l} + \frac{k\pi x}{l}) + \cos(\frac{n\pi x}{l} - \frac{k\pi x}{l}))$$

or

$$\cos(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l}) = \frac{1}{2}(\cos(\frac{n+k}{l}\pi x) + \cos(\frac{n-k}{l}\pi x))$$

The integral on the left of (53) is now easily done:

$$\int_0^l \cos(\frac{n\pi x}{l})\cos(\frac{k\pi x}{l})dx = \frac{l}{2\pi(n+k)}\sin(\frac{n+k}{l}\pi x)|_0^l + \frac{l}{2\pi(n-k)}\sin(\frac{n-k}{l}\pi x)|_0^l \tag{54}$$

which is equal to zero as long as $n \neq \pm k$.

The convergence theorem of chapter 1 was originally proved by Fourier himself. A proof essentially constructed by Dirichlet will now be given.

Theorem - Convergence of Fourier Series. Suppose $f'(x)$ is sectionally continuous on $[-\pi, \pi]$ and that $f(x)$ is 2π-periodic. Then for each x,

$$\frac{f(x+) + f(x-)}{2} = \sum_{k=0}^{\infty} a_k \cos(\frac{k\pi x}{l}) + b_k \sin(\frac{k\pi x}{l}) \tag{55}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx f(x)dx, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx f(x)dx, \quad n = 1, 2, ... \tag{56}$$

and

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{57}$$

Proof. Step 1. Let

$$S_N(x) = \sum_{n=0}^{N} a_n \cos nx + b_n \sin nx \tag{58}$$

(a partial sum of the Fourier series). We shall sum this expression using Dirichlet's technique. We insert the equations from (56)-(57) into (58) to get

$$S_N(x) = \sum_{n=0}^{N} a_n \cos nx + b_n \sin nx \tag{59}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \sum_{n=1}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ny f(y)dy \cos nx + \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ny f(y)dy \sin nx \tag{60}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)[\frac{1}{2} + \sum_{n=1}^{N} \cos ny \cos nx + \sin ny \sin nx]dy \tag{61}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)[\frac{1}{2} + \sum_{n=1}^{N} \cos n(y-x)]dy \tag{62}$$

We shall sum the expression

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^{N} \cos nz \tag{63}$$

Multiply both sides of (63) by $2\sin(z/2)$

$$2\sin(z/2)D_N(z) = \sin\frac{z}{2} + \sum_{n=1}^{N} 2\cos nz \sin\frac{z}{2} \tag{64}$$

$$= \sin\frac{z}{2} + \sum_{n=1}^{N} \sin((n+1/2)z) - \sin((n-1/2)z) \tag{65}$$

The sum telescopes to a single term. Thus

$$D_N(z) = \frac{\sin((N+1/2)z)}{2\sin(z/2)} \tag{66}$$

Thus

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)D_N(y-x)dy \tag{67}$$

<u>Step 2.</u> Define

$$g(y) = \frac{f(x+y) - f(x+)}{2\sin(y/2)}, \quad 0 < y \leq \pi \tag{68}$$

Then $g(y)$ is sectionally continuous (note that $g(0) = f'(x+)$). We shall show that

$$\int_0^{\pi} g(y)\sin((N+1/2)y)dy \to 0, \quad \text{as} \quad N \to \infty \tag{69}$$

First, if $g(x)$ is any continuous function on $[-\pi, \pi]$, let $[a, b]$ be any subinterval of $[-\pi, \pi]$. Then

$$\int_a^b \sin nx dx = \frac{1}{n}(\cos nb - \cos na) \to 0, \quad \text{as} \quad n \to \infty \tag{70}$$

Now, let

$$g_k(x) = \sum_{j=1}^{k} g(x_j)\chi_{[x_j, x_{j+1})}(x) \tag{71}$$

where $\{x_j\}$ is a regular partition of $[-\pi, \pi]$. Then it easy to see that $g_k(x) \to$

$g(x)$ uniformly. Moreover (70) shows that

$$\int_{-\pi}^{\pi} g_k(x)\sin(nx)dx \to 0, \quad \text{as} \quad n \to \infty \tag{72}$$

Now observe

$$\int_{-\pi}^{\pi} g(x)\sin(nx)dx \tag{73}$$

$$= \int_{-\pi}^{\pi} g_k(x)\sin(nx)dx - \{\int_{-\pi}^{\pi} g_k(x)\sin(nx)dx - \int_{-\pi}^{\pi} g(x)\sin(nx)dx\} \tag{74}$$

$$= \int_{-\pi}^{\pi} g_k(x)\sin(nx)dx - \int_{-\pi}^{\pi} (g_k(x) - g(x))\sin(nx)dx \tag{75}$$

Therefore

$$|\int_{-\pi}^{\pi} g(x)\sin(nx)dx| \leq |\int_{-\pi}^{\pi} g_k(x)\sin(nx)dx| + \int_{-\pi}^{\pi} |g_k(x) - g(x)||\sin(nx)|dx \tag{76}$$

$$\leq |\int_{-\pi}^{\pi} g_k(x)\sin(nx)dx| + \int_{-\pi}^{\pi} |g_k(x) - g(x)|dx \tag{77}$$

Given $\epsilon > 0$, k sufficiently large

$$\int_{-\pi}^{\pi} |g_k(x) - g(x)|dx < \epsilon/2 \tag{78}$$

Now let n be large enough so that

$$|\int_{-\pi}^{\pi} g_k(x)\sin(nx)dx| < \epsilon/2 \tag{79}$$

Then for all such n,

$$|\int_{-\pi}^{\pi} g(x)\sin(nx)dx| < \epsilon \tag{80}$$

Since ϵ was arbitrary, this gives (69).[41]

Step 3. A direct computation based on (63) shows that

$$\int_{-\pi}^{\pi} D_N(z)dz = 1 \tag{81}$$

moreover, since (63) is an even function

$$\frac{1}{2} = \int_{0}^{\pi} D_N(z)dz = \int_{-\pi}^{0} D_N(z)dz \tag{82}$$

Step 4.

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)D_N(y-x)dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)D_N(y-x)dy \tag{83}$$

$$= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(t+x)D_N(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)D_N(t)dt \tag{84}$$

Step 5.

$$S_N(x) - \frac{f(x+)+f(x-)}{2} = \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt - \frac{1}{2}f(x-)$$

[41] Actually g being sectionally continuous, requires a slight modification of the proof, breaking up the integral over the intervals of continuity. The conclusion is the same. Moreover, the same conclusion holds for the cosine. The result of step 2 is a version of the "Riemann-Lebesgue Lemma" which holds for much less well-behaved functions g.

$$+\frac{1}{\pi}\int_0^{\pi} f(x+t)D_N(t)dt - \frac{1}{2}f(x+)$$

Now

$$\frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt - \frac{1}{2}f(x-) = \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt - \frac{1}{\pi}\int_{-\pi}^{0} D_N(t)dt f(x-)$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt - \frac{1}{\pi}\int_{-\pi}^{0} f(x-)D_N(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (f(x+t) - f(x-))D_N(t)dt$$

By step 2, the last integral goes to zero as $N \to \infty$. A similar statement holds for

$$\frac{1}{\pi}\int_0^{\pi} f(x+t)D_N(t)dt - \frac{1}{2}f(x+)$$

QED.

Corollary. If $f(x)$ is an odd function, then only the sine terms appear in the expansion. Alternatively, if $f(x)$ is even, then only the cosine terms appear in its Fourier series expansion.

Proof. This a simple consequence of (56), (57) above. In the odd case for example, a_n will clearly be zero, since $\cos nx f(x)$ will be odd, therefore its integral is even. Hence integrating over the symmetric interval must give zero.

Corollary. If $f(x)$ is only defined on $[0, \pi]$, it may be extended as either an even or odd function to $[-\pi, \pi]$. Hence, it can be given a sine or cosine expansion. Moreover in this case, the integrals in (56), (57) can be written as integrals over $[0, \pi]$.

The Fourier series itself is defined on all $(-\infty, \infty)$ and therefore gives a periodic extension of $f(x)$ to all of $\mathbb{R}$.

Corollary. The expansion theorem can be extended to functions satisfying the hypotheses of the theorem on any interval $[-l, l]$. The use of $[-\pi, \pi]$ was merely a convenience to reduce the size of expressions. Exactly the same proof works for any symmetric interval. See page 7 for a version of this more general case (for sine series).

A useful fact which we have (sometimes silently) employed throughout the text is the following.

Theorem.[42] Suppose $u_n(x)$ is a sequence of continuous functions, each defined on the interval $[a, b]$. Suppose further that the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly to a function $u(x)$. Then $u(x)$ is continuous on $[a, b]$.

[42] This theorem may seem obvious, but even the great Cauchy had trouble understanding it.

Moreover, if the functions u_n are differentiable and the series

$$\sum_{n=1}^{\infty} \frac{du_n}{dx}$$

converges uniformly on $[a, b]$ then indeed $\frac{du}{dx}$ exists and

$$\sum_{n=1}^{\infty} \frac{du_n}{dx} = \frac{du}{dx}.$$

In order to understand the proof, we need a clear idea of uniform convergence. We give the following adequate definition:

Definition of Uniform Convergence of a Sequence of Functions. Let $g_n(x)$ be a sequence of functions defined on a set Ω. We say the sequence $\{g_n(x)\}$ converges uniformly to a function $g(x)$ on Ω if for any $\epsilon > 0$ $\exists N$ such that if $n \geq N$ then $|g_n(x) - g(x)| < \epsilon$ for all $x \in \Omega$. Since a series can be thought of in terms of the sequence of its partial sums, the definition also applies to series, i.e. suppose we have a series $\sum u_n(x)$. To say it converges uniformly to $u(x)$ means that for any $\epsilon > 0$ there is a positive integer N such that if $k \geq N$ then

$$|\sum_{n=1}^{k} u_n(x) - u(x)| < \epsilon$$

for all $x \in \Omega$.

Now we construct a proof of the theorem on uniform convergence. First we dispose of the continuity of limits portion. So suppose each $u_n(x)$ is continuous on $[a,b]$ and that $\sum u_n(x)$ converges uniformly to a function $u(x)$ on $[a,b]$. We want to show that $u(x)$ is continuous on $[a,b]$. To this end, suppose $\epsilon > 0$. Let N be large enough so that

$$|\sum_{n=1}^{k} u_n(x) - u(x)| < \epsilon$$

for all $x \in [a,b]$ whenever $k \geq N$. Now each $u_n(x)$ being continuous means that for any $x_o \in [a,b]$ there exists $\delta_n > 0$ so that if $|x - x_o| < \delta_n$ then $|u_n(x) - u_n(x_o)| < \epsilon/k$. So choose x so that $|x - x_o| < \min \delta_n$ Thus

$$|u(x) - u(x_o)| = |u(x) - \sum_{n=1}^{k} u_n(x) + \sum_{n=1}^{k} u_n(x) + \sum_{n=1}^{k} u_n(x_o) - \sum_{n=1}^{k} u_n(x_o) - u(x_o)|$$

$$\leq |u(x) - \sum_{n=1}^{k} u_n(x)| + |u(x_o) - \sum_{n=1}^{k} u_n(x_o)| + |\sum_{n=1}^{k} u_n(x) - \sum_{n=1}^{k} u_n(x_o)|$$

$$< 2\epsilon + \sum_{n=1}^{k} |u_n(x) - u_n(x_o)| < 2\epsilon + \sum_{n=1}^{k} \epsilon/k = 3\epsilon$$

Hence if $|x - x_o| < \min \delta_n \equiv \delta$ then $|u(x) - u(x_o)| < 3\epsilon$. This shows that u is continuous on $[a,b]$.

For the second portion of the proof concerning differentiability it is somewhat simpler to prove the following lemma:

LEMMA. Suppose $f_n(x)$ is a sequence of differentiable functions converging uniformly to $f(x)$ and suppose also that there exists a function $g(x)$ such that $f_n'(x)$ converges uniformly to $g(x)$. Then $f'(x)$ exists and is equal to $g(x)$.

Proof of Lemma. Observe that the mean value theorem implies that for any $x, y \in [a, b]$ there a $z \in [x, y]$ (note that z depends on both n and m) such that,

$$(f_m(x) - f_n(x)) - (f_m(y) - f_n(y)) = (x - y)(f_m'(z) - f_n'(z))$$

If follows that

$$|\frac{f_m(x) - f_m(y)}{x - y} - \frac{f_n(x) - f_n(y)}{x - y}| \leq |f_m'(z) - f_n'(z)|$$

Observe also that

$$|f_m'(z) - f_n'(z)| = |f_m'(z) - g(z) + g(z) - f_n'(z)| \leq |f_m'(z) - g(z)| + |g(z) - f_n'(z)| \leq \epsilon$$

if n, m are sufficiently large, independently of $z \in [a, b]$ by uniform convergence of $f_n'(x)$.

Thus, for any $x, y \in [a, b]$ and n, m sufficiently large

$$|\frac{f_m(x) - f_m(y))}{x - y} - \frac{f_n(x) - f_n(y)}{x - y}| < \epsilon$$

Now let $m \to \infty$ to get

$$|\frac{f(x)-f(y))}{x-y} - \frac{f_n(x)-f_n(y)}{x-y}| < \epsilon$$

Observe that we have

$$|\frac{f(x)-f(y))}{x-y} - g(y)| = |\frac{f(x)-f(y))}{x-y} - g(y) + \frac{f_n(x)-f_n(y)}{x-y} - \frac{f_n(x)-f_n(y)}{x-y}|$$

$$\leq |\frac{f(x)-f(y))}{x-y} - \frac{f_n(x)-f_n(y)}{x-y}| + |g(y) - \frac{f_n(x)-f_n(y)}{x-y}|$$

$$= |\frac{f(x)-f(y))}{x-y} - \frac{f_n(x)-f_n(y)}{x-y}| + |g(y) - f_n'(y) + f_n'(y) - \frac{f_n(x)-f_n(y)}{x-y}|$$

$$\leq |\frac{f(x)-f(y))}{x-y} - \frac{f_n(x)-f_n(y)}{x-y}| + |g(y) - f_n'(y)| + |f_n'(y) - \frac{f_n(x)-f_n(y)}{x-y}|$$

The first term on the right is $< \epsilon$ no matter what n or x or y happen to be as long as n is large. For the second term, if n is large, we can force it to be less that ϵ by uniform convergence. Once such a large n is selected for terms 1 and 2, we can choose $|x-y|$ small enough so that the third term is $< \epsilon$. Thus

$$|\frac{f(x)-f(y))}{x-y} - g(y)| < 3\epsilon$$

for $|x-y|$ close to zero. This shows that not only does $f'(y)$ exist, but it must be equal to $g(y)$, which is the conclusion of the lemma.

Now to apply the lemma, we take $f(x) = u(x)$, $g(x) = \sum u_n'(x)$, $f_n(x) = \sum_{k=1}^{n} u_k(x)$. QED.

An often useful theorem attributed to Weierstrass is the "Weierstrass M-Test." The name arises from the statement of the theorem:

<u>Theorem</u> *Weierstrass M-Test.* Suppose $\sum u_n(x)$ is an infinite series of functions defined on $[a, b]$. Suppose that there is a sequence of constants $\{M_n\}$ with the property $|u_n(x)| \leq M_n$ for each $n = 1, 2, 3, \ldots$. Suppose also that $\sum M_n$ converges (as a series of non-negative constants). Then $\sum u_n(x)$ converges uniformly.

<u>Proof.</u> By the hypotheses, $\sum u_n(x)$ is absolutely convergent and therefore converges. Let

$$U_N(x) = \sum_{n=1}^{N} u_n(x)$$

and let

$$g(x) = \sum_{n=1}^{\infty} u_n(x).$$

The theorem is proved if we can show $U_N(x)$ converges uniformly to $g(x)$ as $N \to \infty$. Let $\epsilon > 0$. Choose K large enough so that

$$\sum_{n=K+1}^{\infty} M_n < \epsilon$$

It follows that

$$\sum_{n=K+1}^{\infty} |u_n(x)| < \epsilon.$$

Now we have

$$|g(x) - U_K(x)| = |\sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{K} u_n(x)| = |\sum_{n=K+1}^{\infty} u_n(x)| \leq \sum_{n=K+1}^{\infty} |u_n(x)| < \epsilon.$$

It follows that $|g(x) - U_N(x)| < \epsilon$ for any $N \geq K$. QED.

Remark 1. The test may be used with functions of more than one variable. The proof is the same.

Remark 2. The theorem on uniform convergence is stated for functions of one variable, but the following more general theorem has essentially the same proof:

General Theorem on Uniform Convergence. Suppose that $u_n(x)$ is a sequence of functions defined on an open subset Ω of $\mathbb{R}^k$. Suppose also that $\sum u_n(x)$ converges uniformly on Ω, and that for some one of the variables of x, say x_i,

$$\sum_{n=1}^{\infty} \frac{\partial u}{\partial x_i}(x)$$

converges uniformly. Then, if we define $u(x) = \sum u_n(x)$, it follows that

$$\frac{\partial u}{\partial x_i} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial x_i}$$

for all $x \in \Omega$.

Proof. The proof consists of reducing this general case to the one variable proof. First, by hypothesis, $\Omega \subseteq \mathbb{R}^k$. Hence $x \in \Omega \implies x = (x_1, x_2, x_3, ...x_k)$. Fix $x \in \Omega$ and consider a small box, $B_\delta = \{y \in \Omega : y_j = -\delta \leq x_j \leq \delta\} \subseteq \Omega$ containing x. In the proof of the one variable case, replace $[a, b]$ by $[-\delta, \delta]$ and x by x_i.

Appendix IV - Limits

The purpose here is to give some brief definitions of useful limit ideas. The most important are

$$\limsup \quad \text{and} \quad \liminf$$

To define these, we need to know what *sup* and *inf* are. Sup stands for *supremum*, inf stands for *infimum*. These in turn are synonyms for *least upper bound* and *greatest lower bound* respectively. It is a fact that any set of real numbers has both a supremum and infimum. For example,

$$\sup\{x|x < 4\} = 4$$

and

$$\inf\{\frac{1}{n}|n = 1, 2, 3, 4, \ldots\} = 0$$

and

$$\sup\{x|x > 0\} = \infty$$

This brings us to the notions of lim sup and lim inf.
Let $N(a, \epsilon) = \{y|0 < |y - a| < \epsilon\}$ then we define

$$\limsup_{x \to a} f(x) = \lim_{\epsilon \to 0} \sup\{f(x)|x \in N(a, \epsilon)\}$$

Observe that when $\epsilon_1 > \epsilon_2$, $\{f(x)|x \in N(a,\epsilon_1)\} \supseteq \{f(x)|x \in N(a,\epsilon_2)\}$. Thus $\sup\{f(x)|x \in N(a,\epsilon_1)\} \geq \sup\{f(x)|x \in N(a,\epsilon_2)\}$. It follows that

$$\limsup_{x\to a} f(x)$$

always exists (∞ or $-\infty$ are allowed possibilities) in contrast to ordinary limits.

Similarly for lim inf,

$$\liminf_{x\to a} f(x) = \lim_{\epsilon\to 0} \inf\{f(x)|x \in N(a,\epsilon)\}$$

whereas

$$\inf\{f(x)|x \in N(a,\epsilon_1)\} \leq \inf\{f(x)|x \in N(a,\epsilon_2)\}$$

whenever $\epsilon_1 > \epsilon_2$.

When more than one variable is involved, the definitions remain essentially the same.

Let $N((a,b),\epsilon) = \{(c,d)||(c,d)-(a,b)| < \epsilon\}$. Then we define

$$\limsup_{(x,t)\to(a,b)} f(x,t) = \lim_{\epsilon\to 0} \sup\{f(x,t)||(x,t) \in N((a,b),\epsilon)\}$$

In one variable, we can define one-sided limsup and liminf:

$$\limsup_{x\to a^-} f(x) = \lim_{y\to a} \sup\{f(x)|y \le x < a\}$$

Observe that

$$\sup\{f(x)|y_1 \le x < a\} \ge \sup\{f(x)|y_2 \le x < a\}$$

whenever $y_1 \le y_2$ because

$$\{f(x)|y_1 \le x < a\} \supseteq \{f(x)|y_2 \le x < a\}$$

Hence as $y \to a$, $\sup\{f(x)|y \le x < a\}$ is decreasing as y increases toward a. So its limit exists, in fact

$$\lim_{y\to a} \sup\{f(x)|y \le x < a\} = \inf\{\sup\{f(x)|y \le x < a\}|y < a\}$$

Similarly for liminf,

$$\liminf_{x\to a^-} f(x) = \lim_{y\to a} \inf\{f(x)|y \le x < a\}$$

and

$$\lim_{y\to a^-} \inf\{f(x)|y \le x < a\} = \sup\{\inf\{f(x)|y \le x < a\}|y < a\}$$

with

$$\liminf_{x \to a^+} f(x) = \lim_{y \to a} \inf\{f(x) | a < x \leq y\}$$

and similarly for

$$\limsup_{x \to a^+} f(x)$$

<u>Theorem</u>

$$\lim_{x \to a} f(x)$$

exists if and only if

$$\limsup_{x \to a} f(x) = \liminf_{x \to a} f(x)$$

The same statement is true for one-sided limits as well as for functions of more than one variable.

Example.

Let

$$f(x) = \sin(\frac{1}{x}), x \neq 0$$

It is clear that

$$\lim_{x \to 0} f(x) \quad \text{does not exist.}$$

However,

$$\limsup_{x \to 0} f(x) = 1, \quad \liminf_{x \to 0} f(x) = -1$$

lim sup and lim inf give information about a function's behavior at a point and can be used to control or estimate such behavior as in Theorem 2 on page 91.

When more than one variable is involved, we don't have a concept of "one-sided" limit exactly, but sometimes it is useful to restrict the idea of lim sup, lim inf in a somewhat similar way. For example,

$$\limsup_{x \to x_o, t \to 0^+} f(x,t)$$

would mean that we choose

$N((x_o,0),\epsilon) = \{(x,t) \mid 0 < |(x,t) - (x_o,0)| < \epsilon, t > 0\}$ and

$$\limsup_{x \to x_o, t \to 0^+} f(x,t) = \lim_{\epsilon \to 0} \sup\{f(x,t) \mid x \in N((x_o,0),\epsilon)\}$$

Promised extended argument from page 9 above:

To extend the argument a bit on uniform convergence of

$$\sum_{n=1}^{\infty} -n^2 (\frac{\pi}{l})^2 c_n \sin\frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}$$

which is the (term by term) derivative of $u(x,t)$, with respect to t, suggests that we consider the following estimate:

$$|-c_n(\frac{\pi}{l})^2 n^2 \sin\frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}| \leq |c_n| (\frac{\pi}{l})^2 n^2 \sin\frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}$$

$$\leq n^2 e^{-k(\frac{n\pi}{l})^2 t}$$

provided n is large enough so that $|c_n|(\frac{\pi}{l})^2 \leq 1$, say for all $n \geq m$ for some m. Now fix $t_o > 0$ and $\delta > 0$. For $t > t_o + \delta$ we claim that for large enough n, that

$$n^2 e^{-k(\frac{n\pi}{l})^2 t} \leq e^{-k(\frac{n\pi}{l})^2 t_o}$$

Let $k(\frac{\pi}{l})^2 = c$. Restating, we have

$$n^2 e^{-cn^2 t} \leq e^{-cn^2 t_o}$$

This is equivalent to

$$n^2 e^{-cn^2(t-t_o)} \leq 1$$

Now note that

$$n^2 e^{-cn^2(t-t_o)} \leq n^2 e^{-cn^2\delta}$$

Observe that l'Hopital's rule implies that

$$\frac{n^2}{e^{cn^2\delta}} \to 0$$

as $n \to \infty$. Thus for sufficiently large m, we have that if $n \geq m$ then

$$\frac{n^2}{e^{cn^2\delta}} \leq 1$$

or

$$n^2 e^{-cn^2(t-t_o)} \leq 1$$

Taking $M_n = e^{-cn^2 t_o}$ gives us the correct hypothesis for the Weierstrass M-test provided we can show that

$$\sum_{n=m}^{\infty} M_n < \infty$$

However l'Hopital's rule again shows that

$$\frac{n^2}{e^{cn^2 t_o}} \to 0$$

as $n \to 0$ and therefore that

$$\frac{1}{n^2} \geq e^{-cn^2 t_o} = M_n$$

for all large n and therefore

$$\sum_{n=1}^{\infty} -n^2 (\frac{\pi}{l})^2 c_n \sin \frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}$$

converges uniformly. Actually we only established the estimate

$$|n^2 (\frac{\pi}{l})^2 c_n \sin \frac{n\pi x}{l} e^{-k(\frac{n\pi}{l})^2 t}| \leq M_n$$

for $n \geq m$ but this is sufficient. Now since t_o and δ were arbitrary, we have term by term differentiability for all t. Also, since the second partial derivative with respect to x of $u(x,t)$ delivers a similar series, we have that both sides of the heat equation are defined for $u(x,t)$ (provided that f satisfies the conditions of the Fourier convergence theorem).