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Average Stability and Decay Properties of Forced Solutions of the Wave Propagation Problems of Classical Physics in Energy and Mean Norms*

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This paper seeks conditions under which the solutions of the wave equations of classical physics with time-dependent coefficients and source terms have solutions which are stable in an average sense. Stability is sought in what in practice are the two senses of "energy" and "mean." The equations for energy are taken as generalizations of the symmetric hyperbolic systems and for the mean sense as generalizations of the d'Alembert equation. It is seen, for example, that solutions have somewhat more stability in the mean for the most general type of system. A fact which makes our approach somewhat unique is that no space or time smoothness of coefficients is required for most of our results and the results are easily extended to domains with boundary with no particular conditions on the shape or boundedness of the domain.

0. INTRODUCTION

One of the long-standing problems of physics is to determine the properties of waves propagating in a medium under the action of prescribed sources. The stability of solutions to the wave propagation problems of classical physics has been well studied in the case where coefficients are time independent. Various special cases of time dependence have been studied as well. In this paper, we will consider the average time stability of solutions to these wave propagation problems, when the source terms decay at a certain rate, in various function spaces both in the sense of the energy of the solutions, and in the somewhat weaker potential sense or what will be roughly the "L_2" sense. In the energy form, Wilcox [26] has

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shown that the wave propagation problems of classical physics may be studied as systems of the sort

$$\frac{\partial u}{\partial t} = E(x, t)^{-1} \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + B(x, t)\, u + \lambda u + f(x, t). \tag{0.1}$$

Here, $f(x, t)$ describes the sources present in the medium and the matrix $B(x, t)$ is a dispersion term. The matrix $E(x, t)$ describes certain properties of the medium peculiar to the physics involved. λ is a parameter which may be real or complex. We shall refer to λ as the frequency of the persistently excited states or simply as the frequency (see Section 1).

In what we have referred to as the L_2 formulation of these problems we take the equation

$$\frac{\partial^2 v}{\partial t^2} = c(x, t)^2 \Delta v + q(x, t)\, v - \lambda v + g(x, t), \tag{0.2}$$

Here it is convenient to study domains which are (possibly) different than all of $\mathbb{R}^n$ and we will study (0.2) in this way (in fact it would be an easy matter to do the same for (0.1) and we shall point this out at the appropriate place below). We will show that both formulations yield similar results in terms of the stability of solutions but that they are not entirely the same. One problem of great interest to us is the case where the coefficients may be discontinuous in time (and space). Our existence results are directed toward this case but others are considered as well. Since we shall consider source terms in all cases, we shall universally assume zero initial conditions.

As a beginning, we will consider the well-known cases where constant coefficients are involved in order to motivate some of the stability and asymptotic problems we consider in the more abstract settings later. This will be handled in Section 1.

In Section 2, we will develop some of the preliminary material necessary for the study of (0.2).

In Section 3 we shall derive an abstract formulation which is applicable to both (0.1) and (0.2). This formalism was also discovered in [2], although our derivation was somewhat different and the use is quite different from that found in [2]. We stumbled on the main idea by considering (0.1) as though t were a "space variable." In fact this is a theme that runs through the results here. This of course is not a new notion and was used by Friedrichs in his classical papers on symmetric systems (see [5], for example—the equations above are not necessarily symmetric, however). Howland has used the same idea for the time-dependent Schrödinger operator [9]. In Section 3 we will use the abstract results to study both equations in the case where the coefficients depend only on the space coordinates.

In Section 4 we consider the case of (0.1), where $B(x, t)$ does depend on t. Two categories of conditions are studied, these may be considered as topological conditions and algebraic conditions (useful since B is a matrix). We employ the formalism of [9] in part here.

In Section 5 we consider (0.2) when $\mathbf{q}$ depends on time.

In Section 6 we allow E and $\mathbf{c}$ to depend on time as well. Here the lack of smoothness for the coefficients requires us to consider the "band limited" solutions to the equations (see [19] for another application). These (stable) solutions always exist whereas the complete solution may fail to be (globally) stable in any of the senses we describe.

An appendix briefly discusses some of the problems of steady-state solutions in the "infrared" range for (0.1). This is related closely to the results of Section 4 and those of [18].

In nearly every case, the parameter λ plays some key role and the asymptotic properties of solutions are studied relative to λ. The scattering theory formalism (see [8], for example) of quantum mechanics is employed in Section 5 to study certain stability and asymptotic properties in time.

In [21, 22] we have remarked on the relationship of (0.1) and (0.2). The correspondence between (0.1) and (0.2) is not direct when B and/or $\mathbf{q}$ is not zero. In any case, In any case, (0.2) is only a special case of the class of problems which may be written in the form of (0.1) and many other particular equations may be cast in a form somewhat similar to that of (0.2) which have counterparts in the energy form of (0.1). These can be analysed with equal success. However, (0.2) is such a commonly occurring form that it was thought that a specific parallel treatment would be worthwhile. The "potentials" B and $\mathbf{q}$ do not in general transform to each other in local form. This makes the study of both problems in terms of the properties of B and $\mathbf{q}$ of interest. We refer the reader also to [6] for further analysis of the relation of general forms of (0.2) to (0.1).

We remark for the initiated that we make no requirements on the symbol of $\sum A_j(\partial/\partial x_j)$ other than symmetry. This allows examples as in [10].

1. The Constant Coefficient Case

Here the two equations take the form

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + \lambda u + f(x, t) \tag{1.1}$$

$$\frac{\partial^2 v}{\partial t^2} = \mathbf{C}^2 \, \Delta v - \lambda v + g(x, t). \tag{1.2}$$

The matrices A_j are always assumed to be symmetric with real entries of dimension $N \times N$, where u is a vector valued function of the n variables x and the "time" t. $f(x, t)$ is some N-vector valued function. v is a scalar function of x and t and $\mathbf{C}$ is a positive constant. λ is some complex number in general but we shall be particularly interested in either real or imaginary values. (The λ imaginary case of (0.1) is in fact (see [19]) the counterpart of the $\lambda > 0$ case for (0.2).)

The principal (spatial) symbol of (1.1) is

$$\sum_{j=1}^{n} A_j p_j \qquad (p = (p_1, ..., p_n) \in \mathbb{R}^n \setminus \{0\}). \tag{1.3}$$

We always write $A(p)$ for this matrix. For the wave propagation problems of physics, the determinant of $A(p)$ is always zero with the dimension of the kernel of $A(p)$ depending of $p/|p|$ in the most general case. If the dimension of the kernel of $A(p)$ is constant, the medium is called strongly propagative, a term coined in Schulenberger and Wilcox [16].

For a treatment of the case where $\det(A(p)) \neq 0$ see Rauch [11].

A few examples of the work which has been done on these problems are [7–30].

The solutions we study will not be of classical type in general. However, in any physical application, a quantity may be and frequently is replaced by its average over a small space–time region. Such an average is well defined for solutions we describe. We shall first consider solutions of (1.1) for λ an imaginary parameter (we shall consider real λ later).

We require the following facts. The Fourier transform of a functon $f(x)$ which is smooth and rapidly decreasing ($f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$) is

$$\Phi f(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \circ p} f(x)\, dx = \hat{f}(p). \tag{1.4}$$

Φ is an isomorphism on $\mathscr{L}$ which extends to an isometry from $L_2(\mathbb{R}^n, \mathbb{C}^N)$ onto itself (the Plancherel theorem). Here $L_2(\mathbb{R}^n, \mathbb{C}^N)$ is the collection of measurable square integrable functions defined on $\mathbb{R}^n$ with values in complex N-space. The adjoint of Φ is determined by $\Phi^* f(p) = \hat{f}(-p)$.

The weighted Sobolev spaces $\mathscr{H}_\alpha^\beta$ are defined by the norm

$$(\| f \|_\alpha^\beta)^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^\alpha |\Phi_x((1 + |y|^2)^{\beta/2} f(y)|^2\, dx \tag{1.5}$$

with $\Phi(\mathscr{H}_\alpha^\beta) = \mathscr{H}_\beta^\alpha$ (see [13]). When one (or both) of α, β is zero we omit it (them). We also note that for $f(x) \in L_1(\mathbb{R}^n, \mathbb{C}^N)$ we have that $f(\hat{p})$ is continuous, and

$$\lim_{|p| \to \infty} \hat{f}(p) = 0. \tag{1.6}$$

In this section and the following sections it will be convenient to consider partially weighted spaces. Write $\mathscr{H}$ for $\mathscr{H}^0_0$.

$L_{2,\alpha}(\mathbb{R}, \mathscr{H})$ is defined by the norm

$$\| f \|_\alpha^2 = \int_{\mathbb{R}} (1 + |t|^2)^\alpha \int_{\mathbb{R}^n} |f(x, t)|^2 \, dx \, dt. \tag{1.7}$$

We note that for $\alpha > \frac{1}{2}$ the Schwarz inequality shows that

$$L_{2,\alpha}(\mathbb{R}, \mathscr{H}) \subseteq L_1(\mathbb{R}, \mathscr{H}) \tag{1.8}$$

and that by the dominated convergence theorem, the Plancherel theorem, and (1.4) for $f \in L_{2,\alpha}(\mathbb{R}, \mathscr{H})$

$$\lim_{|p| \to \infty} \| \hat{f}(\cdot, p) \|_{\mathscr{H}}^2 = \lim_{|p| \to \infty} \int_{\mathbb{R}^n} |\hat{f}(y, p)|^2 \, dy = 0. \tag{1.9}$$

The source terms we consider will be assumed to decrease in energy (in t) sufficiently fast to be members of $L_{2,\alpha}(\mathbb{R}, \mathscr{H})$ for some $\alpha > \frac{1}{2}$. This seems to be a natural class in light of further developments.

It is necessary to establish a notation for certain mathematical objects associated with $A(p)$.

By $\lambda_j(p)$ we denote the eigenvalues of $A(p)$. If these are ordered as

$$\lambda_1(p) \geqslant \lambda_2(p) \geqslant \cdots \geqslant \lambda_N(p) \tag{1.10}$$

it is known that they are continuous in p and homogeneous of order 1 (see [26]). It is also known that if a given λ_j is not identically zero, then it can vanish only on a set of measure zero (see [27]).

We write $\hat{P}_j(p)$ for the $\mathbb{C}^N$ selfadjoint orthoprojectors onto the eigenspaces associated with the λ_j. $\Phi^* \hat{P}_j \Phi = P_j$ is thus a selfadjoint projection on $\mathscr{H}$. It was shown in [27] that the $\hat{P}_j$ can be chosen as measurable functions of p and that they are homogeneous of order zero, and bounded.

The *R local energy* of a homogeneous state $u(x, t)$ is defined as

$$\| u(\cdot, t) \|_R^2 = \int_{|x| < R} |u(x, t)|^2 \, dx. \tag{1.11}$$

The *eventual* local energies of a homogeneous state are defined by (assuming the limit exists)

$$\lim_{t \to \pm\infty} \| u(\cdot, t) \|_R^2. \tag{1.12}$$

The global eventual energies of a homogeneous state are

$$\lim_{R \to \infty} \lim_{t \to \infty} \| u(\cdot, t) \|_R^2. \tag{1.13}$$

The next result says among other things that the residual energy due to excitation of stationary states dies away as the frequency increases.

THEOREM 1.1. *Let $\lambda = i\mu$, μ real. If $GEE_{\pm\mu}$ are the global eventual energies, and $EE_{\pm\mu}$ are the local eventual energies for $f \in L_{2,\alpha}(\mathbb{R}, \mathscr{H})$, $\alpha > \frac{1}{2}$, $\lambda = i\mu$, then*

$$\lim_{|\lambda| \to \infty} EE_{\mu\pm} = 0$$

and

$$\lim_{|\lambda| \to \infty} GEE_{\mu\pm} = 0. \tag{1.14}$$

The eventual energies $EE_{\mu\pm}$ have the asymptotic form

$$u_{\text{periodic}} + u_{\text{decaying}}, \tag{1.15}$$

where u_{periodic} and u_{decaying} are orthogonal and

$$u_{\text{periodic}} = (2\pi)^{-n/2} \sum_{\lambda_j \equiv 0} \int_{\mathbb{R}^n} e^{ix \circ p} \int_0^t e^{-\lambda(s-t)} \hat{f}_j(p, s)\, ds\, dp \tag{1.16}$$

$$u_{\text{decaying}} = (2\pi)^{-n/2} \sum_{\lambda_j \not\equiv 0} \int_{\mathbb{R}^n} e^{ix \circ p} \int_0^t e^{-i(\lambda_j + \lambda)(s-t)} \hat{f}_j(p, s)\, ds\, dp, \tag{1.17}$$

where $\hat{P}_j \hat{f} = \hat{f}_j$.

Proof. Taking the Fourier transform in x of (1.1) we have

$$\frac{\partial \hat{u}}{\partial t} = iA(p)\, \hat{u} + i_\mu \hat{u} + \hat{f}(p, t). \tag{1.18}$$

Applying $\hat{P}_j$ to both sides of (1.16) gives

$$\frac{\partial \hat{u}_j}{\partial t} = i\lambda_j(p)\, \hat{u} + i_\mu \hat{u}_j + \hat{f}_j(p, t) \tag{1.19}$$

or

$$\hat{u}_j(P, t) = e^{i(\lambda_j(p) + \mu)\, t} \int_0^t e^{-i(\lambda_j(p) + \mu)\, s} \hat{f}_j(p, s)\, ds. \tag{1.20}$$

Now, change to polar form $p = \rho\omega$ and assume for the moment that $\hat{f}_j(\rho, \omega, s) = \phi_1(\rho)\, \phi_2(\omega)\, \phi_3(s)$, each factor being smooth with compact support, $\phi_2 = 0$ on a neighborhood of the zero set of $\lambda_j(\rho\omega) = \rho\lambda_j(\omega)$ if it is not identically zero. Now select j, $\lambda_j \not\equiv 0$. Using the Fubini theorem, and Φ^*, we write (1.18) as

$$u_j(x, t) = \int_0^t \int_{|\omega|=1} \int_a^b e^{-i(\rho\lambda_j(\omega)+\mu)(s-t)} \rho^{n-1} \phi_1 \phi_2 \phi_3 e^{ix\circ\rho\omega}\, d\rho\, d\omega\, ds. \tag{1.21}$$

Integrating by parts l times in ρ we obtain

$$u_j(x, t) = \int_0^t \int_{|\omega|=1} \int_a^b \frac{e^{-i(\rho\lambda_j(\omega)+\mu)(s-t)}}{(s-t)^l\,(i\lambda_j(\omega))^l} G(\rho, \omega, s)\, d\rho\, d\omega\, ds. \tag{1.22}$$

Thus

$$|u_j(x, t)| \leqslant \frac{K\,|F(x)|}{\min|s-t|^l}, \qquad s \text{ in the support of } \phi_3. \tag{1.23}$$

Here $F(x)$ is bounded if $|x|$ is bounded. Hence the local energy of such a u_j decays as $t \to \pm\infty$.

Now, finite sums of such functions $\phi_1\phi_2\phi_3$ are dense in $L_{2,\alpha}(\mathbb{R}, \mathscr{H})$. Furthermore, the Plancherel theorem and (1.6) imply that

$$L_{2,\alpha}(\mathbb{R}, \mathscr{H}) = L_{2,\alpha}(\mathbb{R}, \hat{\mathscr{H}}) \subseteq L_1(\mathbb{R}, \mathscr{H}). \tag{1.24}$$

Taking $\{\hat{f}_j^n\}_{n=1}^\infty$ of the form $(\sum_{\text{finite}} \phi_1\phi_2\phi_3)$, we see that if $\hat{f}_j^n \to \hat{f}_j$ in $L_{2,\alpha}(\mathbb{R}, \hat{\mathscr{H}})$ then $u_j^n \to u_j$ by (1.18). Therefore, since finite sums of products of the form $\phi_1\phi_2\phi_3$ are dense in $L_{2,\alpha}(\mathbb{R}, \hat{\mathscr{H}})$ we have by (1.21) and (1.22) that

$$\|u_j(x, t)\|_R^2 \leqslant \|u_j(x, t) - u_j^n(x, t)\|^2 + \|F_n(t)\|_R^2 \tag{1.25}$$

and thus $(\mathrm{EE}_\mu)_j \leqslant \varepsilon$ for all $\varepsilon > 0$. Thus we have proved (1.15). For (1.14) we note that if $\lambda_j \equiv 0$ in (1.19) we have the formula (1.14). We have

$$u(x, t) = \sum_{\lambda_j=0} u_j + \sum_{\lambda_j \not\equiv 0} u_j \tag{1.26}$$

$$\begin{aligned} \sum_{\lambda_j\equiv 0} u_j &= (2\pi)^{-n/2} \sum_j \int_{\mathbb{R}^n} e^{ix\circ p} \int_0^t e^{i\lambda(s-t)} \hat{f}_j(p, s)\, ds\, dp \\ &= e^{i\lambda t} \sum_j \int_0^t e^{-i\lambda s} f_j(x, s)\, ds \\ &\to C_\pm e^{i\lambda t} \qquad \text{as} \quad t \to \pm\infty. \end{aligned} \tag{1.27}$$

It then follows that

$$\mathrm{EE}_{\mu\pm} = \sum_{\lambda_j\equiv 0} \int_{|x|<R} \left| \int_0^{\pm\infty} e^{i\lambda s} f_j(x, s)\, ds \right|^2 dx. \tag{1.28}$$

By (1.7),

$$\lim_{|\mu| \to \infty} \mathrm{EE}_{\mu\pm} = 0. \tag{1.29}$$

From (1.26),

$$\mathrm{GEE}_{\mu\pm} = \sum_{\lambda_j = 0} \int_{\mathbb{R}^n} |\hat{f}_j^{\pm}(x, \mu)|^2 \, dx \tag{1.30}$$

by the Plancherel theorem, $\hat{f}_j$ now referring to the Fourier transform in the $n+1$ variables (x, t), and $f_j^+ = \chi_{(0,\infty)}(s) f_j, f_j^- = \chi_{(-\infty, 0)}(s) f_j$. Thus

$$\lim_{|\mu| \to \infty} \mathrm{GEE}_{\mu\pm} = 0 \tag{1.31}$$

by (1.7). This concludes the proof.

Now we will give similar consideration to the solutions of (1.2) only with the classical condition $\lambda \geqslant 0$. The details are similar but we include them to give a connected treatment.

Here $x \in \mathbb{R}^n$ and $\mathscr{H}_c$ is the Hilbert space defined by the norm $\| f \|$ (often written as $\| f \|_c$):

$$\| f \|^2 = \int_{\mathbb{R}^n} | f(x)|^2 \, c^2 \, dx.$$

$L_{2,\alpha}$ $(\mathbb{R}, \mathscr{H}_c)$ is the set of strongly measurable $\mathscr{H}_c$ valued functions on $(-\infty, \infty)$ which are square itegrable with respect to the weight $(1 + |t|^2)^{\alpha}$, where $\alpha > \frac{1}{2}$. We shall employ the common but somewhat confusing convention of writing our functions with the space variable appearing first, even though it always comes second in the function space notation, also the common convention.

Again, to understand the behavior of $v(x, t)$, we take the Fourier transform of both sides (in x) of (1.2). Thus

$$\partial_t^2 \hat{v} = \mathbf{c}^2 |\, p\,|^2 \, \hat{v} - \lambda \hat{v} + g(p, t), \tag{1.32}$$

where $\hat{v} = \Phi v$ and $\hat{g} = \Phi g$. Since we assume zero initial conditions, we have

$$\begin{aligned} \hat{v}(p, t) = {} & (\mathbf{c}^2 \,|\, p\,|^2 + \lambda)^{-1/2} \, ((\sin(\mathbf{c}^2 \,|\, p\,|^2 + \lambda)^{1/2} \, t) \\ & \times \int_0^t \cos(\mathbf{c}^2 \,|\, p\,|^2 + \lambda)^{1/2} \, t' g(p, t') \, dt' \\ & + \cos(\mathbf{c}^2 \,|\, p\,|^2 + \lambda)^{1/2} \, t \int_0^t \sin(\mathbf{c}^2 \,|\, p\,|^2 + \lambda)^{1/2} \, t' \hat{g}(p, t) \, dt') \end{aligned} \tag{1.33}$$

and

$$v(x, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{v}(p, t) \exp(ix \circ p) \, dp.$$

The integral is well defined (at least in the mean sense) for each t. In the integral we change the $\mathbf{p}$ integration to polar form and assume for the moment that $g(\mathbf{p}, t)=g(\rho\omega, t)=\phi_1(\rho)\,\phi_2(\omega)\,\phi_3(t)$, where $\rho=|\mathbf{p}|$, $\omega=\mathbf{p}/|\mathbf{p}|$ are the polar variables. We also assume that ϕ_1 and ϕ_2 are smooth with compact support. We have as before

$$\begin{aligned} v(x, t) = (2\pi)^{-n/2} \int_0^{\infty} p^{n-1}\, dp \int_{|\omega|=1} (\mathbf{c}^2p^2+\lambda)^{-1/2} \Big\{ &\sin(\mathbf{c}^2p^2+\lambda)^{1/2}\, t \\ &\times \int_0 \cos t'(\mathbf{c}^2p^2+\lambda)^{1/2}\, \phi_1\phi_2\phi_3\, dt' \\ &+ \cos t(\mathbf{c}^2p^2+\lambda)^{1/2} \int_0^t \sin t'(\mathbf{c}^2p^2+\lambda)^{1/2}\, \phi_1\phi_2\phi_3\, dt' \Big\}\, dA_{\omega}. \end{aligned} \tag{1.34}$$

Switching the order of integration so that the ρ integral is innermost, we integrate by parts in ρ to obtain the estimate (t large) as above that

$$|v(x, t)| \leqslant (C_1\,|x|)/|t|. \tag{1.35}$$

It follows that for such $g(x, t)$

$$\lim_{t\to\infty} \int_{|x|<R} |v(x, t)|^2\, dx = 0. \tag{1.36}$$

That is, v decays in local mean sense. For the asymptotic behavior in λ, we look at (1.34). By the Plancherel theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} |v(x, t)|^2\, dx \leqslant \left(\int_{-\infty}^{\infty} (1+|t'|^2)^{-\alpha}\, dt'\right) &\int_{-\infty}^{\infty} (1+|t'|^2)^{\alpha}\, dt' \\ &\times \int_{\mathbb{R}^n} |g(p, t)|^2\, (\mathbf{c}^2\,|p|^2+\lambda)^{-1}\, dp. \end{aligned} \tag{1.37}$$

By Lebesgue's dominated convergence theorem,

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^n} |v(x, t)|^2\, dx = 0 \tag{1.38}$$

uniformly in t for all $g \in L_{2,\alpha}\,(\mathbb{R}, \mathscr{H}_c)$, $\alpha > \frac{1}{2}$. Thus the solution decays in λ in mean sense, uniformly in t. To return to (1.36), we note that the inequality in (1.37) implies that if $g_n(x, t)$ is a sequence of functions in $L_{2,\alpha}$ $(\mathbb{R}, \mathscr{H}_c)$ consisting of finite sums of functions with terms having x Fourier transform like that in (1.34), and $g_n \to g$ in $L_{2,\alpha}$ $(\mathbb{R}, \mathscr{H}_c)$, then

$$\begin{aligned} \int |u(x, t)|^2\, dx \leqslant \Bigg(\left(\int |u(x, t)-u_n(x, t)|^2\, dx\right)^{1/2} \\ + \left(\int |u_n(x, t)|^2\, dx\right)^{1/2}\Bigg)^2 \end{aligned}$$

(integrals taken over $|x|<R$).

The first term on the right is small for n large by (1.37) and the second term is small (R fixed) for $|t|$ large. Thus u decays in local mean sense for all g in the space $L_{2,\alpha}(\mathbb{R}, \mathscr{H}_c)$ as $t \to \infty$.

This concludes our examination of the elementary cases.

2. Domains With Boundary

In this section we shall consider the operator $-\mathbf{c}(x)^2 \Delta$ in a domain Ω of $\mathbb{R}^n$. By $\partial\Omega$ we mean the boundary of Ω. Our treatment follows the method of Wilcox [28].

First, we must establish some boundary conditions on $\partial\Omega$, such that $-\mathbf{c}(x)^2 \Delta$ is selfadjoint on

$$\mathscr{H}_{c,\Omega} = \left\{ f \,\middle|\, \int_\Omega \mathbf{c}(x)^{-2} \, |f(x)|^2 \, dx < \infty \right\}$$

with norm defined as in the $\mathbb{R}^n$ case.

We shall consider acoustically "hard" and "soft" boundaries (Neumann and Dirichlet boundary conditions). We will not require any explicit smoothness assumptions for $\partial\Omega$.

First, we consider the space $C^\infty(\Omega_c)$ as all smooth functions vanishing in a neighborhood of $\partial\Omega$ and the singularities of $\mathbf{c}(x)^{-1}$ and with compact support. Then for any function u in $\mathscr{H}_{c,\Omega}$, $D^\alpha u$ is the function g in $\mathscr{H}_{c,\Omega}$ (or in $L_2(\Omega)$ or $\mathscr{H}_c\beta_{,\Omega}$ in general as is required) such that

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^\alpha \int_\Omega g_\phi \, dx, \qquad \text{for all } \phi \text{ in } C^\infty(\Omega_c).$$

A function $u(x)$ in $\mathscr{H}_{c,\Omega}$ satisfies the Generalized Dirichlet Condition (briefly GDC), if ∇u is square integrable and for all square integrable vector fields $\mathbf{V}$ with divergence in $\mathscr{H}_{c^{-1},\Omega}$,

$$\int_\Omega (\nabla u \circ \bar{\mathbf{V}} + u \nabla \circ \bar{\mathbf{V}}\} \, dx = 0. \tag{2.1}$$

$u(x)$ in $\mathscr{H}_{c,\Omega}$ is said to satisfy the Generalized Neumann Condition (GNC) if ∇u is square integrable and for all v in $\mathscr{H}_{c,\Omega}$ with ∇v in $L_2(\Omega)$ $(\nabla u \in \mathscr{H}_{c^{-1},\Omega})$,

$$\int_\Omega u\bar{v} \, dx + \int_\Omega \nabla u \circ \overline{\nabla v} \, dx = 0 \tag{2.2}$$

$$\mathscr{H}^D_{c,\Omega} = \{u \in \mathscr{H}_{c,\Omega} \mid u \text{ satisfies the GCD}\}$$

$$\mathscr{H}^N_{c,\Omega} = \{u \in \mathscr{H}_{c,\Omega} \mid u \text{ satisfies the GNC}\}.$$

Finally, we define two operators A_D and A_N in $\mathscr{H}_{c,\Omega}$ as

$$\mathscr{D}(A_D)=\{u\in\mathscr{H}_{c,\Omega}\mid \mathbf{c}(x)^2\,\Delta u\in\mathscr{H}_{c,\Omega}\}\cap\mathscr{H}^D_{c,\Omega}$$

and

$$\mathscr{D}(A_N)=\{u\in\mathscr{H}_{c,\Omega}\mid \mathbf{c}(x)^2\,\Delta u\in\mathscr{H}_{c,\Omega}\}\cap\mathscr{H}^N_{c,\Omega},$$

where

$$A_D u=\mathbf{c}(x)^2\,\Delta u,\qquad u\in\mathscr{D}(A_D)$$

$$A_N u=\mathbf{c}(x)^2\,\Delta u,\qquad u\in\mathscr{D}(A_N).$$

We desire to show that these two operators are selfadjoint in $\mathscr{H}_{c,\Omega}$. To do this, we shall apply a well-known result: If A is a densely defined symmetric positive operator in a Hilbert space H and the range of $A+\mathbf{I}$ is all of H, then A is selfadjoint.

To verify the above conditions we first consider that $C^\infty(\Omega_c)$ is dense in both $\mathscr{D}(A_D)$ and $\mathscr{D}(A_N)$. Hence these sets are dense in $\mathscr{H}_{c,\Omega}$ because $C^\infty(\Omega_c)$ is dense in $\mathscr{H}_{c,\Omega}$. Now let $\phi\in C^\infty(\Omega_c)$ and $u\in\mathscr{D}(A_D)$. Then,

$$\int_\Omega \nabla\bar{u}\circ\nabla\phi\,dx=-\int_\Omega \mathbf{c}(x)^2\,\Delta\bar{u}\phi\mathbf{c}(x)^{-2}\,dx=-\int_\Omega \bar{u}\mathbf{c}(x)^2\,\Delta\phi\mathbf{c}(x)^{-2}\,dx.$$

Thus, $\nabla\circ\nabla u\in\mathscr{H}_{c^{-1},\Omega}$ and is equal to Δu. Now take $\mathbf{V}$ in the GDC to be ∇u_1 and $u=u_2$, with u_1 and u_2 in $\mathscr{D}(A_D)$. Switching the roles of u_1 and u_2 shows by the GDC that symmetry holds. Next, if $u\in\mathscr{D}(A_D)$,

$$(-A_D u,u)_{\mathscr{H}_{c,\Omega}}=\int_\Omega|\nabla u|^2\,dx\geqslant 0.$$

Note again that $\nabla u\in L_2(\Omega)$.

Now suppose $f\in\mathscr{H}_{c,\Omega}$ and consider the equation $u-A_D u=f$.

Let $v\in\mathscr{H}^D_{c,\Omega}$. Then $(u,v)_{\mathscr{H}_c}-(A_D u,v)_{\mathscr{H}_c}=(f,v)_{\mathscr{H}_c}$. But then we have

$$(u,v)_{\mathscr{H}_c}+(c\nabla u,c\nabla v)_{\mathscr{H}_c}=(f,v)_{\mathscr{H}_c}.$$

We note that

$$|(f,v)_{\mathscr{H}_c}|\leqslant\|f\|\,\{\|v\|^2_{\mathscr{H}_{c,\Omega}}+\|\nabla v\|^2_{L_2}\}^{1/2}$$

and $\mathscr{H}^D_{c,\Omega}$ is a closed subspace of the Hilbert space defined by the norm $\{\|\cdot\|^2_c+\|\nabla(\cdot)\|^2_{L_2}\}^{1/2}$. Therefore $\mathscr{H}^D_{c,\Omega}$ is a Hilbert space with the inner product defined by this norm. By the Riesz representation theorem, for each f, there is a unique u such that $(u,v)=(f,v)$ $u\in\mathscr{H}^D_{c,\Omega}$. Now for each $\phi\in C^\infty(\Omega_c)$, $\mathbf{V}=\nabla\phi$, we have

$$\int_\Omega \bar{u}\,\Delta\phi\,dx=\int_\Omega \bar{u}\nabla\circ\nabla\phi\,dx=-\int_\Omega\nabla\bar{u}\circ\nabla\phi\,dx=\int_\Omega(\overline{u-f})\,\phi\mathbf{c}^{-2}\,dx.$$

So that $\Delta u = \mathbf{c}^{-2}(u-f)$. For A_N we have symmetry and positivity by the GNC. The rest of the argument is the same as for the GDC. This completes the proof.

3. Temporally Homogeneous Solutions

We shall consider solutions to the equations

$$\frac{\partial u}{\partial t} = E(x)^{-1} \sum_{j=1}^{n} A_j \frac{\partial u}{\partial_j x} + B(x)\, u + \lambda u + f(x, t) \tag{3.1}$$

$$\frac{\partial^2 v}{\partial t^2} = \mathbf{c}(x)^2\, v + \mathbf{q}(x)\, v - \lambda v + g(x, t). \tag{3.2}$$

We first treat (3.1). Again we assume $\lambda = -i\mu$, $\mu \in \mathbb{R}$.

We make the following assumptions which hold throughout this section.

(1) $E(x) \in L_\infty(\mathbb{R}^n, \mathbb{C}^{N\times N})$. (3.3)

(2) For any $v \in \mathbb{C}^N$, there exists a constant $c_1 > 0$, such that for almost all x,

$$c_1(v \circ v) \leqslant (v \circ E(x)\, v) \tag{3.4}$$

$(v \circ u = \sum_{i=1}^{N} v_i \bar{u}_i)$.

(3) $E(x)$ and $iB(x)$ are Hermitian almost everywhere.

(4) $B(x) \in L_\infty(\mathbb{R}^n, \mathbb{C}^{N\times N})$. (3.5)

(5) $E(x)\, B(x) = B(x)\, E(x)$ for almost all x.

We multiply (2.1) by $i = \sqrt{-1}$ to obtain

$$-i\frac{\partial u}{\partial t} = E(x)^{-1}\,(i) \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} - \mu u - iB(x)\, u - if(x, t). \tag{3.6}$$

We shall use the abbreviated forms

$$i\partial_t \text{ for } i\frac{\partial}{\partial t}$$

$$A(D) \text{ for } -i\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} \tag{3.7}$$

$$C(x) \text{ for } iB(x)$$

We omit the i in $if(x, t)$.

Recalling the remarks from the Introduction, we intend to study the solutions of (3.1)–(3.2) in terms of various asymptotic behaviors as in Section 1.

First, we recall some facts from functional analysis (see [12]).

If H_1 and H_2 are two Hilbert spaces, the *tensor product* $H_1 \otimes H_2$ is a Hilbert space defined by the completion of the unitary space $U_\otimes$ whose inner product (,) is given by ($U_1, V_1, \in H_1$; $U_2, V_2 \in H_2$)

$$(U_1 \otimes U_2, V_1 \otimes V_2) = (U_1, V_1)_{H_1} (U_2, V_2)_{H_2} \tag{3.8}$$

and extending by linearity to finite sums of which $U_\otimes$ consists.

If Q_1 and Q_2 are closed operators on H_1 and H_2, respectively, a preclosed operator $Q_1 \otimes Q_2$ is defined on $H_1 \otimes H_2$ by the action in $U_\otimes$ of

$$Q_1 \otimes Q_2(U_1 \otimes U_2) = (Q_1 U_1) \otimes (Q_2 U_2) \tag{3.9}$$

when $U_1 \in \mathscr{D}(Q_1)$, $U_2 \in \mathscr{D}(Q_2)$. Here again the notation $\mathscr{D}(Q)$ is employed for the *domain of Q*.

PROPOSITION 3.1. *For any real* α, $L_{2,\alpha}(\mathbb{R}, \mathscr{H})$ *is unitarily equivalent to*

$$L_{2,\alpha}(\mathbb{R}, \mathbb{C}^N) \otimes L_2(\mathbb{R}^n, \mathbb{C}^N).$$

Here the obvious correspondence,

$$f \otimes g \to fg \tag{3.10}$$

($f(t)\, g(x)$ *is interpreted as*

$$(f_1(t)\, g_1(x), f_2(t)\, g_2(x), ..., f_N(t)\, g_N(x))),$$

gives the appropriate isomorphism.

We define $A(D)$ *as an operator on* $\mathscr{H}$ *by the Fourier transform*

$$(A(D)) = \{f \in | A(p) \hat{f}(p) \in \mathscr{H}\}; \tag{3.11}$$

then for $f \in \mathscr{D}(A(D))$,

$$A(D) f = \Phi^* A(p)\, \Phi f. \tag{3.12}$$

The Hilbert space $\mathscr{H}_E$ *is defined by the norm*

$$\| f \|_E^2 = \int_{\mathbb{R}^n} (f(x), E(x) f(x))\, dx. \tag{3.13}$$

Let us define the operator K_1 *by*

$$K_1 = -i\partial_t - E(x)^{-1} A(D) - C(x). \tag{3.14}$$

It is obvious that K_1 is densely defined in $L_2(\mathbb{R}^{n+1}, \mathbb{C}^N)$. We note that by (2.12) *and* (2.2), (2.4), $\mathscr{H}_E$ *and* $\mathscr{H}$ *are identical sets of (equivalence classes of) functions.*

LEMMA 3.2. *K_1 is a selfadjoint operator on $L_2(\mathbb{R}, \mathscr{H}_E)$.*

Proof. While the proof is elementary, we give one for completeness. Let $u, v \in \mathscr{D}(K_1)$.

$$(K_1 u, v) = \int (-i\partial_t u - E(x)^{-1} A(D)\, u - C(x)\, u, v)_E \, dt$$

$$= \iint_{\mathbb{R}^n} ((-i\partial_t u - E^{-1} A(D)\, u - Cu) \circ Ev)\, dx\, dt$$

$$= \iint_{\mathbb{R}^n} (-i\partial_t u \circ Ev) - (A(D)\, u \circ V) - (Cu \circ \dot{E}v)\, dx\, dt.$$

By integration by parts in the first inner product and using (2.11) and (5) we obtain the symmetry of K_1. Now let $u \in \mathscr{D}(K_1)$, $v \in \mathscr{D}(K_1^*)$. Then

$$(K_1 u, v) = (u, h_v)$$

for some h and all $u \in \mathscr{D}(K_1)$.

By Fourier transform and Parseval's identity,

$$\iint_{\mathbb{R}^n} (\hat{u}(p, \xi) \circ E h_v(p, \xi))\, dp\, d\xi$$

$$= \iint_{\mathbb{R}^n} (\hat{u}(p, \xi) \circ \xi E V(p, \xi) - A(p)\, \hat{v}(p, \xi) - C V(p, \xi))\, dp\, d\xi.$$

Parseval's identity and the density of $\mathscr{D}(K_1)$ give

$$h_v(x, t) = -i\partial_t V(x, t) - E(x)^{-1} A(D)\, v(x, t) - C(x)\, v(x, t).$$

Thus $K_1 = K_1^*$.

Our first result is the following theorem.

THEOREM 3.3. *There exist unique solutions $u_{\pm\mu}(x, t)$ of* (2.1) *in $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)$ for $\beta > \frac{1}{2}$* defined by

$$\lim_{\varepsilon \to 0^+} (K_1 - \mu \pm i\varepsilon)^{-1} f \tag{3.15}$$

for each $f \in L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E)$, $\alpha > \frac{1}{2}$.

Proof. We shall construct the proof in

$$L_{2,\alpha}(\mathbb{R}, \mathbb{C}^N)\otimes L_2(\mathbb{R}^n, \mathbb{C}^N)_E \to L_{2,-\beta}(\mathbb{R}, \mathbb{C}^N)\otimes L_2(\mathbb{R}^n, \mathbb{C}^N)_E$$

instead of $L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E) \to L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)$. K_1 is the closure of

$$-i\partial_t \otimes I_1 - I_2 \otimes (E(x)^{-1} A(D) + C(x)) \tag{3.16}$$

in $L_2(\mathbb{R}, \mathbb{C}^N)\otimes L_2(\mathbb{R}^n, \mathbb{C}^N)_E$.

First, some properties of the operator

$$(-i\partial_t - \mu)^{-1} = (K_0 - \mu)^{-1} \tag{3.17}$$

are required. It is easily shown that

$$(K_0 - \mu)^{-1} f(t) = \begin{cases} i\int_{-\infty}^{t} e^{i\mu(t-s)} f(s)\, ds, & \text{im} \quad \mu > 0 \\ -i\int_{t}^{\infty} e^{i\mu(t-s)} f(s)\, ds, & \text{im} \quad \mu < 0 \end{cases} \tag{3.18}$$

for $f \in L_2(\mathbb{R}, \mathbb{C}^N)$. Suppose $f \in L_{2,\alpha}(\mathbb{R}, \mathbb{C}^N)$.

$$\begin{aligned} \|(K_0-\mu)^{-1} f\|_{-\beta}^2 &= \int_{-\infty}^{\infty} (1+|t|^2)^{-\beta} \left| \int_{-\infty}^{t} e^{i\mu(t-s)} f(s)\, ds \right|^2 dt \\ &\leqslant \int_{-\infty}^{\infty} (1+|t|^2)^{-\beta} \left(\int_{-\infty}^{t} e^{\text{im}\, \mu(t-s)}\, |f(s)|\, ds \right)^2 dt \\ &\leqslant \int_{-\infty}^{\infty} (1+|t|^2)^{-\beta}\, dt \int_{-\infty}^{\infty} (1+|s|^2)^{-\alpha}\, ds \\ &\quad \times \int_{-\infty}^{\infty} (1+|s|^2)^{\alpha}\, |f(s)|^2\, ds < \infty \end{aligned}$$

or

$$\|(K_0-\mu)^{-1} f\|_{-\beta}^2 \leqslant C\, \|f\|_\alpha^2 \tag{3.19}$$

as $\text{im}\, \mu \to 0^+$,

$$(K_0-\mu)_+^{-1} f = i\int_{-\infty}^{t} e^{i\mu(t-s)} f(s)\, ds \tag{3.20}$$

for $f \in L_{2,\alpha}(\mathbb{R}, \mathbb{C}^N)$ by (1.6).

Similar statements are correct for $\text{im}\, \mu \to 0^-$. Thus $(K_0 - \lambda)^{-1}$ is continuous in λ, in fact locally Hölder continuous as a map from $L_{2,\alpha}(\mathbb{R}, \mathbb{C}^N)$ to $L_{2,-\beta}(\mathbb{R}, \mathbb{C}^N)$.

Now write $\mathbb{C}_{\pm}=\{\lambda \mid \pm \operatorname{im} \lambda>0\}$ and

$$dE(\lambda)=(2\pi i)^{-1}\left[(E(x)^{-1}A(D)+C(x)-\lambda)^{-1} - (E(x)^{-1}A(D)+C(x)-\bar{\lambda})^{-1}\right] \tag{3.21}$$

for $\lambda \in \mathbb{C}_{+}$. In the proof of Lemma 3.2 we showed that $E(x)^{-1}A(D)+C(x)$ was selfadjoint on $\mathscr{H}_E$. In the weak sense, $dE(\lambda)\,d\lambda \to dE(\lambda)$ as im $\lambda \to 0$, where $dE(\lambda)$ is the spectral measure for $E(x)^{-1}A(D)+c(x)$ on $\mathscr{H}_E$. Also define

$$dF(\lambda)=\frac{1}{2\pi i}\left((K_0-\lambda)^{-1}-(\bar{K}_0-\lambda)^{-1}\right). \tag{3.22}$$

From the operational calculus, im $\lambda>0$,

$$(K_1-\lambda)^{-1}=i\int_0^{\infty} e^{i\lambda t}e^{-it(-i\partial_t)}e^{+it(E(x)^{-1}A(D)+C(x))}dt. \tag{3.23}$$

This implies, using the formulas ($\xi_i=\operatorname{re}\lambda_i$) (see also [2])

$$e^{it(E(x)^{-1}A(D)+C(x))}=\frac{1}{2\pi i}e^{\operatorname{im}\lambda_1|t|}\int_{-\infty}^{\infty} e^{it\xi_1}\,dE(\lambda_1)\,d\xi_1 \tag{3.24}$$

and a similar formula for $-i\partial_t$, that

$$(K_1-\lambda)^{-1}=\iint \frac{dF(\lambda_1)\otimes dE(\lambda_2)}{\lambda_1+\lambda_2-\lambda}\,d\xi_1\,d\xi_2 \tag{3.25}$$

assuming im λ_i, im $\lambda>0$. Changing variables in (2.24) and using (2.23) in (2.22) we obtain, letting im $\lambda_2 \to 0$,

$$(K_1-\lambda)^{-1}=\int_{-\infty}^{\infty}(K_0-(\lambda-\xi))^{-1}\otimes dE(\xi). \tag{3.26}$$

$dE(\xi)$ is a bounded measure and $(K_0-(\lambda-\xi))^{-1}$ is a bounded continuous function so by [17] the integral makes sense in the strong operator topology. Since $dE(\xi)$ is countably additive in this sense, the dominated convergence theorem [23] implies

$$(K_1-\mu)_{\pm}^{-1}=\int_{-\infty}^{\infty}(K_0-(\mu-\xi))_{\pm}^{-1}\otimes dE(\xi) \tag{3.27}$$

for μ real. But then the theory of [17] may be applied to show that $(K_1-\mu)_{\pm}^{-1}$ is defined by (3.27) in the uniform sense. We therefore conclude that $f \to u_{\mu\pm}$ defines a continuous (in μ) map from $L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E)$ to $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)$ by (3.27).

THEOREM 3.4. (1) *$(K_1-\mu)_{\pm}^{-1}$ is analytic in $\mathbb{C}_{\pm}$.*

(2) *If $u_{\pm\mu}$ is a solution of* (2.1) *given by* (2.26), *then $GEE_{\pm\mu}=0$ in the sense that* $\chi_{(T,\pm\infty)}(t)\, u_{\pm\mu} \to 0$ in $L_{2,-\beta}$ as $T \to \pm\infty$.

(3) *If $f(x,t) \in L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E)$ then $\lim_{|\mu|\to\infty} u_{\pm\mu}=0$ in $L_{2,-\beta}$.*

Proof. For complex μ, $(K_1-\mu)^{-1}=L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E) \to L_2(\mathbb{R}, \mathscr{H}_E)$ and the first resolvent equation applies. This gives analyticity. The second part (which is much weaker than the result in Section 1) is trivial.

For (3) we note that for $f(x,t)$ of the special form $f_1(t) \otimes f_2(x)$ the

$$\left\| \int_{-\infty}^{\infty} (K_0-(\mu-\xi))_{\pm}^{-1} \otimes dE(\xi)(\chi_c \otimes I) f(x,t) \right\| \to 0 \tag{3.28}$$

by the dominated convergence theorem and the Riemann–Lebesgue lemma.

A simple density argument now shows the final result.

It seems reasonable to suppose that if $E(x) \to I$ as $|x| \to \infty$ then some result similar to Theorem 1.1 for $EE_{\mu\pm}$ and $GEE_{\mu\pm}$ would be correct. We do not pursue this here, but we consider it in Section 4.

Remark. The requirement that $E(x)$ be almost everywhere bounded is not necessary. The results of this section remain true if $E(x)$ is allowed to grow in the sense that

$$\int_{|x-y|<1} |E(x)|^2\, dx < c(1+|x|)^m$$

for some $m>0$. Under these conditions selfadjointness prevails (see Schechter [14]).

We now give the parallel treatment of (3.2).

With the assumption that $\mathbf{q}(x)$ is bounded above,

$$-\mathbf{c}(x)^2 \Delta - \mathbf{q}(x)$$

is bounded below. In any case, we will assume it is selfadjoint on $\mathscr{H}_{c,\Omega}$. Again, for vector valued functions $f(t)$, $f\colon \to \mathscr{H}_{c,\Omega}$, f belongs to $L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$ if and only if $\|f\|_{0,c} < \infty$, where

$$\|f\|_{0,c} = \left(\int_{\Omega} \|f(t)\|_c^2\, dt \right)^{1/2}. \tag{3.29}$$

The weighted spaces $L_{2,\alpha}$ $(\mathbb{R}, \mathscr{H}_{c,\Omega})$ are defined by the norm $\|\cdot\|_{\alpha,c}$:

$$\|f\|_{\alpha,c} = \left(\int (1+|t|^2)^{\alpha}\, \|f(t)\|_c^2\, dt \right)^{1/2}. \tag{3.30}$$

We note that $L_{2,\alpha}(\mathbb{R}, \mathscr{H}_{c,\Omega}) \simeq L_{2,\alpha}(\mathbb{R}) \otimes \mathscr{H}_{c,\Omega}$. By $\bar{K}_0$ we mean ∂_t^2 in the space $L_2(\mathbb{R})$. We write $\bar{K}_1$ for the operator $\bar{K}_0 - c(x)^2 \Delta - \mathbf{q}(x)$.

PROPOSITION. *$\bar{K}_1$ defines a selfadjoint operator in $L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$. If the spatial part is bounded below then there exist solutions $u \in L_{2,-\beta}(\mathbb{R}, \mathscr{H}_{c,\Omega})$ for the equation $\bar{K}_1 u = f - \lambda u$, $\lambda \in \mathbb{R}_+$, for any $f \in L_{2,\alpha}(\mathbb{R}, \mathscr{H}_{c,\Omega})$, for $\beta > \frac{1}{2}$ and λ sufficiently large. We abuse notation by writing $\mathbf{q}(x)$ for the operator of multiplication generated by $\mathbf{q}(x)$, etc.*

Following the argument above we can write (λ real)

$$(\bar{K}_1 + (\lambda \pm i\varepsilon))^{-1} = \int (\bar{K}_0 + (\lambda \pm i\varepsilon + \mu))^{-1} \otimes dE(\mu). \tag{3.31}$$

Here $dE(\ )$ is the spectral measure for $-\mathbf{c}(x)^2 \Delta - \mathbf{q}(x)$. To proceed further, we require some information about $(\bar{K}_0 + \lambda)^{-1}$. It is easily shown that ($\operatorname{im} \sqrt{\lambda} > 0$)

$$(\bar{K}_0 + \lambda)^{-1}(t) = \frac{2i}{\sqrt{\lambda}} \int_{-\infty}^{\infty} \exp(i\sqrt{\lambda}\, |t-s|) f(s)\, ds. \tag{3.32}$$

Thus as a map from $L_{2,\alpha}(\mathbb{R})$ to $L_{2,-\beta}(\mathbb{R})$ we obtain

$$\|(\bar{K}_0 + \lambda)^{-1} f\|_{-\beta} \leqslant 2C\, |\lambda|^{-1/2}\, \|f\|_{\alpha}, \tag{3.33}$$

where C depends on β. The same bound is correct when we let $\operatorname{im} \sqrt{\lambda} \to 0^+$. We obtain the maps $(\bar{K}_0 + \lambda)_{\pm}^{-1} : L_{2,\alpha}(\mathbb{R}) \to L_{2,-\beta}(\mathbb{R})$, where $\pm \operatorname{re} \sqrt{\lambda} > 0$. By the dominated convergence theorem [23],

$$(\bar{K}_1 + \lambda)^{-1} = \int_{-\infty}^{\infty} (\bar{K}_0 + \lambda + \mu)^{-1} \otimes dE(\mu) \qquad (\lambda < 0) \tag{3.34}$$

and

$$(\bar{K}_1 + \lambda)_{\pm}^{-1} = \int_{-\infty}^{\infty} (\bar{K}_0 + \lambda + \mu)_{\pm}^{-1} \otimes dE(\mu) \qquad (\lambda > 0). \tag{3.35}$$

The integrals in (3.35) converge in the uniform topology in the subspace $I \otimes E(0, \infty) L_{2,\alpha}(\mathbb{R}, \mathscr{H}_{c,\Omega})$ with range in $I \times E(0, \infty) L_{2,-\beta}(\mathbb{R}, \mathscr{H}_{c,\Omega})$. The integral in (3.33) converges in the uniform sense on the subspace $I \otimes E(-\infty, 0) L_{2,\alpha}(\mathbb{R}, \mathscr{H}_{c,\Omega})$ (see [17] for the appropriate definition of integral here). If the spectrum of $-\mathbf{c}(x)^2 \Delta - \mathbf{q}(x)$ is absolutely continuous, then the singularity $|\lambda|^{-1/2}$ is integrable, and $(\bar{K}_1 + \lambda)_{\pm}^{-1}$ exists for all $\lambda \in \mathbb{R}$. If $-\mathbf{c}(x)^2 \Delta - \mathbf{q}(x)$ has negative eigenvalues, $\mu_j < 0$ but no negative continuous spectrum, we have well-defined boundary values for $(\bar{K}_1 + \lambda)^{-1}$ along $\mathbb{R}_+$, except for $\lambda = -\mu_j$. This is easy to see by simply breaking up the integral (3.35) into parts over $(-a, 0)$ and $(0, \infty)$. One easily obtains the associated spectral projections as well. Suppose C is the contour in Fig. 1. Then,

$$\oint_C (\bar{K}_1+\lambda)^{-1}\, d\lambda = \sum_{i,k} \int_{C_k} (\bar{K}_0+\lambda+\mu_1)^{-1} \otimes E(\mu_i)\, d\lambda$$
$$= \sum \int_{C_{ii}} (\bar{K}_0+\lambda+\mu_i) \otimes E(\mu_i)\, d\lambda$$
$$= \int_{C_0} (\bar{K}_0+e^{i\theta})^{-1}\, ie^{i\theta}\, d\theta \otimes \sum E(\mu_i). \qquad (3.36)$$

The kernel of $\int (\bar{K}_0+e^{i\theta})^{-1}\, ie^{i\theta}\, d\theta$ is $-4 \sin |s-t|/k\, |s-t|$ (which defines a bounded operator from $L_{2,\alpha}(\mathbb{R})$ to $L_{2,-\beta}(\mathbb{R})$).

We have proved the following.

THEOREM 3.5. *Let $u(x, t, \lambda)$ be a solution of* (3.2) *defined by Proposition* 2.1. *Then u decays in the norm of $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_{c,\Omega})$ as $\lambda \to \infty$.*

Proof. It is a matter of applying the dominated convergence theorem [23] to the integral in (3.35) using the inequality (3.33).

To obtain a result concerning time decay, we need some further discussion on the behavior of the spatial part of $\bar{K}_1$. This will be postponed to Section 5.

4. TIME-DEPENDENT DISPERSIONS IN THE ENERGY SETTING

Here we shall consider solutions to the equation

$$\frac{\partial u}{\partial t} = E(x)^{-1} \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + B(x, t)u + \lambda u + f(x, t). \qquad (4.1)$$

As usual, $f(x, t)$ will be considered to lie in $L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E)$. We shall consider both real and imaginary values for λ. The assumptions of Section 3 shall hold for $E(x)$. We shall also assume the existence of a matrix $B(x)$ satis-

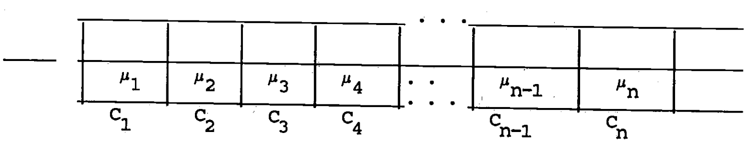

FIG. 1. The contour of integration for the eigenprojectors of K_1.

fying the criteria of Section 3. This matrix will satisfy certain assumptions relative to $B(x, t)$. These are (not necessarily in force at the same time)

$$|B(x, t) - B(x)| \leqslant C(1 + |t|)^{-\gamma} \qquad \text{(a.e. } x \text{ and } t\text{, where } \gamma \geqslant 0). \tag{4.2}$$

γ will take different values depending on the result desired.

$$B(x, t) \text{ is antisymmetric and } B(x, t) \text{ and } E(x) \text{ commute (a.e.).} \tag{4.3}$$

Under both the conditions (4.2) and (4.3) nearly ideal conditions prevail. First, however, we will employ just condition (4.2).

To begin, we will define the operator K_2 formally by

$$i\frac{\partial u}{\partial t} - (E(x)^{-1} A(D)\, u + C(x, t)\, u + \mu u + f(x, t)). \tag{4.4}$$

Here we have suppressed the i in $f(x, t)$ and we write $\lambda = -i\mu$. As before, $C(x, t) = iB(x, t)$. A certain continuity in t will occasionally be assumed for $C(x, t)$. For λ imaginary, a somewhat weaker stability condition holds for u under just the hypothesis (4.2). It is necessary that the constant C in (4.2) be relatively small. This is in part because $A(D)$ fails to be an elliptic operator. For further remarks on this problem in the steady state, see the Appendix. The following theorem holds.

THEOREM 4.1. *Suppose condition* (4.2) *holds for some* $\gamma > 1$ *and* C *in* (4.2) *is less than the reciprocal of the maximum for*

$$\|(K_1 - \mu)^{-1}\|_{L_{2,\alpha} \to L_{2,-\beta}}$$

(*a quantity which is uniformly bounded in* μ). *Then a solution* μ *of Eq.* (4.1) *exists in the space* $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)$. *Furthermore, this solution decays in* $L_{2,-\beta}$ *as* $|\mu| \to \infty$.

Proof. Let us write $D(x, t)$ for $C(x, t) - C(x)$. Then if u is a solution to (4.1)

$$(K_1 - \mu)^{-1} (K_2 - \mu)\, u = (K_1 - \mu)^{-1} f,$$

so

$$(I + (K_1 - \mu)^{-1} D)\, u = (K_1 - \mu)^{-1} f.$$

Under the hypothesis of the theorem $D\colon L_{2,-\beta} \to L_{2,\alpha}$ provided α and β are greater than $\frac{1}{2}$ but sufficiently close to $\frac{1}{2}$. Our work in Section 3 shows that $(I + (K_1 - \mu)^{-1} D)$ is uniformly invertible on $L_{2,-\beta}$. Thus, the Riemann–Lebesgue lemma and the dominated convergence theorem imply the conclusion of the theorem.

If $D(x, t)$ is a continuous map (in t) from $\mathscr{H}_E$ to itself, then the following result holds.

THEOREM 4.2. *Suppose that* (4.2) *holds and D is continuous as noted. If $\gamma > 1$ then there exists a solution u of* (4.1) *which lies in $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)$. Furthermore, this solution decays as $|\mu| \to \infty$ for each f in $L_{2,\alpha}$, provided α and β are sufficiently close to $\frac{1}{2}$.*

Proof. Let c_n be a bounded neighborhood locally finite covering consisting of abutting compact intervals such that in each c_n, $C(x, t)$ deviates from its operator maximum (in norm) less than the bound required on C in Theorem 4.1. Then by Theorem 4.1 there exists a solution u_n to the equation

$$i\frac{\partial u}{\partial t} = E(x)^{-1} A(D)\, u + C(x, t)_n\, u + \mu u + f(x, t),$$

where $C(x, t)_n$ is defined to be $C(x, t)$ on c_n and operator max $C(x, t)$ in c_n elsewhere, with $C(c) = \max C(x, t)_n$ a scalar. As c_n moves to $\pm\infty$ $(n \to \infty)$, (4.2) dictates that the length of c_n may be arbitrarily large. This implies that $u = \sum u_n \chi_{cn}$ is in $L_{2,-\beta}$ and is a distributional solution to (4.1). Since each component u_n decays as $\lambda \to \infty$, this completes the proof.

Now we discuss the energy stability when λ is real.

THEOREM 4.3. *Let* (4.2) *hold* $(\gamma \geqslant 0)$ *and let λ be a real parameter. For $|\lambda|$ sufficiently large, there exists a solution u to* (4.1) *which lies in $L_2(\mathbb{R}, \mathscr{H}_E)$ and decays as $|\lambda|$ goes to infinity.*

Proof. Following the proof of Theorem 4.1 and substituting $i\lambda$ for μ and using the fact that $\|(K_1 - i\lambda)^{-1}\| \leqslant |\lambda|^{-1}$ gives the result.

We note that for the conclusion of Theorem 4.3 it is only necessary that f belong to $L_2(\mathbb{R}, \mathscr{H}_E)$.

In order to obtain results like Theorem 4.3 for λ imaginary and to study the *pointwise* time decay properties of solutions to (4.1), we now impose (4.3). Under this condition, K_2 is a selfadjoint operator in $L_2(\mathbb{R}, \mathscr{H}_E)$ and so solutions exist for real λ by the general elementary theory of selfadjoint operators. We use some of the terminology from the theory of potential scattering in quantum mechanics (see [8, 9]).

Consider the Cauchy problem for

$$-i\frac{\partial u}{\partial t} = iE(x)^{-1} A(D)\, u + B(x, t)\, u. \tag{4.5}$$

We define a solution by means of the propagator $U(s, t)$:

$$\frac{\partial u}{\partial t} U(s, t) = A(t)\, U(s, t) \tag{4.6}$$

$$A(t) = -iE(x)^{-1} A(D) + B(x, t).$$

Assume (4.2) holds with $\gamma > \frac{1}{2}$. $A(t)$ is selfadjoint on $\mathscr{H}_E$ for each t.

For $f \in L_2(\mathbb{R}, \mathscr{H}_E)$,

$$f(t) \in \mathscr{H}_E$$

and

$$U_2(s, t) f(t) = (\exp(-(s-t)\, iK_2) f)(s). \tag{4.7}$$

We have that

(1) $U(s, s) = I$,

(2) $U(s, t)\, U(t, v) = U(s, v)$, $s, t, v, \in \mathbb{R}$,

(3) $\| U(s, t) \| = 1$,

(4) $\mathscr{D}(A(t)) = \mathscr{D}(A(s))$ all $s, t \in \mathbb{R}$.

It follows that wave operators connecting and intertwining $A(t)$ and $E(x)^{-1} A(D) + iB(x)$ exist if and only if the strong limits

$$\lim_{\sigma \to \pm\infty} \exp(\sigma i K_2) \exp(-\sigma i K_1) \tag{4.8}$$

exist. The Cook–Kuroda method may be employed to show that this is the case. We merely have to verify that

$$\int_{-\infty}^{\infty} \| (K_2 - K_1) \exp(-\sigma i K_1) f \|_{L_2(\mathfrak{R}, \mathscr{H}_E)}\, d\sigma < \infty \tag{4.9}$$

for a dense set of f. We note that a somewhat weaker condition on $B(x, t) - B(x)$ is required than in Theorems 4.1–4.3. We have ($\exp(\sigma \partial_t)$ is translation by σ)

$$\int_{-\infty}^{\infty} \| (K_2 - K_1) \exp(-\sigma i K_1) \|_{L_2}\, d\sigma$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} | B(x, t) - B(x) |^2 \, | \exp((\sigma - t)\, i(E(x)^{-1} A(D)) + iB) f(t - \sigma) |_E^2 \, dx\, dt \right\}^{1/2} d\sigma. \tag{4.10}$$

Let us assume $f(x, t) = f_1(x) f_2(t)$, $f_2 \in L_1(\mathbb{R}, \mathbb{C}^N)$, $f_1 \in L_2(\mathbb{R}^n, \mathbb{C}^N)$. Then (4.10) is less than or equal to

$$C \int_{-\infty}^{\infty} |f_1(\sigma)|\, d\sigma \left\{ \int_{-\infty}^{\infty} (1 + |t - \sigma|)^{-1-\varepsilon}\, dt \right\}^{1/2} < \infty \tag{4.11}$$

and finite sums of the form $f_1 f_2$ are dense in $L_2(\mathbb{R}, \mathscr{H}_E)$.

We write

$$W_{\pm}(s) = \lim_{t \to \pm\infty} U_2(t, s)\, U_1(t - s).$$

We note that the addition of the parameter $\lambda = i\mu$ in (4.6) simply adds the factor $e^{i\mu t}$ to $U(s, t)$, etc. Furthermore, if $f(x, t) \in L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E)$, then so does $U(s, t) f$.

In case the action of $E(x)$ and $B(x)$ also "diminishes," it is possible to prove that the energy of solutions of (4.1) decays locally uniformly in $\mathscr{H}_E$ in time, for a restricted class of source terms $f(x, t)$. One way to show this is to examine solutions to the homogeneous equations

$$-i\partial_t u = E(x)^{-1} A(D)\, u + \mu u + C(x)\, u, \qquad u(x, 0) = u_0(x) \tag{4.12}$$

$$-i\partial_t u_1 = A(D)\, u_1 + \mu u_1, \qquad u_1(x, 0) = u_{00}(x). \tag{4.13}$$

We assume here that for some $\delta > 0$, $C_1 > 0$, $C_2 > 0$,

$$|E(x) - I| \leqslant C_1 (1 + |x|)^{-1-\delta}, \qquad x \in \mathbb{R}^n \tag{4.14}$$

and

$$|C(x)| \leqslant C_2 (1 + |x|)^{-1-\delta}, \qquad x \in \mathbb{R}^n.$$

Using somewhat modified versions of the arguments found in [1], the existence of the strong limits

$$\tilde{W}^{\pm}(s) = \lim_{t \to \pm\infty} U(s - t)\, J U_0(t - s)\, P, \tag{4.15}$$

where U_0 is the unitary group on $\mathscr{H}$ generating solutions to (4.13) and U likewise generating solutions to (4.12) in $\mathscr{H}_E$. Here $J: \mathscr{H} \to \mathscr{H}_E$ is the identification map (recall $\mathscr{H} = \mathscr{H}_E$ as sets and have equivalent topologies), and $P = I - P_0$ is the $\mathscr{H}$ orthogonal projection into the complement of the null space of $A(D)$. Clearly, $\tilde{W}^{\pm}(s)$ are bounded maps from $P\mathscr{H}$ into $\mathscr{H}_E$. The solutions of (4.1) for the analogues of (4.12), (4.13) with zero initial conditions may be represented as

$$u_1(x, t) = \int_0^t U_0(t - s) f_1(s)\, ds \tag{4.16}$$

$$u(x, t) = \int_0^t U(t - s) f(s)\, ds, \tag{4.17}$$

the integrals converging absolutely in the norms of $\mathscr{H}$ and $\mathscr{H}_E$, respectively. Thus

$$\| u - Ju_1 \|_E \leqslant \int_0^t \| U(t-s) f(s) - JU_0(t-s) f_1(s) \|_E \, ds \tag{4.18}$$

$$\leqslant \int_0^{\pm\infty} \| f(s) - U(s-t) \, JU_0(t-s) f_1(s) \|_E \, ds. \tag{4.19}$$

Now suppose $f(s) = \tilde{W}^+(s) f_1(s)$ or $\tilde{W}^-(s) f_1(s)$. Then the dominated convergence theorem (observe that the integrand in (4.19) is bounded by $2 \| f_1(s) \|_E \in L_1(-\infty, \infty)$ if $f_1 \in (L_{2,\alpha}(\mathbb{R}, \mathscr{H}_E))$ shows that

$$\mathrm{EE}_{\pm\mu} = 0$$

for such solutions of (4.1).

Now let $u_2(x, t)$ be a solution of (4.1) under the hypothesis (4.2) ($\gamma > \frac{1}{2}$) and (4.3). Then for f in (4.1) equal to f_2,

$$\begin{aligned} &\| u_2 - u \|_E \\ &\qquad \leqslant \int_0^t \| U(s, t) f_2(s) - U(t-s) f(s) \| \, ds \\ &\qquad \leqslant \int_0^{\pm\infty} \| f_2(s) - U(t, s) \, U(t-s) f(s) \| \, ds \\ &\qquad = \int_0^{\pm\infty} \| W^\pm(s) f(s) - U(t, s) \, U(t-s) f(s) \| \, ds \to 0 \qquad \text{as} \quad t \to \infty. \end{aligned}$$

Therefore, provided f_2 is in the range of $W^\pm(s) \, \tilde{W}^\pm(s)$, the solutions of (4.1) decay locally in t.

THEOREM 4.4. *If $f(s)$ in* (4.1) *lies in the range of $W^\pm(s) \, \tilde{W}^\pm(s)$, then the solution of* (4.1) *decays locally in t ($EE_{\pm\mu} = 0$).*

The value of such a result depends in part on knowing the ranges of the operators $\tilde{W}^\pm(s)$ and $\tilde{W}^\pm(s)$. This is a difficult problem for two reasons. First, the fact that the $\lambda_j'(\omega)$ may vanish for some ω and not others creates difficulties in any "stationary" attempt to characterize the ranges of the operators $\tilde{W}^\pm(s)$. (See [30] for a proof in the strongly propagative case when $B = 0$. See also [24] in case the λ_j have certain smoothness properties, $B = 0$.) Second, a rather nasty circumstance can take place when $B \neq 0$. *This is that the set* $\{\lambda(x) | \det(C(x) - \lambda(x) I) = 0\}$ *may be injected into the spectrum of $E(x)^{-1} A(D)$ through the null space of $A(D)$.* (It may be in modified form.) The key step in a stationary theory of the $W^\pm$ is the proof of a *limiting absorption principle.* This injection of singular continuous spectrum (and possibly eigenvalues) makes such a principle unlikely for small values of μ. For some examples and a general result see the Appendix.

For $W^{\pm}(s)$ the characterization of the range depends on solving (4.1) for imaginary λ. This means either Theorem 4.1 or Theorem 4.2 must be employed. Then Theorem 4.1 of [14] may be used together with (4.7).

We summarize the results of this section in Table I.

5. Mean Properties of Solutions In Media Containing Temporally Inhomogeneous Barriers

We wish to study the existence and asymptotic properties of solutions to the equation

$$\frac{\partial^2 v}{\partial t^2} = \mathbf{c}(x)^2 \Delta v + \mathbf{q}(x, t)\, v - \lambda v + f(x, t). \tag{5.1}$$

Table I

Conditions	$L_2(t)$	$L_{2,-\beta}(t)$	$L_2^{\text{loc}}(t)$	Decays as $\lambda \to \infty$	Decays as $\mu \to \infty$	Local decay in t	Property f must have for local decay of u
Constant coefficients f in $L_{2,\alpha}$	If λ not pure imag.	OK	OK	OK	OK	OK	$Pf = 0$
Coefficients depend only on ×	λ real	OK	OK	OK	OK	If $E(x)$ and $B(x)$ decay	$Pf = 0$
B satisfies (4.2) only, $\gamma > 1$	λ real and large	λ real and large or C small	λ real and large or C small	OK	If C is small	?	?
B satisfies (4.2), $\gamma > 1$ continuous (t)	λ real and large	OK	OK	OK	OK	?	?
B satisfies (4.2), $\gamma \geqslant 0$ continuous (t)	λ real and large	λ real and large	OK	OK	OK	?	?
B obeys (4.2), (4.3), $\gamma > \frac{1}{2}$	λ real	OK	OK	OK	OK	OK	f in range of $W_{\pm}\, \tilde{W}_{\pm}$

We shall write $\bar{K}_2$ for the formal operator $\partial_t^2 - \mathbf{c}(x)^2 \Delta - \mathbf{q}(x, t)$ (see [9]). We shall consider $\bar{K}_2$ as an operator on $L_2(\mathbb{R}, \mathscr{H}_c)$ or $L_2(\mathbb{R}, \mathscr{H}_{c, \Omega})$ with appropriate boundary conditions as in Section 2 above. We assume that there is a real valued function $\mathbf{q}(x)$ for which $\mathbf{c}(x)^2 \Delta + \mathbf{q}(x)$ is selfadjoint and bounded above and that $\mathbf{q}(x, t)$ oscillates about such a function in that there exist positive constants δ and C such that

$$|\mathbf{q}(x, t) - \mathbf{q}(x)| \leqslant C(1 + |t|)^{-1-\delta} \tag{5.2}$$

for almost all x and t. If $\mathbf{q}(x, t)$ is real valued then (5.2) implies that $\bar{K}_2$ is essentially selfadjoint.

We denote the quantity $\mathbf{q}(x, t) - \mathbf{q}(x)$ by $\mathbf{B}$. Then the following chain of implications holds:

$$\begin{aligned} (\bar{K}_2 + \lambda)\, u &= f \\ (\bar{K}_1 + \lambda)^{-1} (\bar{K}_2 + \lambda)\, u &= (\bar{K}_1 + \lambda)^{-1} f \\ (I + (\bar{K}_1 + \lambda)^{-1} \mathbf{B})\, u &= (\bar{K}_1 + \lambda)^{-1} f \\ u &= (I + (\bar{K}_1 + \lambda)^{-1} \mathbf{B})^{-1} (\bar{K}_1 + \lambda)^{-1} f. \end{aligned} \tag{5.3}$$

As before we consider $f \in L_{2, \alpha} (\mathbb{R}, \mathscr{H}_c)$ for some $\alpha > \frac{1}{2}$, and we wish to check conditions under which $(I + (K_1 + \lambda)^{-1} \mathbf{B})$ is invertible on $L_{2, -\beta}(\mathbb{R}, \mathscr{H}_c)$. We know from Section 1 that

$$\|(\bar{K}_0 + \lambda + \mu)^{-1}\|_{L_{2,\alpha} \to L_{2,-\beta}} \leqslant \frac{c}{|\lambda + \mu|} \frac{1}{2}.$$

Therefore for $\lambda > (2a + C_1)^2$, $(I + (\bar{K}_2 + \lambda)^{-1} \mathbf{B})$ is invertible on $L_{2, -\beta}(\mathbb{R}, \mathscr{H}_c)$.

THEOREM 5.1. *For α, $\beta > \frac{1}{2}$ but sufficiently close to $\frac{1}{2}$ so that $-1 + \delta + \beta \leqslant -\alpha$ [δ from (5.2)] we have that for each $f \in L_{2, \alpha}(\mathbb{R}, \mathscr{H}_c)$, there exist $u_\pm \in L_{2, -\beta}(\mathbb{R}, \mathscr{H}_c)$ such that $u_\pm$ are solutions to (5.1) when $\lambda > (2a + C_1)$ as above. $\mathbf{q}(x, t)$ may have complex values. Furthermore, $u_\pm$ decay (in the mean sense) as $\lambda \to \infty$.*

The proof is contained in the discussion above.

For real valued $\mathbf{q}$, a stronger statement can be made regarding the two operators $\bar{K}_1$ and $\bar{K}_2$. Suppose $\bar{K}_2$ is selfadjoint on $L_2(\mathbb{R}, \mathscr{H}_{c, \Omega})$ with either of the boundary conditions of Section 3 assumed to hold.

We wish to show the existence of the *wave operators* for K_1 and K_2 in $L_2(\mathbb{R}, \mathscr{H}_{c, \Omega})$. These operators are defined by the strong limits [4], [8] again

$$W_\pm = \lim_{\sigma \to \pm\infty} \exp(iK_2 \sigma) \exp(-iK_1 \sigma). \tag{5.4}$$

We need only show that these limits exist for a dense set of functions in $L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$. To demonstrate the existence of these operators we shall apply the Cook criterion again. We shall attack the important problem of characterizing the ranges of W_+ and W_- by another method based on the limiting absorption principle as in Section 4. We shall establish that under certain conditions the following two integrals are finite for a certain choice of $f(x, t)$ forming a dense set in $L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$:

$$\int_T^\infty \|(K_1 - K_2)\, e^{-i\sigma K_1} f\|\, d\sigma$$

$$\int_{-\infty}^T \|(K_1 - K_2)\, e^{-i\sigma K_1} f\|\, d\sigma.$$

We consider the first integral briefly. It may be written as

$$\int_T^\infty \left\{ \int_{-\infty}^\infty \int_{\mathbb{R}^n} |\mathbf{q}(x, t) - \mathbf{q}(x)|^2\, |(\exp(-i\sigma\, K_1 f))(x, t)|^2\, dx\, dt \right\}^{1/2} d\sigma.$$

We note that

$$\exp(-i\sigma K_1) = \exp(-i\sigma K_0)\exp(-i\sigma(-A_{D,N} - \mathbf{q})).$$

Let us assume that $f(x, t) = f_1(x) f_2(t)$. Write $S(\sigma)$ for $\exp(-i\sigma K_0)$. It is well known that

$$(S(\sigma) f)(t) = (4\pi\sigma i)^{-1/2} \int_{-\infty}^\infty \exp(-(t-s)^2\, (4i\sigma)^{-1}) f_2(s)\, ds.$$

By (5.2), we may bound the integral with lower limit T by

$$\int_T^\infty \left\{ \int_{-\infty}^\infty ((1 + |t|)^{-1-\delta})^2\, |S(\sigma) f_2(t)|^2\, dt \right\}^{1/2} d\sigma. \tag{5.5}$$

For $f_2(t)$ we take the special function $\exp(-\gamma(t-\omega)^2)$, where $\gamma > 0$ and ω are arbitrary. A direct computation reveals that for this f_2,

$$S(\sigma) f_2 = (1 + 4i\gamma\sigma)^{-1/2} \exp(-\gamma(t-\omega)^2\, (1 + 4i\gamma\sigma)^{-1}).$$

We have

$$\int_T^\infty \int_{-\infty}^\infty ((1 + |t|)^{-1-\sigma})^2\, (1 + 16y^2\sigma^2)^{-1/2} \exp(-2y(t-\omega)^2$$
$$\times (1 + 16\gamma^2\sigma^2)^{-1})\, dt^{1/2}\, d\sigma.$$

We integrate by parts twice in the t integral to obtain (write g for $((1+|t|)^{-1-\delta})^2$)

$$\int_T^\infty \int_{-\infty}^\infty \left(\iint g\right) 16t^2\gamma^2(1+16\gamma^2\sigma^2)^{-5/2} \exp(-2\gamma(t-\omega)^2 \\ \times (1+16\gamma^2\sigma^2)^{-1})\, dt^{1/2}\, d\sigma.$$

This is finite if $\delta > \frac{3}{2}$ in (5.2). The argument for the other T integral is identical. Thus, noting that functions of the form $\exp(-\gamma(t-\omega)^2)$ are dense in $L_2(\mathbb{R})$ and finite linear combinations of functions $f_1 f_2$ are dense in L_2 $(\mathbb{R}, \mathscr{H}_{c,\Omega})$ $(f_2 \in \mathscr{H}_{c,\Omega})$, we obtain the existence of the operators $W_\pm$. A direct consequence of Theorem 4.1 and Theorem 5.1 of [5] is the following result:

THEOREM 5.2. *Let E_1 and E_2 be the spectral measures associated with K_1 and K_2, respectively. The notation E_i^{ac} means that part of E_i which is absolutely continuous. Λ is a compact subset of $\lambda > (2a+C)^2$ and $\delta > \frac{3}{2}$. Then the limits*

$$(W_\pm(\Lambda) f, g) = \lim_{\sigma \to \pm\infty} (\exp(i\sigma K_2) \exp(-i\sigma K_1)\, E_1^{\mathrm{ac}}(\Lambda) f, g) \tag{5.6}$$

exist and define isometries from $E_1^{\mathrm{ac}}(\Lambda)\, L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$ onto $E_2^{\mathrm{ac}}(\Lambda)$ $L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$. The operators $W_\pm(\Lambda)$ define similarity transformations intertwining $\bar{K}_1$ and $\bar{K}_2$ (or at least the parts in $E_i^{\mathrm{ac}}(\Lambda)$). Furthermore, $W_\pm(\Lambda)$ is just the restriction of $W_\pm$ to $E_i^{\mathrm{ac}}(\Lambda)$.

If Ω has special properties, for example, it may be an "exterior" domain with boundary having the "local compactness property" (see [28]), then we can consider $A_{D,N} + \mathbf{q}(x, t)$ as perturbations of Δ in $\mathbb{R}^n$ *provided* $\mathbf{c}(x)$ *is bounded away from zero* and $\mathbf{q}(x)$ and $\mathbf{c}(x) - 1$ vanish at $|x| = \infty$ at an appropriate rate. Results similar to those above may be obtained. We shall not consider this problem here.

It is possible to relate solutions given in Theorem 5.1 (in the selfadjoint case) to those (defined by the group generated by K_2) of the equation

$$i\partial_\sigma v = K_2 v + f(x, t)\, e^{i\sigma\lambda}.$$

The distributional Fourier transform (in σ) of this equation is exactly (5.1). $\Phi(v)$ then defines a solution of (4.1) which decays in a weak sense in t: there exists a sequence $\{t_n\} \to \infty$ such that $\Phi(v)$ decays in mean. This follows from the fact that $\Phi(v) \in L_2(\mathbb{R}, \mathscr{H}_{c,\Omega})$.

Finally, we point out that the results in this section hold when $\Omega = \mathbb{R}^n$, that is, when $\mathscr{H}_{c,\Omega} = \mathscr{H}_c$.

We note that the idea of Theorem 4.2 above may be applied to show that when $\mathbf{q}(x, t)$ has the appropriate operator continuity in t the restrictions on λ in the theorem may be relaxed so that we just require $\lambda > 0$. The details are left to the reader.

6. Time-Dependent Energy Coefficients and Time-Dependent Sound Speeds

In case the coefficients E, B, c, q are smooth in t, then the existence of solutions in L_2^{loc} for (0.1)–(0.2) may be shown for $\lambda = 0$. However, the methods generally used in such problems are quite different than ours, relying mostly on the well-known results of Kato (see [7], [25] for a nice example of these methods). With the hypothesized lack of smoothness for the coefficients it is difficult to obtain solutions in the standard sense. Instead, we shall consider the "band limited" signals u and v with cut-off frequencies a, b, $-\infty < a < b < \infty$, where $\Lambda = (a, b)$ (see [19]), to the equations

$$\frac{\partial u}{\partial t} = E(x, t)^{-1} \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + B(x)\, u + \lambda u + f(x, t) \tag{6.1}$$

$$\frac{\partial^2 v}{\partial t^2} = \mathbf{c}(x, t)^2\, v + \mathbf{q}(x)\, v - \lambda v + g(x, t). \tag{6.2}$$

We suppose that the functions $\mathbf{c}(x)$, $\mathbf{q}(x)$, $E(x)$, and $B(x)$ satisfy the boundedness and symmetry requirements used previously in Section 3. Write $E(\Lambda)$ for the spectral mesure of $E(x)^{-1} A(D) + C(x)$ on $\mathscr{H}_E$. The reader is warned not to confuse the two E's!

For (6.1) the band limitation problem is the requirement that $f \in I \otimes E(\Lambda)\, L_2(\mathbb{R}, \mathscr{H}_E)$ for some bounded set Λ and u is sought in the same space.

For (6.1) we shall apply the following computation, considering

$$i \frac{\partial u}{\partial t} - E(x, t)^{-1} A(D)\, u - C(x)\, u - i\lambda u = K_3 u - i\lambda u. \tag{6.3}$$

If $K_3 u - i\lambda u = f$ under the band limitation then

$$\begin{aligned} &I \otimes E(\Lambda)(K_1 - i\lambda)^{-1} (K_3 - i\lambda)\, I \otimes E(\Lambda)\, u \\ &\quad = I \otimes E(\Lambda)(K_1 - i\lambda)^{-1}\, I \otimes E(\Lambda) f \end{aligned}$$

and thus,

$$\begin{aligned} &(I \otimes E(\Lambda) + I \otimes E(\Lambda)(K_1 - i\lambda)^{-1} (E(x, t)^{-1} - I \otimes E(x)^{-1}) \\ &\quad \times I \otimes E(\Lambda)\, A(D))\, u = (K_1 - i\lambda)^{-1}\, I \otimes E(\Lambda) f. \end{aligned} \tag{6.4}$$

For brevity, we write f_Λ for $I \otimes E(\Lambda) f$, etc. Now consider

$$(K_1 - i\lambda)_\Lambda^{-1} \,(E(x, t)^{-1} - E(x)^{-1})\, E(x)\, E(x)^{-1}\, A(D)_\Lambda. \tag{6.5}$$

In order to make our formal computation rigorously reversible, we must define the action of this operator, and make some determination of its dependence on λ. The operator $A(D)_\Lambda$ (that is, $A(D)\,E(\Lambda)$) is bounded on $\mathscr{H}_E$ (we remark that *in case $A(D)$ is an elliptic operator, this implies that band limited solutions are very smooth*). To see this, note that $(E(x)^{-1} A(D) + C(x))\, E(\Lambda)$ is bounded and $A(D)_\Lambda = E(x)(E(x)^{-1} A(D) + C(x))\, E(\Lambda) - E(x)\, C(x)\, E(\Lambda)$. We apply this fact by considering the operator

$$A(D)(K_1 - i\lambda)_\Lambda^{-1}. \tag{6.6}$$

This is equivalent to

$$-E(x)(K_0 + i\lambda)_\Lambda^{-1}, \tag{6.7}$$

which can be written as

$$I \otimes (-E(x))(I \otimes I - (K_0 - i\lambda) \otimes I \int_\Lambda (K_0 - i\lambda + \mu)^{-1} \otimes dE(\mu))$$

or after a short computation

$$I \otimes (-E(x))\left(\int_\Lambda \mu (K_0 - i\lambda + \mu)^{-1} \otimes dE(\mu)\right). \tag{6.8}$$

The integrand may be written as the trivial pseudodifferential oprator

$$\Phi^*(\mu(1/(p + i\lambda + \mu)))\, \Phi \otimes I, \tag{6.9}$$

which is clearly bounded as an operator valued function of μ if the range of values for μ is bounded (p is the dummy variable in Φ^*) and $\lambda \neq 0$ (this indicates why stability fails for the complete solution). Thus, $(K_0 - i\lambda)_\Lambda^{-1}$ defines a uniformly bounded map from $L_2(\mathbb{R}, \mathscr{H}_E)$ to $L_2(\mathbb{R}, \mathscr{H}_E)$. If $(E(x, t)^{-1} - E(x)^{-1})$ maps $L_{2,-\beta}$ to $L_{2,\alpha}$ we may allow λ to take pure imaginary values and the left-hand side of (6.4) will make sense as an operator on $L_{2,-\beta,\Lambda}$ by (3.19) and (6.11). Looking at

$$I \otimes (-E(x)_\Lambda)(\Phi^* \otimes I \int_\Lambda (\mu/(p - i\lambda + \mu))) \otimes dE(\mu)\, \Phi \otimes I \tag{6.10}$$

we see that as $\lambda \to \pm\infty$, the integral vanishes.

THEOREM 6.1. *If*

$$|E(x,t)^{-1}-E(x)^{-1}|\leqslant C(1+|t|)^{-\gamma} \tag{6.11}$$

then band limited solutions to (6.1) exist for λ sufficiently large and $\gamma\geqslant 0$. These solutions belong to $L_2(\mathbb{R}, \mathscr{H}_E)$ and therefore satisfy stability criteria considered already. Furthermore, they decay for $\lambda\to\pm\infty$. If λ takes imaginary values, solutions exist in $L_{2,-\beta,\Lambda}$ for C sufficiently small and $\gamma>1$.

THEOREM 6.2. *Band limited solutions to the equation*

$$(K_4-i\lambda)\,u=f, \qquad f\in L_2(\mathbb{R}, \mathscr{H}_E)_\Lambda \tag{6.12}$$

exist for λ sufficiently large where

$$i\frac{\partial u}{\partial t}-E(x,t)^{-1}\,A(D)\,u-C(x,t)\,u=K_4 u \tag{6.13}$$

and $C(x,t)$ satisfies (4.2) *with $\gamma\geqslant 0$. If the C in* (4.2) *and* (6.5) *is small and $\gamma>1$, then band limited solutions exist in $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_E)_\Lambda$* to (6.1) *when λ is imaginary.*

The proof is similar to the previous argument and will not be given here. *If $E(x,t)$ and $B(x,t)$ satisfy the fall-off conditions* (4.2) *and* (6.5) ($\gamma>1$) *and are continous as operator valued functions of t on $\mathscr{H}_E$, then distributional band limited solutions for $\lambda=i\mu$ may still be constructed in $L_{2,-\beta}$ following the method of Theorem* 4.2. Since the details are similar, we omit them here. Solutions exist in L_2^{loc} if $1\geqslant\gamma>0$. The method of proof for the following is completely analogous to the previous results except that the estimates required are those related to v from Section 3.

THEOREM 6.3. *Suppose that*

$$|\mathbf{c}(x,t)-\mathbf{c}(x)|\leqslant C_1(1+|t|)^{-\gamma_1}$$

$$|\mathbf{q}(x,t)-\mathbf{q}(x)|\leqslant C_2(1+|t|)^{-\gamma_2}$$

then for γ_1 and $\gamma_2>1$ and for $\lambda>0$ sufficiently large, there exist band limited solutions to (0.2) *which lie in $L_{2,-\beta}(\mathbb{R}, \mathscr{H}_{c,\Omega})_\Lambda$, when $f\in L_{2,\alpha}(\mathbb{R}, \mathscr{H}_{c,\Omega})_\Lambda$. Furthermore, these solutions decay as $\lambda\to\infty$.*

The time decay of solutions to the equations considered in this section is not apparent except under very special conditions. However, solutions to (6.1) for λ real do decay *on the average* like $t^{-\varepsilon}$ where $\varepsilon>\frac{1}{2}$. If the variation of E and B shuts off after a finite time and becomes constant in x as well, then the elementary results of Section 1 imply time decay for any solution for appropriate f and decay for $\lambda = -i\mu$ as $\mu \to \pm\infty$. Some further results can be given if smoothness conditions are imposed on the coefficients. We do not consider this here.

APPENDIX: REMARKS ON THE SPECTRUM OF $E^{-1}A(D)+C$

A recurring, very important phenomenon in these systems is the existence of *spectral barriers.* This property does not exist for elliptic systems (see [3]), but it becomes important for the study of both steady-state and transient solutions in the equations of classical physics, for example, the Maxwell equations in a domain with finite conductivity. Simply stated, the phenomenon arises because the spectrum of B may be injected into the spectrum of the full operator through the null space of $A(D)$. Our intent is to give a few examples of this in a number of different settings and then state some more general results for systems.

First, to indicate that spectral barriers occur in many different problems, consider the ordinary differential system

$$T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^2}{dx^2} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defined on $(0, \pi)$ with zero boundary conditions. (This can be reduced to a first-order system easily.)

This operator, in the eigenvalue equation, reduces to

$$v = \lambda u$$

$$v'' + u = \lambda v$$

or

$$\lambda_n = \frac{-n^2 + \sqrt{n^4 + 4}}{2}, \qquad n = 1, 2, \ldots \text{ (see Fig. 2).}$$

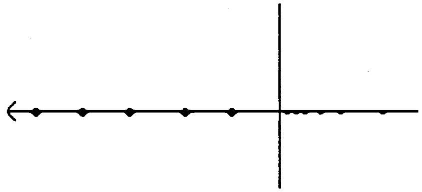

FIG. 2. Eigenvalues of T.

To illustrate a second sort of behavior, consider

$$T_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^2}{dx^2} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of T_1 are

$$\lambda = \frac{-n^2 + \sqrt{n^2 - 4}}{2}, \qquad n = 1, 2, 3, \ldots,$$

which are real except at $n = 1$, where we have the eigenvalues of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ perturbed by rotations: see Fig. 3.

A third example shows a more drastic injection of spectrum:

$$T_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \frac{d^2}{dx^2} + \phi(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -\infty < x < \infty.$$

T_2 acts on $L_2(\mathbb{R}, \mathbb{C}^2)$.

Here $T_2\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \lambda \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ reduces to

$$\phi(x)\, u = \lambda u$$
$$-v'' + \phi(x)\, v = \lambda v.$$

λ is in the resolvent set of T_2 only when

$$T_2 \begin{pmatrix} u \\ v \end{pmatrix} - \lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

has unique L_2 solutions ${}^t(u, v)$ for ${}^t(f, g)$ (here ${}^t(\ )$ indicates transpose) in L_2. Taking $g = 0$ shows that the spectrum of T_2 includes *the essential range of* $\phi(x)$. See Fig. 4. (Here we assume $\phi(x) \to 0$ as $|x| \to \infty$.)

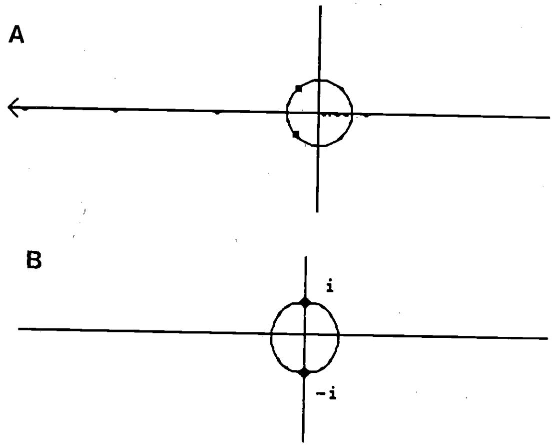

FIG. 3. (A) Eigenvalues of T_1. (B) Eigenvalues of perturbation.

The operator T_2 bears some closer scrutiny. If ϕ is real valued, it may be hoped that a spectral theory for T_2 can be developed analogous to that of Schrödinger operators. A first step in this process would be the proof of a *limiting absorption principle.* This means the extension of the resolvent $\lambda \to (T_2 - \lambda I)^{-1}$ to the absolutely continuous spectrum in some appropriate topology. Here

$$(T_2 - \lambda I)^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \dfrac{f}{\phi(x)-\lambda} \\ R(\lambda)\, g \end{pmatrix}.$$

$R(\lambda)$ is the resolvent of the Schrödinger equation $-v'' + qv - \lambda v = g$.

Consider

$$\lim_{\varepsilon \to 0^+} \int_A^B (T_2 - (\lambda + i\varepsilon))^{-1}\begin{pmatrix} f \\ g \end{pmatrix} - (T_2 - (\lambda - i\varepsilon))^{-1}\begin{pmatrix} f \\ g \end{pmatrix} d\lambda,$$

which defines the spectral resolution of T. This can be written as

$$\lim_{\varepsilon \to 0^+} \int_A^B \begin{pmatrix} \dfrac{f}{\phi - (\lambda + i\varepsilon)} - \dfrac{f}{\phi - (\lambda - i\varepsilon)} \\ R(\lambda + i\varepsilon)\, g - R(\lambda - i\varepsilon)\, g \end{pmatrix} d\lambda.$$

Thus if $E(\lambda)$ is the spectral resolution of T_2 and assuming q is real valued,

$$\left(E(A, B)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)\pi$$

$$\varepsilon \lim_{\varepsilon \to 0} \int_A^B \begin{pmatrix} \dfrac{u_1}{\phi - (\lambda + i\varepsilon)} - \dfrac{u_1}{\phi - (\lambda - i\varepsilon)} \\ R(\lambda + i\varepsilon)\, u_2 - R(\lambda - i\varepsilon)\, u_2 \end{pmatrix} \circ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dx$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} u_1(x)\, \overline{v_1(x)} \int_A^B \frac{2i\varepsilon}{(\phi(x) - \lambda)^2 + \varepsilon^2}\, d\lambda\, dx$$

$$+ \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_A^B (R(\lambda + i\varepsilon)\, u_2 - R(\lambda - i\varepsilon)\, u_2)(v(x))\, d\lambda\, dx$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} -2iu_1(x)\, \overline{v_1(x)}\, \chi_{(A, B)}(\phi(x))\, dx$$

$$+ \int_{-\infty}^{\infty} E_1(A, B)\, u_2(x)\, \overline{v_2(x)}\, dx,$$

E_1 being the spectral resolution for $R(\lambda)$.

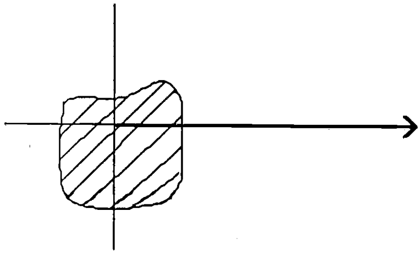

FIG. 4. Essential spectrum of T_2 for complex phi.

It follows that the spectral resolution for T_2 essentially reduces to

$$\chi_{(A,B)}(\phi(x)) = \begin{cases} 1, & x \in \{x \mid A < \phi(x) < B\} \\ 0, & \text{otherwise} \end{cases}$$

on the subspace $\binom{f}{0}$. This will not be absolutely continuous if ϕ takes a set of positive measure to a set of measure zero, or to a nonmeasurable set.

We note that while the spectral resolution is well defined, in general *no limiting absorption principle is known except on the subspace* $\binom{0}{g}$.

Without belaboring the point we give one more illustration to show that disentangling the spectrum may be more complex. Define

$$T_3 = \begin{pmatrix} 0 & -0 \\ 0 & -1 \end{pmatrix} \frac{d^2}{dx^2} + \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ 0 & 0 \end{pmatrix}, \qquad -\infty < x < \infty.$$

Let $R(x, y, \lambda)$ be the resolvent kernel of $-d^2/dx^2$. Then

$$(T_3 - \lambda I)^{-1} \begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{cases} \dfrac{f(x) - \phi_2(x) \int_{-\infty}^{\infty} R(x, y, \lambda)\, g(z)\, dy}{\phi_1(x) - \lambda} \\ \displaystyle\int_{-\infty}^{\infty} R(x, y, \lambda)\, g(y)\, dy \end{cases}.$$

Here, there is no limiting absorption principle in the essential range of ϕ_1 (this set contains the singular spectrum of T_3).

Generally then, a *spectral barrier* occurs when the perturbation injects singular continuous spectrum or many eigenvalues into the perturbed operator. The perturbation may inject eigenvalues into the system when it (the perturbation) has stationary points in its range. In T_3 this would occur, for example, when ϕ_1 takes a constant value on some neighborhood.

It is abundantly clear that some rather special hypotheses are required for the spatial part of nonelliptic systems like (2.1) to satisfy a limiting absorption principle (at least for low frequencies). Such hypotheses must involve both the structure of $A(D)$ and B. The situation is actually simpler when B is nonlocal. However, we are interested in the local case here.

We shall require certain auxiliary spaces. Define (α real)

$$L_{2,\alpha}(\mathbb{R}^n, \mathbb{C}^m) = \left\{ f \,\middle|\, \int_{\mathbb{R}^n} (1+|x|^2)^\alpha \, |f(x)|^2 \, dx < \infty, f : \mathbb{R}^n \to \mathbb{C}^m \right\}.$$

Below, $\mathbb{C}^\pm = \{\lambda \mid \pm \text{im}\, \lambda > 0\}$.

We write $\mathscr{H}_1 = PL_2(\mathbb{R}^n, \mathbb{C}^m)$, $\mathscr{H}_0 = P_0 L_2(\mathbb{R}^n, \mathbb{C}^m)$, $\mathscr{H} = L_2(\mathbb{R}^n, \mathbb{C}^m)$, etc. Then we may also write $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. An operator B is said to decompose on $\mathscr{H}$ if $B = B_1 \oplus B_0$, $PB_1P = B_1$, $P_0 B_0 P_0 = B_0$. Thus if B decomposes, $B = B_{11} \oplus B_{00}$.

THEOREM A. *The following is a sufficient condition for the operator* $T_4 = E^{-1}(x)\, A(D) + E^{-1}(x)\, B(x)$ *to verify a limiting absorption principle. B decomposes and* $B_0 \equiv 0$ *with B bounded, E uniformly positive definite and bounded, and* $|B(x)| = O(|x|^{-1-\varepsilon})$, $|I - E(x)| = O(|x|^{-1-\varepsilon})$ *as* $|x| \to \infty$ *for some* $\varepsilon > 0$. *It is assumed that* $A(D)$ *is strongly propagative.*

Proof. We start by studying the resolvent of the operator in question, just as in the examples. Suppose

$$T_4 u - \lambda u = E(x)^{-1} A(D)\, u + E(x)^{-1} B(x)\, u - \lambda u = f.$$

Then

$$A(D)\, u + B(x)\, u - \lambda E(x)\, u = E(x) f.$$

Operate on both sides with $(A(D) - \lambda)^{-1}$ to get

$$[I + \lambda (A(D) - \lambda)^{-1} (I - E + B/\lambda)]\, u = (A(D) - \lambda)^{-1} Ef.$$

Now we study

$$[I + \lambda (A(D) - \lambda)^{-1} (I - E + B/\lambda)]$$

to determine when it is invertible. First, note that

$$(P_0 + P)(A(D) - \lambda)^{-1} = -P_0/\lambda + P(A(D) - \lambda)^{-1}.$$

Thus we can rewrite (1.2) as

$$[I - P_0(I - E + B/\lambda) + \lambda P(A(D) - \lambda)^{-1} (I - E + B/\lambda)].$$

The asymptotics on $I - E$ and B show that $(I - E + B/\lambda)\, L_{2,-\beta}(\mathbb{R}^n, \mathbb{C}^m) \subset L_{2,\alpha}(\mathbb{R}^n, \mathbb{C}^m)$ for some α and β greater than $\frac{1}{2}$ (but close to $\frac{1}{2}$). From (2.9) of [29] it is easily seen ($\alpha, \beta > \frac{1}{2}$) that $P(A(D) - \lambda)^{-1}$, thought of as mapping $L_{2,\alpha} \to L_{2,-\beta}$, is continuous (Hölder continuous in fact) in $\mathbb{C}^\pm$ with compact values and has continuous extensions $P(A(D) - \lambda)^{-1}_{\pm}$ to $\overline{\mathbb{C}^+}$

and $\overline{\mathbb{C}^-}$, except for $\lambda = 0$. $P(A(D)-\lambda)_{\pm}^{-1}$ assumes compact values. This all implies that

$$\lambda P(A(D)-\lambda)_{\pm}^{-1}\,(I-E+B/\lambda)\colon L_{2,-\beta} \to L_{2,-\beta}$$

is compact for each $\lambda \neq 0$, and

$$\lambda \to \lambda P(A(D)-\lambda)_{\pm}^{-1}\,(I-E+B/\lambda)$$

is a continuous compact operator valued map. The first resolvent equation shows that it is even analytic in $\mathbb{C}^{\pm}$. To show (1.3) is invertible for λ real it remains to check that (see [19])

$$I - P_0(I-E+B/\lambda) \text{ is invertible.}$$

This we can write as

$$I - P\left(I - E + \frac{(\operatorname{Re}\lambda)\,B}{|\lambda|^2} - \frac{(\operatorname{im}\lambda)\,B}{|\lambda|^2}\right).$$

Now suppose (1) is true. Then (1.4) reduces to

$$I - P_0(I-E),$$

which is boundedly invertible on $L_{2,-\beta}$ (see [19]). By the analytic Fredholm theory, (1.3) is invertible, except at a discrete set of λ in $\mathbb{C}^{\pm}$ and a set which is nowhere dense and of linear measure zero in $\mathbb{R}$.

The reader will note that we have not assumed $E^{-1}(x)\,A(D) + (E^{-1}(x)\,B(x))$ is selfadjoint. Besides, we know from the examples that selfadjointness will not help particularly.

We point out a fault in the proof above. It is crude from the standpoint of determining the resolvent set. It automatically excludes a ball of radius

$$\frac{\|P_0 B\|}{\|I - P_0(I-E)\|}$$

from consideration, when for selfadjoint operators $\mathbb{C}^{\pm}$ is contained in the resolvent set (see Fig. 5).

The operator $A = E(x)^{-1}\,A(D) + E(x)^{-1}\,B(x)$ will be selfadjoint on $\mathscr{H}_E$ if $E(x)\,B(x) = B(x)\,E(x)$ and $B(x)^* = B(x)$ for elmost all x. Suppose that B is positive. Then $I - P_0(I-E+B/\lambda)$ is invertible for all $\lambda < 0$ since then we have that $E - B/\lambda$ is positive and the argument noted above in Theorem A holds. *Hence the limiting absorption principle is true for almost all* $\lambda < 0$. A similar statement is true for $B < 0$ and $\lambda > 0$. We could also show that the discrete exceptional points correspond to true L_2 eigenfunctions. We shall not discuss this here. This of course supports our previous intuitive state-

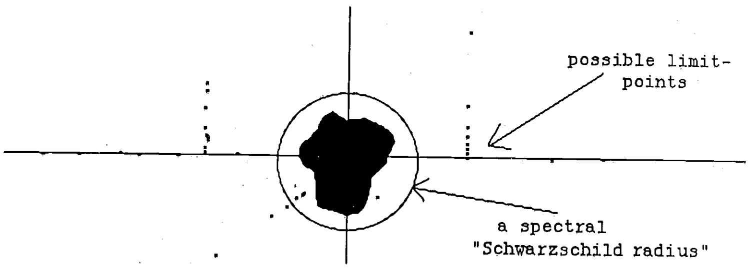

FIG. 5. The nonabsolutely continuous spectrum of A.

ment that the spectrum of B is injected into the spectrum of A through the null space of $A(D)$.

Following the technique of [14] (Theorem 4.1) we can prove the *local* existence and completeness of the wave operators intertwining $A(D)$ and A outside of the spectral barrier.

Finally, we note that according to Theorem A, *no spectral barrier exists when* $A(D)$ *is elliptic.*

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