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Wave Motion in a Conducting Fluid with a Boundary Layer. I. Hilbert Space Formulation

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The wave motion of MHD systems can be quite complicated. In order to study the motion of waves in a perfectly conducting fluid under the influence of an external magnetic field with a boundary layer, we make the simplifying assumption that the pressure is constant (to first order). This is classical "cold plasma" approximation from the physical literature. This is still an interesting system and is not strongly propagative. Alfven waves are still present. The system is further simplified by assuming that the external field is either orthogonal or parallel to the boundary layer. While it may seem presumptuous to claim anything new about this problem the method introduced here is cumulative: it may be extended to more complex problems of the same type. In Part I the appropriate energy preserving boundary conditions are studied. For the parallel field case there are just two possible boundary conditions which preserve energy. For the orthogonal case, there are two one parameter families of energy preserving boundary conditions. One of the boundary conditions for the parallel case is selected and from this boundary condition, it is shown that the relevant operator is selfadjoint and data which propagates is characterized. A crucial density result is then proved.

0. INTRODUCTION

This paper is devoted to the study of linearized equations of magnetohydrodynamics in the setting of media filling a half-space with the presence of a boundary layer. The pressure is assumed to be constant to first order. This is the "cold plasma" condition. It is well known in the MHD literature. Our object in treating this system is to develop tools for more

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complex related problems. Part I (the present work) is devoted to establishing some necessary facts about the system and introducting some methods useful for boundary layer problems of similar type. The results may therefore be of independent interest. This work is based on [S1] in which a problem is treated involving one plane boundary. We also gratefully acknowledge the use of facts developed in [GS], as well as unpublished work by these authors. In the present work, two plane boundaries are present. The structure of [S1] is extended to the present problem. This system is considerably simpler than that of [S1] although it still preserves many of the unique features of that system, including the existence of unbounded slowness surfaces [W]. The hyperbolic part of the linearized equations of MHD *with the assumption of constant pressure* can be written as [S1]

$$\mu \frac{\partial H}{\partial t} = \nabla \times (v \times H_0)$$
$$\rho \frac{\partial v}{\partial t} = (\nabla \times H) \times H_0. \tag{0.1}$$

$\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. The fluid is assumed to fill a half-space written as $\mathbf{R}^3_{-a} = \{x \in \mathbf{R}^3\colon x_3 > -a\}$, $0 < a < \infty$, where the boundary $\{x_3 = -a\}$ is energy preserving while the density and magnetic permeability are given by ρ, ρ_0 and μ, μ_0 in the layer $\mathbf{R}^2 \times (-a, 0)$ and the half-space $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3\colon x_3 > 0\}$, respectively. To formulate this problem, let E_0 and E denote the 6×6 matrices

$$E_0 = \operatorname{diag}(\rho_0 I_{3\times3}, \mu_0 I_{3\times3}), \qquad E = \operatorname{diag}(\rho I_{3\times3}, \mu I_{3\times3}) \tag{0.2}$$

and define $\mathscr{E}(x) = \chi_+(x_3)E_0 + \chi_-(x_3)E$, where χ_+ and χ_- are the characteristic functions of $R_+ = \{x\colon x > 0\}$ and $(-a, 0)$, respectively. Let $A(D)$, $D_j = -i\partial/\partial x_j$, $j = 1, 2, 3$, be the 6×6 matrix differential operator

$$A(D) = \Sigma_j A_j D_j.$$

We have studied the systems closely related to (0.1) in [S1–S4]. It will be useful to write (0.1) in matrix form as

$$\mathscr{E} \frac{\partial u}{\partial t} = \sum_{j=1}^{3} A_j \frac{\partial u}{\partial x_j}. \tag{0.3}$$

The matrices A_j are given by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -h_2 & -h_3 \\ 0 & 0 & 0 & 0 & h_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -h_2 & h_1 & 0 & 0 & 0 & 0 \\ -h_3 & 0 & h_1 & 0 & 0 & 0 \end{bmatrix} \tag{0.4}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & h_2 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & 0 & -h_3 \\ 0 & 0 & 0 & 0 & 0 & h_2 \\ h_2 & -h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_3 & h_2 & 0 & 0 & 0 \end{bmatrix}. \tag{0.5}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & -h_1 & -h_2 & 0 \\ h_3 & 0 & -h_1 & 0 & 0 & 0 \\ 0 & h_3 & -h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{0.6}$$

1. Boundary Conditions

In order to formulate and the solve the initial-boundary problem for system (0.3) it is now necessary to discover those boundary conditions which are energy preserving for this system. Even though this system is not as complex as the ones studied in [S1, S2], it will be necessary to simplify it even further so that more complex computations and results may be carried out later. With this in mind, we shall consider the two special cases $H_0 = (0, 0, h_3)$ and $H_0 = (0, h_2, 0)$ for system (0.3). In the problems we are considering, these two cases correspond to the external magnetic field being orthogonal and parallel to the boundary layer, respectively.

DEFINITION 1.1. A subspace $S(\mathbf{n})$ of $\mathbf{R}^6$ is a maximal conservative boundary space for $A(D)$ in $\mathbf{R}^3_+$ if and only if $\zeta_0 A(\mathbf{n})\zeta = 0$ for all ζ in $S(\mathbf{n})$ and $S(\mathbf{n})$ is maximal with respect to this property. Here $\mathbf{n}$ is to be interpreted as the unit normal to the boundary surface.

We interpret the boundary as a plane parallel to the x_1, x_2 plane. $A(\mathbf{p})$ is the symbol of the matrix operator $A(D)$ and therefore the meaning of $A(\mathbf{n})$ is $A((0, 0, 1))$. The solutions of the equation $\det(A(\mathbf{n}) - \lambda I) = 0$ are needed. These will depend on which of the two choices we make for H_0. We need only consider the value $h_2 = 1$, since each entry of $A(\mathbf{n})$ is a multiple of h_2. The same is also true of h_3. $A_2(\mathbf{n})$ will be the matrix associated with $H_0 = (0, h_2, 0)$. $A_3(\mathbf{n})$ will stand for the other choice. The eigenvalues and a set of corresponding eigenvectors for $A_2(\mathbf{n})$ are:

$$
\begin{aligned}
-1 &\to (0, 0, 1, 0, 1, 0) \\
1 &\to (0, 0, 1, 0, -1, 0) \\
0 &\to (0, 0, 0, 0, 0, 1), \qquad (0, 0, 0, 1, 0, 0), \\
&\ (0, 1, 0, 0, 0, 0), \qquad (1, 0, 0, 0, 0, 0).
\end{aligned}
$$

For $A_3(\mathbf{n})$ we have

$$
\begin{aligned}
-1 &\to (-1, 0, 0, 1, 0, 0), \qquad (0, -1, 0, 0, 1, 0) \\
1 &\to (0, 1, 0, 0, 1, 0), \qquad (1, 0, 0, 1, 0, 0) \\
0 &\to (0, 0, 0, 0, 0, 1), \qquad (0, 0, 1, 0, 0, 0).
\end{aligned}
$$

By the positive eigenvectors we shall mean those which correspond to positive eigenvalues.

LEMMA. *Let $N(A(\mathbf{n}))$, $X(\mathbf{n})$, $Y(\mathbf{n})$ denote, respectively, the null space of $A(\mathbf{n})$, the subspace spanned by the positive eigenvectors of $A(\mathbf{n})$, and the subspace spanned by the negative eigenvectors of $A(\mathbf{n})$. Let ζ_j be any orthonormal base of $N(A(\mathbf{n}))$. Let ξ_j be any base of $X(\mathbf{n})$ which is orthonormal with respect to $A(\mathbf{n})$, i.e., $\xi_i \circ A(\mathbf{n})\xi_j = \delta_{ij}$, and let ν_j be any base of $Y(\mathbf{n})$ orthonormal with respect to $-A(\mathbf{n})$. Then $S(\mathbf{n}) = \operatorname{span}\{\zeta_j, \xi_j + \nu_j\}$ is a maximal conservative boundary space for $A(D)$ and any such boundary space may be constructed in this way.*

To classify these spaces, consider any basis of $X \oplus Y$ for, say, $A_3(\mathbf{n})$. We have $-1 \to \nu_j$, $1 \to \xi_j$ $(j = 1, 2)$. Let e_{i-} and e_{i+} be any such fixed basis.

Then

$$\nu_i = d_{i1}e_{1-} + d_{i2}e_{2-}$$

$$\xi_i = c_{i1}e_{1+} + c_{i2}e_{2+}.$$

In order that the orthonormality conditions be satisfied, it must be that $c_{i1}c_{j1} + c_{i2}c_{j2} = \delta_{ij}$ and the same for $[d_{ij}]$. Hence $[d_{ij}]$ and $[c_{ij}]$ are orthogonal matrices. By letting $C = [c_{ij}]$ and $D = [d_{ij}]$ run through all possible such matrices, we obtain every basis of a maximal conservative boundary space for $A_3(D)$. Two possible orientations exist for such a boundary space. These are determined by the sign of $\det(CD) = \pm 1$.

THEOREM 1.2. *Suppose that* $k_{01}, k_{02}, k_{1+}, k_{2+}, k_{1-}, k_{2-}$ *are orthonormal (in the* $\mathbf{R}^6$ *sense) eigenvectors of* $A_3(\mathbf{n})$ *spanning the nullspace of* A_3, X, *and* Y, *respectively. Then the subspace of* $\mathbf{R}^6$ *given by* $S_3(\mathbf{n}) = \operatorname{span}\{k_{01}, k_{02}, \xi_i + \nu_i, i = 1, 2\}$ *where*

$$\nu_i = d_{i1}k_{1-} + d_{i2}k_{2-}$$

$$\xi_i = c_{i1}k_{1+} + c_{i2}k_{2+}$$

and $[c_{ij}]$ and $[d_{ij}]$ belong to the orthogonal group of dimension 2, $\mathbf{O}(2)$ is a maximal conservative boundary space for A_3, and every such boundary space is obtained by letting C and D run through $\mathbf{O}(2)$.

THEOREM 1.3. *Suppose that* $k_{01}, k_{02}, k_{03}, k_{04}, k_{1+}, k_{1-}$ *are orthonormal (in the* $\mathbf{R}^6$ *sense) eigenvectors of* $A_2(\mathbf{n})$ *spanning the nullspace of* A_2, X, *and* Y, *respectively. Then the subspaces of* $\mathbf{R}^6$ *given by* $S_2(\mathbf{n}) = \operatorname{span}\{k_{01}, k_{02}, k_{04}, k_{1+} \pm k_{1-}\}$ *are maximal conservative boundary spaces for* $A_2(\mathbf{n})$.

The next step is to obtain the associated boundary conditions for these boundary spaces. We start with A_2. Let P_0, P_1, and P_{-1} be the orthogonal projections onto $N(A_2(\mathbf{n}))$, X, and Y, respectively. The projection $Q(\mathbf{n})$ onto $S_2(\mathbf{n})$ is then given by

$$Q(\mathbf{n}) = P_0 + 2^{-1}(k_{1+} + \nu_{1-}) \otimes (k_{1+} + \nu_{1-}).$$

Here we take

$$k_{1+} = \pm 2^{-1/2}(0, 0, 1, 0, -1, 0)$$

and

$$k_{1-} = \pm 2^{-1/2}(0, 0, 1, 0, 1, 0).$$

The kernel of the projection $I - Q(\mathbf{n})$ will determine a boundary operator for $A_2(D)$. Since $I = P_0 + P_1 + P_{-1}$,

$$I - Q(\mathbf{n}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $u_5 = 0$ at the boundary is a correct boundary condition. Also, $u_3 = 0$ is a correct boundary condition.

These conditions are not new; however, this method applies to more complex related systems. We state the result for reference.

THEOREM 1.4. *The energy preserving boundary conditions for $A_2(D)$ at the plane boundary $x_3 = -a$ are $u_5 = 0$ or $u_3 = 0$.*

Now we consider the operator $A_3(D)$. We have that $I - Q(\mathbf{n}) = 2^{-1}(P_1 + P_2 + P_{-1} + P_{-2}) - 2^{-1}(\cos\phi(k_{1+} \otimes k_{1-}) + (-1)^{n+m}\cos\phi(k_{2+} \otimes k_{2-}) + (-1)^{n+1}\sin\phi(k_{2+} \otimes k_{1-}) + (-1)^m \sin\phi(k_{1+} \otimes k_{2-}) +$ transpose of last 4 terms). Here we have taken

$$C = \begin{bmatrix} \cos\theta & (-1)^{n+1}\sin\theta \\ \sin\theta & (-1)^n\cos\theta \end{bmatrix}, \qquad D = \begin{bmatrix} \cos\psi & (-1)^{m+1}\sin\psi \\ \sin\psi & (-1)^m\cos\psi \end{bmatrix}.$$

Now the result is somewhat different from Theorem 1.4. Set $\phi = \theta - \psi$.

THEOREM 1.5. *There are two possible structures for $S_3(\mathbf{n})$ determined by the orientation of $X \otimes Y$. These are given as the kernels of one or the other of two projections, first for $n + m$ even,*

$$\begin{bmatrix} (1+\cos\phi)/2 & 0 & 0 & 0 & (-1)^m(\sin\phi)/2 & 0 \\ 0 & (1+\cos\phi)/2 & 0 & -(-1)^m(\sin\phi)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(-1)^m(\sin\phi)/2 & 0 & (1+\cos\phi)/2 & 0 & 0 \\ (-1)^m(\sin\phi)/2 & 0 & 0 & 0 & (1+\cos\phi)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and then for n + m odd,

$$\begin{bmatrix} (1+\cos\phi)/2 & (-1)^m(\sin\phi)/2 & 0 & 0 & 0 & 0 \\ (-1)^m(\sin\phi)/4 & (1+\cos\phi)/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+\cos\phi)/2 & -(-1)^m(\sin\phi)/2 & 0 \\ 0 & 0 & 0 & -(-1)^m(\sin\phi)/2 & (1+\cos\phi)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Without loss of generality, we assume that m is odd in the projections above. The ratio $(1 - \cos\phi)/\sin\phi$ is set to γ. Then the boundary operators are

First form:

$$B^1_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^1_\gamma = \begin{bmatrix} \gamma & 0 & 0 & 0 & -1 & 0 \\ 0 & \gamma & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The second form:

$$B^2_\gamma = \begin{bmatrix} -1 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 1 & 0 \end{bmatrix}$$

$$B^2_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

2. Technical Matters

The linearized equations of constant pressure magnetohydrodynamics in the layered half-space may be written now as

$$i\partial_t u(x, t) = \mathscr{E}^{-1}(x)A(D)u(x, t) \equiv \Lambda'(D)u(x, t). \tag{2.1}$$

$u(x, t)$ is the six-dimensional vector (v, H), v is the fluid velocity, and H is the magnetic field vector.

We will write $\Lambda'(D)$ as

$$\Lambda'(D) = \chi_+\Lambda^0(D) + \chi_-\Lambda(D) \equiv \chi_+E_0^{-1}A(D) + \chi_-E^{-1}A(D). \quad (2.2)$$

The problem now is to find a solution of (2.1) which is square-summable on $\mathbf{R}^3_{-a}$ for each t and satisfies the initial and boundary conditions

$$u(x, 0) = u_0(x) \quad (2.3)$$

$$Bu(x_1, x_2, -a, t) = 0, \quad (2.4)$$

where B is a boundary operator which preserves energy. In order to explicitly identify B for the rest of this work, we will make the assumption that the external magnetic field is of the form (0, 1, 0), i.e., $h_2 = 1$, $h_1 = 0$, $h_3 = 0$ so that the external field is parallel to the boundary layer.

Here we give some properties of Λ^0 and Λ. These will be stated for Λ. The corresponding properties for Λ^0 may be obtained by just affixing the index ^{0} to all quantities containing the density and sound speed parameters. The transpose of a matrix M is denoted by ${}^tM$ and the conjugate transpose by ${}^{\bar t}M$ while the adjoint of an operator is denoted by M^*. The matrices E and E_0 generate equivalent inner products in $\mathbf{C}^6$ (Complex 6-space) by the rule ${}_E(x, y) = Ex \circ y = {}^{\bar t}xEy$. In this inner product, the symbol $\Lambda(p)$ of $\Lambda(D)$ is given by $E^{-1}A(p)$ with

$$A(p) = \begin{bmatrix} 0 & 0 & 0 & p_2 & -p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_3 & p_2 \\ p_2 & 0 & 0 & 0 & 0 & 0 \\ -p_1 & 0 & -p_3 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 0 & 0 & 0 \end{bmatrix}. \quad (2.5)$$

$A(p)$ is selfadjoint. $\Lambda(p)$ has the eigenvalues ($p = (p_1, p_2, p_3)$ not zero in $\mathbf{R}^3$)

$\lambda_0(p) = 0$ (multiplicity two) [the stationary speed];

$\lambda_{\pm 1}(p) = \pm p_2/(\sqrt{\rho}\sqrt{\mu})$ [Alfven speeds];

$\lambda_{\pm 2}(p) = \pm |p|/(\sqrt{\rho}\sqrt{\mu})$ [magnetospeeds].

Here the notation $|p|^2$ is used for $p_1^2 + p_2^2 + p_3^2$. Let us define $(\sqrt{\rho}\sqrt{\mu}) = c^{-1}$. The nonzero eigenvalues above each have multiplicity one for almost all p. Both the Alfven speed and the magnetospeed may coincide for some

p; hence this system is not uniformly propagative. We see that it also fair to be strongly propagative, since $\lambda_{\pm 1}(p)$ may vanish for nonzero p. *There exists a set of Lebesgue measure zero β such that when $p \in \mathbf{R}^3\backslash\beta$, the eigenvalues $\lambda_0(p)$, $\lambda_{\pm 1}(p)$, $\lambda_{\pm 2}(p)$ do not coincide* [S1]. We shall generally assume $p \in \mathbf{R}^3\backslash\beta$. Associated with each of the eigenvalues, there are mutually orthogonal eigenprojectors $P_{\pm j}(p)$, $P_0(p)$ with respect to the E inner product. They generate the resolution of the identity for $\Lambda(p)$:

$$\begin{aligned} I &= P_0(p) + P_1(p) + P_{-1}(p) + P_2(p) + P_{-2}(p) \\ \Lambda(p) &= \lambda_1(p)P_1(p) + \lambda_{-1}(p)P_{-1}(p) + \lambda_2(p)P_2(p) \\ &\quad + \lambda_{-2}(p)P_{-2}(p). \end{aligned} \tag{2.6}$$

The $P_{\pm j}(p)$ satisfy the identities

$$\begin{aligned} {}^{t}(EP_{\pm j}) &= EP_{\pm j}, \qquad \delta_{jk}P_k = P_jP_k \\ \Lambda(p)P_{\pm j}(p) &= \lambda_{\pm j}(p)P_{\pm j}(p). \end{aligned} \tag{2.7}$$

The P will be needed explicitly. We write these so that when "z" occurs in it we agree to substitute $\pm\lambda_j = \lambda_{\pm j}$ as appropriate ($|p|_1^2$ will stand for $p_1^2 + p_3^2$)

$$P_{\pm 2z}(p) = \frac{1}{2\mu|p|_1^2|p|^2}$$
$$\begin{bmatrix} \mu p_1^2|p|^2 & 0 & \mu p_1p_3|p|^2 & \mu^2p_1^2p_2z & -\mu^2p_1|p|_1^2z & \mu^2p_1p_2p_3z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mu p_1p_3|p|^2 & 0 & \mu p_3^2|p|^2 & \mu^2p_1p_2p_3z & -\mu^2p_3|p|_1^2z & \mu^2p_2p_3^2z \\ \mu p_1^2p_2\rho z & 0 & \mu p_1p_2p_3\rho z & \mu p_1^2p_2^2 & -\mu p_1p_2|p|_1^2 & \mu p_1p_2^2p_3 \\ -\mu p_1|p|_1^2\rho z & 0 & -\mu p_3|p|_1^2\rho z & -\mu p_1p_2|p|_1^2 & \mu|p|_1^4 & -\mu p_2p_3|p|_1^2 \\ \mu p_1p_2p_3\rho z & 0 & \mu p_2p_3^2\rho z & \mu p_1p_2^2p_3 & -\mu p_2p_3|p|_1^2 & \mu p_2^2p_3^2 \end{bmatrix} \tag{2.8}$$

$$P_{\pm 1z}(p) = \frac{1}{2p_2|p|_1^2}\begin{bmatrix} p_2p_3^2 & 0 & -p_1p_2p_3 & \mu p_3^2z & 0 & -\mu p_1p_3z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_2p_3 & 0 & p_1^2p_2 & -\mu p_1p_3z & 0 & \mu p_1^2z \\ p_3^2\rho z & 0 & -p_1p_3\rho z & p_2p_3^2 & 0 & -p_1p_2p_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_3\rho z & 0 & p_1^2\rho z & -p_1p_2p_3 & 0 & p_1^2p_2 \end{bmatrix} \tag{2.9}$$

$$P_0(p) = \frac{1}{|p|^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |p|^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1^2 & p_1p_2 & p_1p_3 \\ 0 & 0 & 0 & p_1p_2 & p_2^2 & p_2p_3 \\ 0 & 0 & 0 & p_1p_3 & p_2p_3 & p_3^2 \end{bmatrix}. \tag{2.10}$$

Let $\mathcal{H}$ be the space $L_2(\mathbf{R}^3, \mathbf{C}^6)$, with the E inner product: $(f, g)_{\mathcal{H}} = \langle f, Eg\rangle$ [$\langle \cdot, \cdot\rangle$ will represent the usual L_2 inner product in this work]. $\mathcal{S}'$ will denote the dual of the space $\mathcal{S} = \mathcal{S}(\mathbf{R}^n, \mathbf{C}^6)$ of rapidly decreasing, smooth functions. The Fourier transform

$$\Phi_n f(p) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} \exp(-ip \circ x) f(x)\, dx$$

is an automorphism of $\mathcal{S}$ with inverse $\Phi_n^* f(p) = \Phi_n f(-p)$ which extends by continuity to an automorphism of $L_2(\mathbf{R}^3, \mathbf{C}^6)$ and by duality to an automorphism of $\mathcal{S}'$. For any f in $\mathcal{H}$ the quantity $\Lambda(D)f \in \mathcal{S}'$, and operator with domain $\mathfrak{D}(\Lambda) = \{f \in \mathcal{H}: \Lambda f \in \mathcal{H}\}$ is selfadjoint with resolvent $I(z) = (\Lambda - zI)^{-1}$ given by

$$I(z)f(x) = \int_{\mathbf{R}^3} I(x, y; z) f(y)\, dy.$$

$\mathrm{Im}(z) \neq 0$ and $I(x, y; z) = I(x - y; z)$ is the fundamental solution

$$[\Lambda(D) - zI]I(x; z) = \delta(x)I, \tag{2.11}$$

where I may be obtained from (in $\mathcal{S}'$)

$$I(\cdot; z) = (2\pi) - 3/2\Phi^*[\Lambda(\cdot) - zI]^{-1}\Phi. \tag{2.12}$$

$[\Lambda(\cdot) - zI]^{-1}$ may be determined by the spectral theorem from

$$[\Lambda(p) - zI]^{-1} = \Sigma\, [\lambda_{\pm k}(p) - z]^{-1} P_{\pm k}(p). \tag{2.13}$$

To proceed, we shall require explicit formulas for I on hyperplanes orthogonal to the x_3 axis. To get these formulas, write $p = (\xi, \eta)$, $\xi \in \mathbf{R}^2$, and extend $\lambda_{\pm k}(p)$ to complex η by the requirement $\pm\mathrm{re}\lambda_{\pm k}(p) \geq 0$ upon replacing p by (ξ, τ) where $\tau = \eta + i\kappa$. Then $\lambda_{\pm k}(p)$ becomes $\lambda_{\pm k}(\xi, \tau)$ with $P_{\pm k}(\xi, \tau)$ satisfying

$$\Lambda(\xi, \tau)P_{\pm k}(\xi, \tau) = (\lambda_{\pm k}(\xi, \tau)P_{\pm k}(\xi, \tau).$$

For ξ not zero define

$$\tau_{\pm 2}(\xi, z) = c^{-1}(z^2 - c^2|\xi|^2)^{1/2}, \tag{2.14}$$

where $\pm \mathrm{im}\tau_{\pm k} \geq 0$ in the z-plane with branch cuts $(-\infty, -c|\xi|)$, $(c|\xi|, \infty)$. Note that $\lambda_{\pm 1}(\xi, \tau)$ *is a constant function of* τ. Observe that

$$\tau_{+2}(\xi, z) = -\tau_{-2}(\xi, z), \qquad \bar{\tau}_{\pm 2}(\xi, \bar{z}) = -\tau_{\pm 2}(\xi, z). \tag{2.15}$$

The matrix $[\Lambda(\xi, \tau) - zI]^{-1}$ is regular in τ except for poles in the upper (lower) half plane at the zeros of $\det[\Lambda(\xi, \tau) - zI]$, that is, at $\tau_{+k}(\tau_{-k})$, and in a neighborhood of these poles

$$[\Lambda(\xi, \tau) - zI]^{-1} = \Sigma[\lambda_{\pm k}(\xi, \tau) - z]^{-1}P_{\pm k}(\xi, \tau). \tag{2.16}$$

If we now apply Φ_2 to both sides of (2.12), we obtain, in $\mathscr{S}'$, the relation

$$\Phi_2 I(\xi, x_3; y; z) = (2\pi)^{-2}e^{-iy'\xi}\int_{\mathbf{R}} e^{i(x_3-y_3)}[\Lambda(\xi, \tau) - zI]^{-1}\, d\tau. \tag{2.17}$$

In order to simplify the notation in the evaluation of this integral, we shall from here on employ the definitions $\tau \equiv \tau_+$, $\tau^0 \equiv \tau_+^0$. An elementary computation gives

$$\begin{aligned} \Phi_2 I(\xi, -a, y; z) &= i(2\pi)^{-1}c^{-2}e^{-iy'\xi}e^{i\tau(a+y_3)}z\tau^{-1}P(\xi, z, -\tau) \\ \Phi_2 I(\xi, 0, y; z) &= i(2\pi)^{-1}c^{-2}e^{-iy'\xi}e^{i\tau y_3}z\tau^{-1}P(\xi, z, -\tau), \qquad -a < y_3 < 0 \\ \Phi_2 I(\xi, 0, y; z) &= i(2\pi)^{-1}c_0^{-2}e^{-iy'\xi}e^{i\tau^0 y_3}z\tau^{0-1}P^0(\xi, z, -\tau^0), \qquad 0 < y_3. \end{aligned} \tag{2.18}$$

Here $P(\xi, z, \tau)$ is given explicitly by (2.8) [p_3 is replaced by τ]. In actual fact, a trivial term must be added to these functions to account for the projections $P_{\pm 1}$. In order to simplify the treatment, we shall leave this term off, and account for the action of $P_{\pm 1}$ in Section 5.

$P(\xi, z, \tau)$ is a solution of

$$\Lambda(\xi, \tau)P(\xi, z, \tau) = zP(\xi, z, \tau). \tag{2.19}$$

$P^0(\xi, z, \tau^0)$ is obtained from (2.8) by replacing ρ and c with ρ_0 and c_0, respectively. It satisfies

$$\Lambda^0(\xi, \tau)P^0(\xi, z, \tau^0) = zP^0(\xi, z, \tau^0). \tag{2.20}$$

The identities

$$^{t}(EP(\xi, \bar{z}, \tau(\xi, \bar{z}))) = EP(\xi, z, -\tau(\xi, z)) \tag{2.21}$$

$$^{\bar{t}}(E_0P^0(\xi, \bar{z}, \tau^0(\xi, \bar{z}))) = E_0P^0(\xi, z, -\tau^0(\xi, z)) \tag{2.22}$$

hold.

3. Selfadjoint Operators

The operator Λ' is a selfadjoint operator in the space consisting of functions f, g in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6$ with the $\mathscr{E}$ inner product:

$$(f, g)_{\mathscr{E}} = \langle f, \mathscr{E}g\rangle = \int_{\mathbf{R}^3_{-a}} {}^{\bar{t}}f(x)\mathscr{E}(x)g(x)\, dx.$$

The operator Λ' is a Hilbert space operator whose action is defined by the differential operator $\Lambda'(D)$ with the boundary condition (Theorem 1.4)

$$Bf(x', -a) = 0 \tag{3.1}$$

$$B = [0, 0, 0, 0, 1, 0] \qquad \text{or} \qquad f_5(x', -a) = 0.$$

The construction of the operator A in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ from $A(D)$ (see (0.2)) is sufficient to define Λ' (with the same boundary condition) and will done now. We introduce here the required definitions and spaces for this task. The notation $L_2(\mathbf{R}_{-a}, \mathbf{H}^s)$ denotes the space of functions square-integrable on $\mathbf{R}_{-a} = (-a, \infty)$ with values in $\mathbf{H}^s$, $s \in \mathbf{R}$, the Sobolev space of order s. $\mathfrak{D}(\Omega)$, $\Omega \subseteq \mathbf{R}^n$, is the Schwartz space of compactly supported C^∞ functions on Ω. $\mathfrak{D}(\overline{\mathbf{R}}^3_{-a})$ is the set of C^∞ functions on $\overline{\mathbf{R}}^3_{-a}$ with bounded support in $\overline{\mathbf{R}}^3_{-a}$; $\mathfrak{D}(\overline{\mathbf{R}}^n_{-a}, \mathbf{H}^s)$ is the set of C^∞ functions on $\overline{\mathbf{R}}^n_{-a}$ with values in $\mathbf{H}^s$, having bounded support in $\overline{\mathbf{R}}^n_{-a}$; $C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^s)$ is the space of bounded, continuous functions from $\overline{\mathbf{R}}_{-a}$ to $\mathbf{H}^s$ with norm

$$|f|_{C_b} = \sup\{|f(t)|_{\mathbf{H}^s}: t \in \overline{\mathbf{R}}_{-a}\}.$$

$W(\overline{\mathbf{R}}_{-a}) = \{f: f \in L_2(\mathbf{R}_{-a}, \mathbf{H}^0), \partial_t f \in L_2(\mathbf{R}_{-a}, \mathbf{H}^{-1})\}$ is the Hilbert space with norm

$$|f|^2_W = (|f|_{L_2(\mathbf{R}_{-a}, \mathbf{H}^0)})^2 + (|\partial_t f|_{L_2(\mathbf{R}_{-a}, \mathbf{H}^{-1})})^2$$

[LM, p. 13]. Recall that $\mathbf{H}^1(\mathbf{R}_{-a})$ is isomorphic to $\{f: f \in L_2(\mathbf{R}_{-a}, \mathbf{H}^1), \partial_t f \in L_2(\mathbf{R}_{-a}, \mathbf{H}^0)\}$. The notation j_ε will refer to the operator of tangential mollification: $j \in \mathfrak{D}(\mathbf{R}^2)$, $j \geq 0$, $j(x') = j(-x')$, $\operatorname{supp}(j) \subseteq \{x': |x| \leq 1\}$,

$\int_{\mathbf{R}^2} j = 1$; for $\varepsilon > 0$, let $j_\varepsilon(x') = \varepsilon^{-2} j(\varepsilon^{-1}x')$, and for $f \in L_2(\mathbf{R}^3_{-a})$, define $J_\varepsilon f$ to be $j_\varepsilon * f$, the convolution of j and f. Writing $x \in \mathbf{R}^3_{-a}$ as (x', t)

$$J_\varepsilon f(x', t) = \int_{\mathbf{R}^2} j_\varepsilon(x' - z') f(z', t)\, dz' \tag{3.2}$$

whenever this is meaningful pointwise; otherwise $J_\varepsilon f$ is that element of $L_2(\mathbf{R}^3_{-a})$ defined by $\langle J_\varepsilon f, \phi\rangle = \langle f, j_\varepsilon * \phi\rangle$ for all ϕ in $\mathbf{H}^1(\mathbf{R}^3_{-a})$. It will be necessary to state the following elementary embedding result (see [LM] for example):

LEMMA 3.1. *The set $\mathfrak{D}(\overline{\mathbf{R}}_{-a}, \mathbf{H}^0)$ is dense in $W(\mathbf{R}_{-a})$ and $\mathfrak{D}(\overline{\mathbf{R}}_{-a}, \mathbf{H}^1)$ is dense in $\mathbf{H}^1(\mathbf{R}^3_{-a})$. Further, $f \in W(\mathbf{R}_{-a})$ implies $f \in C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^{-1/2})$ and $f \in \mathbf{H}^1(\mathbf{R}^3_{-a})$ implies $f \in C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^{1/2})$. These are continuous embeddings.*

COROLLARY 3.2. *If $f \in W(\mathbf{R}_{-a})$ and $\psi \in \mathbf{H}^1(\mathbf{R}^3_{-a})$, then*

$$i[f(-a), \psi(-a)] = \int_{-a}^{\infty} [D_3 f(t), \psi(t)]\, dt - \langle f, D_3\psi\rangle, \tag{3.3}$$

where the $[\cdot, \cdot]$ on the left is the $[\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}]$ duality bracket, while in the integrand it is the $[\mathbf{H}^{-1}, \mathbf{H}^1]$ duality bracket. Also from $f \in W(\mathbf{R}_{-a})$ it follows $J_{1/n} f = f_n \in W(\mathbf{R}_{-a})$ and $f_n \to f$ in $W(\mathbf{R}_{-a})$ as $n \to \infty$; hence on any hyperplane, $x_3 = c$ also $f_n(\circ, c) \to f(\circ, c)$ in $\mathbf{H}^{-1/2}$.

Let B be the kernel of the matrix operator B in $\mathbf{C}^6$. By (0.6)

$$A_3\beta = {}^t(0, 0, -\beta_5, 0, -\beta_3, 0), \beta \in \mathbf{C}^6.$$

From Section 1 above, we know that $[A_3 B]^\perp = B$. Now if $f \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, then $Af \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ means that there is an element $g \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ such that $\langle g, \phi\rangle = \langle f, A\phi\rangle$ for all $\phi \in \mathfrak{D}(\overline{\mathbf{R}}^3_{-a}, \mathbf{C}^6)$ $[g \equiv Af]$. If both f and Af belong to $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ then

$$A_3 D_3 f = Af - A_1 D_1 f - A_2 D_2 f \in L_2(\mathbf{R}^3_{-a}, \mathbf{H}^{-1})$$

so that $A_3 f$ is in $W(\mathbf{R}_{-a})$ and therefore in $C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^{-1/2})$ by Lemma 3.1. We now discuss the so-called weak and strong versions of A $[A_w, A_s]$:

DEFINITION 3.3. $f \in \mathbf{D}(A_w)$ [domain of A_w] if and only if f and Af belong to $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ and $A_3 f(\circ, -a)$ belongs to $B^\perp$ in $\mathbf{H}^{-1/2}$; that is, $[A_3 f(\circ, -a), \phi] = 0$ for all $\phi \in \mathbf{H}^{1/2}$ such that $B\phi = 0$.

DEFINITION 3.4. The operator A_s is $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ is defined as the graph closure of A^0, the operator $A(D)$ on

$$\mathbf{D}(A^0) = \{f \in \mathfrak{D}\overline{\mathbf{R}}^3_{-a}, \mathbf{C}^6): Bf(x', -a) = 0\}. \tag{3.4}$$

LEMMA. A_w is a selfadjoint operator in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$.

Proof. We first note the following facts:

(1) $u \in W(\mathbf{R}_{-a})$ implies $J_{1/n}u = u_n \in W(\mathbf{R}_{-a})$ and $u_n \to u$ in $W(\mathbf{R}_{-a})$ as $n \to \infty$. (see above).

(2) $(v, au) = (w, u)$ for all u such that u and $Au \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ implies that $v_n, Av_n \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, $(v_n, au) = (w_n, u)$, and $Av_n = J_{1/n}Av \to Av = w$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$.

(3) $u \in \mathbf{D}(A_w)$ implies $u_n \in \mathbf{D}(A_w)$.

It follows from the definition of derivative $\partial_t u$ in $L_2(\mathbf{R}_{-a}, \mathbf{H}^{-1})$ that $\partial_t u_n = J_{1/n}\partial_t u$. $\partial_t u_n$ is defined as an element of $\mathfrak{D}(\mathbf{R}_{-a}, \mathbf{H}^{-1})$, i.e., as a linear map from $\mathfrak{D}(\mathbf{R}_{-a})$ into $\mathbf{H}^{-1}$, by $\partial_t u_n(\psi) = -u_n(\partial_t\psi)$ for all ψ in $\mathfrak{D}(\mathbf{R}_{-a})$. Thus, if $\psi \in \mathfrak{D}(\mathbf{R}_{-a})$, $\phi \in \mathbf{H}^{-1}$, then for almost all t, $[\partial_t u_n(t)(\psi(t)), \phi] = -[u_n(t)(\partial_t\psi(t)), \phi] = -(u(t)(\partial_t\psi(t)), \phi_n] = [\partial_t u(t)(\psi(t)), \phi_n] = [J_{1/n}\partial_t u(t)(\psi(t)), \phi]$. Furthermore, $|J_{1/n}\partial_t u(t)|^2_{\mathbf{H}^{-1}} \leq |\partial_t u(t)|^2_{\mathbf{H}^{-1}}$ so that $\partial_t u_n = J_{1/n}\partial_t u \in L_2(\mathbf{R}_{-a}, \mathbf{H}^{-1})$ and $\lim_{n\to\infty} \int_{-a}^{\infty} |J_{1/n}\partial_t u(t) - \partial_t u(t)|^2_{\mathbf{H}^{-1}}\, dt = 0$ by the dominated convergence theorem since $g \in \mathbf{H}^s$ implies $J_{1/n}g \to g$ in $\mathbf{H}^s$ for any s. This shows (1). To check (2), note that if $\phi \in \mathfrak{D}(\mathbf{R}_{-a})$, then $(w_n, \phi) = (w, \phi_n) = (v, A\phi_n) = (v, J_{1/n}A\phi) = (v_n, A\phi)$ so that $Av_n = w_n$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, and $Av_n = J_{1/n}Av$ ($Av = w$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$). Repeating this with u in place of ϕ gives $(v_n, Au) = (w_n, u)$. On the basis of (2) it need only be shown that $Au_n(\circ, -a) \in B^\perp$ in $\mathbf{H}^{-1/2}$. Let $\phi \in \mathbf{H}^{1/2}$ and $\{\xi_k\}$ ($k = 1, \ldots, 5$) a fixed basis (see Sect. 1, above) for B. Then $[A_3u_n(\circ, -a), \phi\xi_k] = [A_3u(\circ, -a), \phi_n\xi_k] = 0$, $k = 1, \ldots, 4$, since $A_3u(\circ, -a) \in B^\perp$. Hence $A_3u_n(\circ, -a) \in B^\perp$ and so $u_n \in \mathbf{D}(A_w)$. Since $\mathfrak{D}(\mathbf{R}_{-a}) \subseteq \mathbf{D}(A_w)$, A_w is densely defined. We now check that $A_w \subseteq A_w^*$. Let $u, f \in \mathbf{D}(A_w)$. Define u_k, f_j as $J_{1/k}u$ and $J_{1/j}f$, respectively. By (3), u_k, $f_j \in \mathbf{D}(A_w)$ and A_3u_k, $A_3f_j \in \mathbf{H}^1(\mathbf{R}^3_{-a}, \mathbf{C}^6)$. Therefore, $\mathfrak{u}_k$, $\mathfrak{f}_j$ the components of u_k, f_j in the range of A_3, are in $\mathbf{H}^1(\mathbf{R}^3_{-a})$. For $i = 1, 2$,

$$\langle A_iD_iu_k, f_j\rangle = \int_{-a}^{\infty} [A_iD_iu_k, f_j]\, dt = \int_{-a}^{\infty} [u_k, A_iD_if_j]\, dt = \langle u_k, A_iD_if_j\rangle.$$

Thus by Corollary 3.2,

$$\begin{aligned}\langle Au_k, f_j\rangle &= \langle A_3D_tu_k, f_j\rangle + \langle A_1D_1u_k, f_j\rangle + \langle A_2D_2u_k, f_j\rangle \\ &= \langle D_tA_eu_k, \mathfrak{f}_j\rangle + \langle u_k, A_1D_1f_j\rangle + \langle u_k, A_2D_2f_j\rangle \\ &= i[A_3u_k(-a), \mathfrak{f}_j(-a)] + \langle u_k, Af_j\rangle = \langle u_k, Af_j\rangle\end{aligned} \tag{3.5}$$

since $A_eu_k(\circ, -a) \in B^\perp$ and $\mathfrak{f}_j(\circ, -a) \in B$. Passing to the limits $j \to \infty$ and $k \to \infty$ gives $\langle Au, f\rangle = \langle u, Af\rangle$. It remains to show that $A_w \supseteq A_w^*$. First,

$v \in \mathbf{D}(A_w^*)$ if and only if there exists w in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ such that $\langle v, Au\rangle = \langle w, u\rangle$ for all $u \in \mathbf{D}(A_w)$. $\mathfrak{D}(\mathbf{R}_{-a}) \subseteq \mathbf{D}(A_w)$, so that thus $Av = w$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$. Hence, $A_3 v \in W(\mathbf{R}_{-a})$. By (2), $v_n = J_{1/n}v \in \mathbf{D}(A_w^*)$ and $Av_n = w_n = J_{1/n}w$ so that $A_3 v_n \in \mathbf{H}^1(\mathbf{R}^3_{-a}, \mathbf{C}^6)$. Setting $u_k = J_{1/k}u \in \mathbf{D}(A_w)$ and repeating (3.5) gives

$$\begin{aligned}\langle w_n, u_k\rangle &= \langle v_n, Au_k\rangle\\ &= i[A_3 v_n(-a), u_k(-a)] + \langle Av_n, u_k\rangle\\ &= i[A_3 v_n(-a), u_k(-a)] + \langle w_n, u_k\rangle,\end{aligned}$$

and thus $[A_3 v_n(-a), u_k(-a)] = 0$. Since $A_3 v \in W(\mathbf{R}_{-a})$ and $A_3 v_n \to A_3 v$ in $W(\mathbf{R}_{-a})$ by (1), it follows that $A_3 v_n(-a) \to A_3 v(-a)$ in $\mathbf{H}^{-1/2}$. Therefore, $[A_3 v(-a), u_k(-a)] = 0$ for all $u \in \mathbf{D}(A_w)$. Take $u(x', t) = \psi(t)\phi(x')\xi_i$ where $y \varepsilon D(R)$, $\psi(0) = 1$, $\phi \in \mathbf{H}^1$, and $\{\xi_i\}$ is the base for B; it is clear that $u \in \mathbf{D}(A_w)$. Thus

$$0 = [A_3 v(-a), u_k(-a)] = [A_3 v(-a), \phi_k \bar{\xi}_i] = [A_3 v(-a), \phi_k \bar{\xi}_i],$$

$\phi_k = J_{1/k}\phi$, for any $\phi \in \mathbf{H}^1$. The set $\{\phi_k : \phi \in \mathbf{H}^1\}$ is dense in $\mathbf{H}^{1/2}$. Therefore $A_3 v(-a) \in B^\perp$ in $\mathbf{H}^{-1/2}$. Thus, $v \in \mathbf{D}(A_w)$ and for all $u \in \mathbf{D}(A_w)$, $\langle A_w^*, v, u\rangle = \langle v, A_w u\rangle = \langle A_w v, u\rangle$, This completes the proof of the lemma.

It will now be shown that $A_s = A_w$.

LEMMA 3.5. *If $f, g \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ and $\langle f, A\phi\rangle = \langle g, \phi\rangle$ for all $\phi \in \mathbf{D}(A^0)$, then $\langle Af, f\rangle = \langle f, Af\rangle$ ($AF = g$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$).*

Proof. As above, set $f_n = J_{1/n}f$, and $g_n = J_{1/n}g$. Then $Af_n = J_{1/n}Af = g_n$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ implies that $A_3 D_3 f_n = g_n - A_1 D_1 f_n - A_2 D_2 f_n \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ and hence $A_3 f_n \in \mathbf{H}^1(\mathbf{R}^3_{-a}, \mathbf{C}^6)$. Thus $A_3 f_n$ and f_n belong to $C_b(\bar{\mathbf{R}}_{-a}, \mathbf{H}^{1/2})$. If $\phi \in \mathbf{D}(A^0)$, then also $\phi_n \in \mathbf{D}(A^0)$, and so by Corollary 3.2,

$$\begin{aligned}\langle g_n, \phi\rangle = \langle g, \phi_n\rangle = \langle f, A\phi_n\rangle &= \langle f_n, A\phi\rangle\\ &= i[A_3 f_n(\circ, -a), \phi(\circ, -a] + \langle g_n, \phi\rangle,\end{aligned}$$

and so $A_3 f_n(\circ, -a) \in B^\perp$ and hence $f_n(\circ, -a) \in B$. By Corollary 3.2,

$$\langle Af_n, f_n\rangle = i[A_3 f_n(\circ, -a), f_n(\circ, -a)] + \langle f_n, Af_n\rangle = \langle f_n, Af_n\rangle,$$

Now let $n \to \infty$.

DEFINITION 3.6. Let $J: L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6) \to \mathcal{H}$ where $\mathcal{H}$ is the space $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ with the $\mathcal{E}$ inner product, J being the identification map, $Jf = f$. Define Λ' in $\mathcal{H}$ by $\mathbf{D}(\Lambda') = J\mathbf{D}(A_w)$ and for $f \in \mathbf{D}(A_w)$, $\Lambda' f = \mathcal{E}^{-1} Af$.

PROPOSITION 3.7. $A_s = A_w$.

Proof. This will be shown by proving that A_s is a selfadjoint operator. To this end, we establish that the deficiency indices def$(A_s \pm iI) = (0, 0)$. First we show that $R(A_s \pm iL)$, the range of $A_s \pm iI$, is closed. Let g belong to the closure of $R(A_s \pm iI)$. Then there is a sequence $\{\phi_n\}$ contained in $\mathbf{D}(A_s)$ such that $(A_s \pm iI)\phi_n \to g$. Now, will $\|\circ\|$ as the norm in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$,.

$$\begin{aligned}\|\phi_n - \phi_m\| &= 2^{-1}|\langle (A_s \pm iI)(\phi_n - \phi_m), \phi_n - \phi_m\rangle \\ &\quad - \langle \phi_n - \phi_m, (A_s \pm iI)(\phi_n - \phi_m)\rangle| \\ &\leq \|\phi_n - \phi_m\| \, \|(A_s \pm iI)(\phi_n - \phi_m)\|\end{aligned}$$

so that $\phi_n \to \phi$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, and so $A_s\phi_n \to \psi$ in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$. This implies that $\phi \in \mathbf{D}(A_s)$ and $A_s\phi = \psi$, so $g = (A_s \pm iI)\phi = \psi \pm i\phi$ is in $R(A_s \pm iI)$. Next, $R(A_s \pm iI)$ is dense in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$: if $f \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ is orthogonal in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ to $R(A_s \pm iI)$, then $0 = \langle f, (A_s \pm iI)\phi\rangle$ and thus $\langle f, A\phi\rangle = \langle \pm if, \phi\rangle$ for all ϕ in $\mathbf{D}(A^0)$. By Lemma 3.5, $0 = \langle Af, f\rangle - \langle f, Af\rangle = +2i\|f\|^2$, so that $f = 0$. This shows that the deficiency indices are both zero and therefore that A_s is selfadjoint. A_s is therefore maximally symmetric and the conclusion follows.

THEOREM 3.8. *Λ' is a selfadjoint operator in $\mathcal{H}$, the space $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ with the $\mathcal{E}$* inner product.

Proof. For $f, g \in \mathbf{D}(\Lambda')$, $(f, \Lambda' g)_{\mathcal{H}} = \langle f, Ag\rangle = \langle Af, g\rangle = (\Lambda' f, g)_{\mathcal{H}}$ is therefore a symmetric operator. It must be maximally symmetric, otherwise A would have a symmetric extension and this is impossible. This completes the proof.

If f, $Af \in L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, then $A_3 f = {}^t(0, 0, -f_5, 0, -f_3, 0)$ is in $C_b(\mathbf{R}_{-a}, \mathbf{H}^{-1/2})$. Thus $A_3 f$ is continuous across any plane $x_3 =$ constant $> -a$, and in particular across the interface $x_3 = 0$. Hence for $Lf = {}^t(f_3, f_5)$,

$$Lf(\circ, 0-) = Lf(\circ, 0+) \tag{3.6}$$

in the sense of $\mathbf{H}^{-1/2}$. Note that this is also implied byu the fact that Λ' is selfadjoint,

$$0 = (\Lambda' f, \phi)_{\mathcal{H}} - (f, \Lambda'\phi)_{\mathcal{H}} = i[\{A_3 f(\circ, 0+) - A_3 f(\circ, 0-)\}, \phi(\circ, 0)]$$

for $f \in \mathbf{D}(\Lambda')$, $\phi \in \mathcal{D}(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, and thus $A_3 f(\circ, 0+) = A_3 f(\circ, 0-)$ in the sense of $\mathbf{H}^{-1/2}$. If f also is smooth (say, $f \in \mathbf{D}(A^0)$) then (3.6) holds pointwise.

It will be necessary to characterize data in $N(\Lambda')^\perp$, the orthogonal complement of the null space of Λ'. A dense set in $N(\Lambda') = JN(A)$ will be sufficient.

LEMMA 3.9. *The set* $S = \{{}^t(0, \phi, 0, \nabla\psi): \psi, \phi \in \mathfrak{D}(\overline{\mathbf{R}}^3_{-a})\}$ is dense in $N(A)$. ($\nabla f = {}^t(D_1 f, D_2 f, D_3 f)$.)

Proof. Suppose that $f \in N(A)$, and $f \perp S$. Then $f_2 = 0$, $D_1 f_1 + D_3 f_3 = 0$, and thus [$Af = 0$ shows $D_2 f_1 = D_2 f_3 = 0$] $D_2(f_1, f_2, f_3) = 0$. Since f_1, f_2, f_3 are in L_2, it follows that $(f_1, f_2, f_3) = 0$ in L_2. $f \in N(A)$ then further implies that $D_1 f_5 - D_2 f_4 = 0$, and $D_3 f_5 - D_2 f_6 = 0$. $f \perp S$ implies that $D_1 f_4 + D_2 f_5 + D_3 f_6 = 0$. If follows then that $\Delta f_5 = 0$. Now because $Bf = 0$, $f_5(\circ, -a) = 0$ in $\mathbf{H}^{-1/2}$. Then we have $f_5 \in \mathbf{H}^1$ [LM, p. 151] and so $f_5(\circ, -a) \in \mathbf{H}^{1/2}$ but then $f_5 \in \mathbf{H}^1(\mathbf{R}^3_{-a})$ [LM, p. 161]. This implies that $f_5 = 0$ and thus $f_4 = f_6 = 0$. This completes the proof.

PROPOSITION 3.10. *A function* $f = {}^t(f_1, f_2, f_3, f_4, f_5, f_6, f_7)$ in $\mathcal{H}$ *is orthogonal in* $\mathcal{H}$ *to* $N(\Lambda')$ *if and only if*

$$f_2 = 0 \text{ and } \operatorname{div}(f_4, f_5, f_6) = 0 \text{ in } L_2(\mathbf{R}^3_+, \mathbf{C}^6) \text{ and in } L_2(\mathbf{R}^2 \times (-a, 0), \mathbf{C}^6) \tag{3.7}$$

and

$$(\mathscr{E}f)_6(x', 0-) = (\mathscr{E}f)_6(x', 0+) \tag{3.8}$$

$$(\mathscr{E}f)_6(\circ, -a) = 0 \tag{3.9}$$

in $\mathbf{H}^{-1/2}$.

Proof. A function f in $\mathcal{H}$ belongs to $N(\Lambda')^\perp$ if and only if $\mathscr{E}f = g$ is orthogonal in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ to S. Thus we must show that g in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ is orthogonal to S if and only if (3.7)–(3.9) are satisfied. First we show that (3.7)–(3.9) are necessary conditions. g is orthogonal in $L_2(\mathbf{R}^3_{-a}, \mathbf{C}^6)$ to S implies that $\operatorname{div}(g_4, g_5, g_6) = 0$, and $g_2 = 0$. In other works, $\langle g, {}^t(0, 0, 0, \operatorname{grad}(\psi))\rangle = 0$ and therefore γ_6 is in $C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^{-1/2})$. It follows that $g_6(x', 0-) = g_6(x', 0+)$. This implies (3.8). Now since $\psi \in \mathfrak{D}(\overline{\mathbf{R}}^3_{-a})$, and $(g_4, g_5, g_6) \perp \operatorname{grad}(\psi)$ we have $J_{1/n}(g_4, g_5, g_6) = ((\mathscr{E}f_n)_4, (\mathscr{E}f_n)_5, (\mathscr{E}f_n)_6) \perp \operatorname{grad}(\psi)$. Since $\operatorname{div}(J_{1/n}(g_4, g_5, g_6)) = J_{1/n}\operatorname{div}(g_4, g_5, g_6) = 0$ we have $(\mathscr{E}f_n)_6 \in \mathbf{H}^1(\mathbf{R}^3_{-a})$. Therefore Corollary 3.2 shows that

$$\begin{aligned}\langle((\mathscr{E}f_n)_4, (\mathscr{E}f_n)_5, (\mathscr{E}f_n)_6), \operatorname{grad}(\psi)\rangle &= -i[(\mathscr{E}f_n)_6(\circ, -a), \psi(\circ, -a)] \\ &\quad + \langle \operatorname{div}(J_{1/n}(g_4, g_5, g_6)), \psi\rangle, \\ &= -i[(\mathscr{E}f_n)_6(\circ, -a), \psi(\circ, -a)].\end{aligned}$$

Letting $n \to \infty$ gives (3.9).

Now suppose that (3.7)–(3.9) are satisfied. Let ${}^t(0, \phi, 0, \text{grad}(\psi))$ be in S, and let $f_n = J_{1/n}f$. Equations (3.7)–(3.9) hold for f_n. Equation (3.7) implies that $(f_n)_6 \in \mathbf{H}^1(\mathbf{R}^3_+)$ and $(f_n)_6 \in \mathbf{H}^1(\mathbf{R}^2 \times (-a, 0))$ [see the proof of Lemma 3.9 above]. Therefore,

$$
\begin{aligned}
&\langle((\mathscr{E}f_n)_4, (\mathscr{E}f_n)_5, (\mathscr{E}f_n)_6), \text{grad}(\psi)\rangle \\
&= \int_{\mathbf{R}^3_+} {}^{\bar{t}}[((E_0f_n)_5, (E_0f_n)_6)](x) \, \text{grad}(\psi(x)) \, dx \\
&\quad + \int_{\mathbf{R}^2\times(-a,0)} {}^{\bar{t}}[((Ef_n)_4, (Ef_n)_5, (Ef_n)_6)](x) \, \text{grad}(\psi)(x)) \, dx \\
&= \mu \int_{\mathbf{R}^2_+} \overline{\text{div}}((f_n)_4, (f_n)_5, (f_n)_6)\psi(x) \, dx \\
&\quad + \mu \int_{\mathbf{R}^2\times(-a,0)} \overline{\text{div}}((f_n)_4, (f_n)_5, (f_n)_6)\psi(x) \, dx \\
&\quad + i[\{(E_0f_n)_6(x', 0+) - (Ef_n)_6(x', 0-)\}, \psi(x', 0)] \\
&\quad + i[(Ef_n)_6(x', -a)\}, \psi(x', -a)] = 0.
\end{aligned}
$$

Now let $n \to \infty$ and this gives that $\langle \mathscr{E}f, {}^t(0, \phi, 0, \text{grad}(\psi))\rangle = 0$ and completes the proof.

We write from now on, $\overline{\mathscr{H}}$ for $\mathscr{H} \cap N(\Lambda')^{\perp}$. In examining the conditions (3.7)–(3.9), we see that they are satisfied by any function in the range of $\Lambda'(D)$ provided such a function vanishes near the plane $x_3 = 0$. That there are other functions which satisfy these conditions is perhaps of independent interest. Define the function Z^R by

$$
\begin{aligned}
\Phi_2 Z^R(\xi, x_3; \alpha) = \chi_R(|\xi|)\{&\mu\chi_+(x_3)\alpha(\xi)\exp(-|\xi|(x_3 + a))^t(0, 0, 0, d(\xi)) \\
&+ \mu_o\chi_-(x_3)e(\xi)\alpha(\xi)[\exp(-|\xi|(x_3 + a)\ {}^t(0, 0, 0, d(\xi)), \\
&+ \exp(|\xi|(x_3 + a))^t(0, 0, 0, \bar{d}(\xi))]\}.
\end{aligned}
\tag{3.10}
$$

The functions $\alpha(\xi)$, $e(\xi)$, $\chi_R(|\xi|)$, and $d(\xi)$, are defined as:

$$
\begin{aligned}
&\alpha(\xi) \text{ is square summable in neighborhoods of zero and} \\
&e^{-a|\circ||\circ|^{1/2}}\alpha \text{ is square summable in neighborhoods of } \infty;
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
d(\xi) &= (\xi_1, \xi_2, i|\xi|) \\
e(\xi) &= (1 - e^{2|\xi|a})^{-1} \\
\chi_R(|\xi|) &= \chi_{0,R)}(|\xi|).
\end{aligned}
$$

Then define

$$Z(x; \alpha) = \text{L.I.M.}_{R\to\infty} Z^R(x; \alpha). \tag{3.12}$$

Note that the components of Z are harmonic functions in the sense of $\mathfrak{D}(\mathbf{R}^3_+)$ and $\mathfrak{D}(\mathbf{R}^2 \times (-a, 0))$. The components are therefore ordinary harmonic functions in $\mathbf{R}^3_{-a} - \{x_3 = 0\}$ and are therefore smooth in $\mathbf{R}^3_{-a} - \{x_3 = 0\}$. $Z(x; \alpha)$ satisfy (3.7)–(3.9) and are square-summable by conditions (3.11).

DEFINITION 3.11.

$$T = \{\Lambda'(D)\phi \colon \phi \in \mathfrak{D}(\mathbf{R}^3_{-a}, \mathbf{C}^6),\ \operatorname{supp}(\phi) \cap \{x_3 = 0\} = \varnothing\} \quad (3.13)$$

and $\overline{T}$ is the closure in $\mathcal{H}$ of the set T. $[Z]$ is the set of all functions of the form (3.10). $\overline{\Lambda}$ is the part of Λ' in $\mathcal{H}$, $\mathbf{D}(\overline{\Lambda}) = \mathbf{D}(\Lambda') \cap \mathcal{H}$.

COROLLARY 3.12. *$\mathcal{H}$ is the $\mathcal{E}$-orthogonal direct sum of* $[Z]$ *and* $\overline{T}$. *Furthermore,* $\mathbf{D}(\overline{\Lambda}) \cap [Z] = \{0\}$.

Proof. First we check that $[Z]$ is orthogonal to T. Let $\phi \in \mathfrak{D}(\mathbf{R}^3_{-a}, \mathbf{C}^6)$, $\operatorname{supp}(\phi) \cap \{x_3 = 0\} = \varnothing$. Then $i(\Lambda'(D)\phi, Z(\circ; \alpha)) = i\,\langle A(D)\phi, Z(\circ; \alpha)\rangle = \lim_{R\to\infty} \langle A(D)\phi, Z^R(\circ; \alpha)\rangle$. Now note that we may replace $Z^R(\circ; \alpha)$ by a perfect gradient. This shows that $\langle A(D)\phi, Z^R(\circ; \alpha)\rangle = 0$ by Proposition 3.10. Suppose now that $f \in \mathcal{H}$ is orthogonal to T. Then it is easily seen that as in the proof of Lemma 3.9, $f_j, j = 4, 5, 6$, must be harmonic in the sense of $L_2(\mathbf{R}^2 \times (-a, 0))$ and $L_2(\mathbf{R}^3_+)$. It follows that

$$\Phi_2 f_j(\xi, x_3) = c^+(\xi)\exp(-|\xi|(x_3 + a)) \qquad \text{in } \mathbf{R}^3_+$$

$$\Phi_2 f_j(\xi, x_3) = c^-(\xi)\exp(-|\xi|(x_3 + a)) + d(\xi)\exp(|\xi)(x_3 + a)),$$
$$\text{in } \mathbf{R}^2 \times (-a, 0), j = 4, 5, 6.$$

By the above remarks, f has the form $[Z]$.

Finally, it is important to have a smooth core of functions in $\mathbf{D}(\overline{\Lambda})$.

PROPOSITION 3.13. *If* $f \in \mathbf{D}(\overline{\Lambda})$, then there exists a sequence $\{f_n\}$ in $\mathbf{D}(\overline{\Lambda})$ converging of f in graph norm with the properties.

$$Lf_n \in C(\overline{\mathbf{R}}^3_{-a}) \cap \mathbf{H}^1(\mathbf{R}^3_{-a}) \quad (3.14)$$

and

$$(f_n)_6 \in C_b(\mathbf{R}^2 \times [-a, 0)) \cap C_b(\overline{\mathbf{R}}^3_+) \cap \mathbf{H}^1(\mathbf{R}^2 \times (-a, 0)) \cap \mathbf{H}^1(\mathbf{R}^3_+) \quad (3.15)$$

has a finite jump across $x_3 = 0$ given pointwise by (3.8).

Proof. Define $f_n = J_{1/n}f$. Since $f_n \in \mathbf{D}(\overline{\Lambda})$, it follows that Af_n and hence Lf_n are in $\mathbf{H}^1(\mathbf{R}^3_{-a})$. There $(\mathcal{E}f_n)_6$ and Lf_n are in $C_b(\overline{\mathbf{R}}_{-a}, \mathbf{H}^{1/2})$. Now write

$f_n(x', t) = J_{1/n}f(x', t) = [J_{1/n}(x' - \circ), f(\circ, t)]$ where $[\circ, \circ]$ is the $\mathbf{H}^{1/2}$, $\mathbf{H}^{-1/2}$ duality. Now since $\overline{\Lambda}f_n = J_{1/n}\overline{\Lambda}f$, $|f - f_n| + |\overline{\Lambda}f - \overline{\Lambda}f_n| \to 0$ as $n \to \infty$.

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