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The Cauchy Problem for Electromagnetic Wave Propagation in Globally Perturbed Nonselfadjoint Media

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0. INTRODUCTION

The Cauchy problem for perturbations of the system of partial differential equations known as Maxwell's equations is the subject of this paper.

The purpose of our investigation is at least two-fold. First, the perturbation of the Maxwell equations is rarely considered directly (for good reasons the reader will appreciate if he dare continue) so we take up this challenge, and, second, perturbations of the type considered here, which are not locally acting, offer a generality in formulation not available in the local case. In this connection a word about motivation is in order. The choice of the problem was dictated by an inspiration: is it possible, in general systems of equations, to formulate perturbation problems in terms of how propagation modes (for those trained to think in terms of waves) are forced to interact with each other by the perturbation. The Maxwell system is the simplest nontrivial setting in which to study this question. A plethora of mathematical uncertainties arise here and no doubt we have failed to address many of them in the most elegant fashion. All this is meant to point out that the problem we study is not motivated by any particular physical model (though it is not difficult to think of some) but rather by a "new" mathematical idea. Hence the body of the paper is analysis, not model justification. The question of whether the mathematics done here is "pure" or "applied" then may be answered differently by different persons. We think it "applied" in the best sense of the word. After the disclaimer it may be wise to *speculate* on physical interpretation. As to

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what kinds of systems may be viewed in the way discussed here, we leave that to the reader, which is where the hard part is always left. The interpretation of the *effect* of perturbations like those studied below is made explicit by the mathematics and does not require, nor would it be possible to give, explanation here. It is worth noting now though that an extrapolation from the physical understanding yielded in the past by various perturbation problems for the Maxwell equations seems to tell us that the same kind of new insight may occur here.

What forms such understanding may take is not part of our study but is a matter to be established by experimentation, observation, and perhaps the rethinking of "old" problems, etc.

The work here is to be regarded as a natural continuation of [3], which depends in turn upon several articles by J. R. Schulenberger *et. al.* (see [6]).

The informed reader will recognize on sight that the problem studied here is likely to be ill-posed in general and in very fact we are able to show (construct) solutions for only restricted classes of initial data. It is a little unsatisfying but there evidently is no help for this.

Our methodology is to use the nonstandard spectral theory of certain operators related to our stated PDE problem together with standard semigroup techniques. Again, the chief idea behind our construction is that the perturbation does not "mix" the stationary and antistationary data in arbitrary fashion but in a controlled way about which we have very specific quantitative information. This paper is organized into two sections beyond this one (which contains a brief review of required material from [3]) with Section 1 a discussion of the problem of eigenfunctions and Section 2 the Cauchy problem proper.

For convenience we review some of the material from [3]. We study

$$A(D, x)\, u = -iA(D)\, u + B(u) \tag{0.1}$$

where $A(D)$ is Maxwell's operator:

$$A(D)\, u = \begin{bmatrix} i \operatorname{rot} u^2 \\ -i \operatorname{rot} u^1 \end{bmatrix}; \tag{0.2}$$

u_1 and u^2 represent the field vectors (electric and magnetic). The symbol $A(p)$ of $A(D)$ is a 6×6 Hermitian matrix of rank 4. We shall use Φ and Φ^* to stand for the Fourier transformation and its adjoint, respectively. Φf is also denoted by $\hat{f}$ and $\check{f}$ denotes $\Phi^* f$ as well. $A(D)$ may be defined on $L^2(\mathbb{R}^3, \mathbb{C}^6)$ (the square integrable 6-space valued functions on $\mathbb{R}^3$) as

$$A(D) = \Phi^* A(p)\, \Phi. \tag{0.3}$$

The space $H = L^2(\mathbb{R}^3, \mathbb{C}^6)$ has the orthogonal direct sum decomposition $H = H_1 \oplus H_0$, where $H_0 = P_0 H$ is the null space of (0.3) and $H_1 = PH$ is its orthocomplement.

For Maxwell's equations, H_0 is nontrivial and the same is true of the other wave propagation phenomena of classical physics, which therefore never have elliptic spatial part.

In [3] we studied the steady-state wave propagation problem for

$$A(D)+PB \tag{0.4}$$

on H_1, where B was assumed to be an integral operator in momentum space with a kernel of Carleman type. There exist maps σ, σ^*: [6]

$$\begin{aligned}\sigma &: BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2) \to H\\ \sigma^* &: H \to BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)\end{aligned} \tag{0.5}$$

which have the property that

$$P\sigma = \sigma \qquad \text{and} \qquad \sigma^* P = \sigma^*. \tag{0.6}$$

Here $BL(\mathbb{R}^3, \mathbb{C}^2)$ consists of the completion of $\mathscr{S}(\mathbb{R}^3, \mathbb{C}^2)$ in the Hilbert space norm

$$\| f \|_{BL}^2 = \int_{\mathbb{R}^3} |p|^2 \, |(\Phi f)(p)|^2 \, dp \tag{0.7}$$

$\mathscr{S}(\mathbb{R}^3, \mathbb{C}^2)$ denoting the rapidly decreasing functions on $\mathbb{R}^3$ with values in $\mathbb{C}^2$.

Explicitly

$$P_0 = \Phi^* \hat{P}_0(p)\, \Phi \tag{0.8}$$

$$P = \Phi^* \hat{P}(p)\, \Phi = \Phi^* \hat{P}_+(p)\, \Phi + \Phi^* \hat{P}_-(p)\, \Phi \tag{0.9}$$

$$\sigma = \Phi^* \hat{\sigma}(p)\, \Phi \tag{0.10}$$

$$\sigma^* = \Phi^* \hat{\sigma}^*(p)\, \Phi \tag{0.11}$$

with

$$P_0(p) = \begin{bmatrix} \dfrac{p \otimes p}{|p|^2} & 0_{3\times 3} \\ 0_{3\times 3} & \dfrac{p \otimes p}{|p|^2} \end{bmatrix} \tag{0.12}$$

$$\hat{P}_\pm(p) = \begin{bmatrix} -\dfrac{(p_\wedge)}{|p|^2} & \pm\dfrac{p_\wedge}{|p|^2} \\ \pm\dfrac{p_\wedge}{|p|^2} & -\dfrac{(p_\wedge)^2}{|p|^2} \end{bmatrix} \tag{0.13}$$

$$2^{1/2}\hat{\sigma}(p)=(2(w_1^2+w_2^2))^{-1/2}\begin{bmatrix} a(w)\,|p| & 0_{3\times 2} \\ 0_{3\times 2} & ib(w) \end{bmatrix} \tag{0.14}$$

$$2^{1/2}\hat{\sigma}^*(p)=(2(w_1^2+w_2^2))^{-1/2}\begin{bmatrix} \dfrac{a(w)^*}{|p|} & 0_{2\times 3} \\ 0_{2\times 3} & -ib(w)^* \end{bmatrix} \tag{0.15}$$

where $(w=p/|p|=(w_1, w_2, w_3))$ and

$$a(w)=\begin{bmatrix} -w_1 w_3 & -w_2 \\ -w_2 w_3 & w_1 \\ w_1^2+w_2^2 & 0 \end{bmatrix} \tag{0.16}$$

$$b(w)=\begin{bmatrix} w_2 & -w_1 w_3 \\ -w_1 & -w_2 w_3 \\ 0 & w_1^2+w_2^2 \end{bmatrix}. \tag{0.17}$$

In (0.12), (0.13)

$$p\otimes p=\begin{bmatrix} p_1^2 & p_1 p_2 & p_1 p_3 \\ p_1 p_2 & p_2^2 & p_2 p_3 \\ p_1 p_2 & p_3 p_2 & p_3^2 \end{bmatrix} \tag{0.18}$$

and

$$p_\wedge=\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}. \tag{0.19}$$

We assumed in [3] that the "kernel" of B, $\tilde{B}(x, y)$ had the property that $\hat{P}$ images of its columns are $\hat{\sigma}$ images and that the $\hat{\sigma}$ preimage has certain asymptotic properties:

(1) $\sigma^* PBu=(\Phi^* K)f$, where $K\in L^2(\mathbb{R}^3, \mathbb{C}^{16})$ and $\sigma^* u=f$,

(2) $\{(\Phi^* K)(x)(1+|x|)^{1/2+\delta}\}\in L^2(\mathbb{R}^3, \mathbb{C}^{16})$ and

(3) $k_{ij}(x)\in L^1(\mathbb{R}^3)$ for $i=1, 2$, $1\leqslant j\leqslant 4$, where $K=(k_{ij})$.

It was shown in [3] that on H_1, $A(D)+B$ may have eigenvalues scattered throughout the plane with the real axis containing the essential spectrum where $1\notin\sigma(\mathscr{K})$ with

$$\mathscr{K}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}. \tag{0.20}$$

(See Theorem 2.11 of [3].) As noted in [3] $\overset{*}{\hat{K}}$ factors as $K_2 K_1 = K_1 K_2$ with

$$\sup_x \int_{\mathbb{R}^3} |x-y|^{-2} |K_1(y)|^2 \, dy < \infty, \tag{0.21}$$

and $K_2 \in L^2(\mathbb{R}^3, \mathbb{C}^{16})$.

For purposes of our work here we shall further assume that

(4) $k_{ij}(x) \in L^1(\mathbb{R}^3)$ $1 \leqslant i \leqslant 4$, $j = 3, 4$, and $K_2 \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{C}^{16})$, and $\overset{*}{\hat{K}}(H^2 \oplus H^1) \subseteq H^1 \oplus L^2$ where H^1 and H^2 are the usual Sobolev spaces.

Under Fourier transformation,

$$R_0(\xi) = \sigma^* \{A(D) - \xi I\}^{-1} \tag{0.22}$$

may be written as

$$\hat{R}(p, \xi) \overset{*}{\sigma} = i \begin{bmatrix} \frac{\xi i}{|p|^2 - \xi^2} & 0 & \frac{-1}{|p|^2 - \xi^2} & 0 \\ 0 & \frac{\xi i}{|p|^2 - \xi^2} & 0 & \frac{-1}{|p|^2 - \xi^2} \\ \frac{|p|^2}{|p|^2 - \xi^2} & 0 & \frac{\xi i}{|p|^1 - \xi^1} & 0 \\ 0 & \frac{|p|^2}{|p|^2 - \xi^2} & 0 & \frac{\xi i}{|p|^2 - \xi^2} \end{bmatrix} \sigma^*. \tag{0.23}$$

(im $\xi \neq 0$).

In the following sections we will be working under the assumption that $P_0 B$ is a compact operator. This is true, for example, if $\hat{P}_0(p)\, \Phi_p \tilde{B}(\cdot, y)$ is square integrable jointly in p and y. In any case we shall assume that for $f \in H$,

$$P_0 B f = \sum_{i=1}^{\infty} \mu_i (f, \psi_i)\, \phi_i \tag{0.24}$$

where $\{\psi_i\}$ and $\{\phi_i\}$ are orthonormal sequences and $\{\phi_i\} \subseteq H_0$.

1. Eigenfunction Expansions

We note that if im $\lambda \neq 0$, then

$$(A(D) - \lambda I)^{-} f = u \tag{1.1}$$

is represented as (on H_1)

$$u(x, \lambda) = \int_{\mathbb{R}^3} \Omega(x-y, \lambda)\, f(y)\, dy = \Omega_* f \tag{1.2}$$

and

$$\Omega_* f = \Phi^* \hat{\Omega} \Phi f \tag{1.3}$$

with

$$\hat{\Omega}(p, \lambda) = \frac{2}{|p|^2 - \lambda^2} \begin{bmatrix} -\dfrac{\lambda(p_\wedge)^2}{|p|^2} & -\dfrac{p_\wedge}{|p|^2} \\ \dfrac{p_\wedge}{|p|} & -\dfrac{\lambda(p_\wedge)^2}{|p|^2} \end{bmatrix}. \tag{1.4}$$

Let $\lambda_j(p)$ stand for the three eigenvalues (each of multiplicity two) of $A(p)$, $j = -1, 0, 1$. Define the functions

$$\Psi_j^0(x, p) = (2\pi)^{-3/2} e^{ix \circ p} \hat{P}_j(p). \tag{1.5}$$

The Ψ_j^0 are then improper eigenfunctions for $A(D)$ since

$$(A(D) - \lambda_j(p))\, \Psi_j^0(p, x) = 0. \tag{1.6}$$

Defining

$$\Phi_j = \hat{P}_j \Phi \tag{1.7}$$

then

$$\Phi_j^* \Phi_k = \delta_{jk} P_k. \tag{1.8}$$

These facts immediately show that

$$\Phi_j A \subset \lambda_j(p)\, \Phi_j, \qquad j = -1, 0, 1. \tag{1.9}$$

Since

$$\Psi_j^*(x, p) = (2\pi)^{-3/2} e^{-ix \circ p} P_j(p), \tag{1.10}$$

we have the expansions

$$f = \sum \Phi_j^* \hat{f}_j; \qquad \hat{f}_j = \Phi_j f \tag{1.11}$$

which can be written

$$\hat{f}_j(p) = \underset{M \to \infty}{\text{l.i.m.}} \int_{|x| \leqslant M} \Psi_j^0(x, p)^* f(x)\, dx \tag{1.12}$$

and

$$f(x) = \sum_{j=-1}^{1} \underset{M \to \infty}{\text{l.i.m.}} \int_{|p| \leqslant M} \Psi_j^0(x, p)\, \hat{f}_j(p)\, dp. \tag{1.13}$$

Equation (1.13) implies that

$$(Pf)(x) = \sum_{j \neq 0} \underset{M \to \infty}{\text{l.i.m.}} \int_{|p| \leqslant M} \Psi_j^0(x, p)(Pf)_j(p)\, dp. \tag{1.14}$$

To construct the eigenfunctions for the perturbed problem we seek first matrix-valued functions $\Psi_j(x, p, \lambda)$ which are (formally) solutions to

$$\Psi_j(x, p, \lambda) = \Psi_j^0(x, p) - \int_{\mathbb{R}^3} \Omega(x - y, \lambda)\, PB\Psi_j(\cdot, p, \lambda)\, dy \tag{1.15}$$

with limits ($j = -1, 1$)

$$\Psi_j^{\pm}(x, p) = \lim_{\varepsilon \to 0} \Psi_j(x, p, \lambda_j(p) \pm i\varepsilon). \tag{1.16}$$

It is apparent that the limits (1.16) will fail to exist when $\lambda_j(p) \in \mathscr{S}$ (see Lemma (3.1) of [3]). Thus at best we can obtain an expansion valid in a subspace of H_1. This subspace is an analog of the "absolutely continuous" subspace for a selfadjoint operator; however, it will fail to be a closed subspace in general. The setting of Eq. (1.15) is difficult to deal with especially since we shall require adjoint eigenfunctions to furnish any sort of expansion analogous to (1.13).

Let

$$L = \sigma^* A(D)\, \sigma + \sigma^* PB\sigma \tag{1.17}$$

on $BL(\mathbb{R}^3, \mathbb{C}^2) \oplus L^2(\mathbb{R}^3, \mathbb{C}^2)$. Then the resolvent of L, $R(\lambda)$, can be written as

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)\, \sigma^* PB\sigma R(\lambda), \tag{1.18}$$

leading to the "kernel" equation

$$R(x, y, \lambda) = R_0(x, y, \lambda) - \int_{\mathbb{R}^3} R_0(x, y, \lambda)\, \sigma^* PB\sigma R(z, y, \lambda)\, dz. \tag{1.19}$$

Here $R_0(x, y, \lambda)$ is given by $\Phi^* \hat{R}(\cdot, \lambda)$ (see (0.23) above), and (1.18) holds by an appeal to the proof of (2.7) of [3], and assumption (4).

$R_0(x, y, \lambda)$ may be decomposed as

$$R_0(x, y, \lambda) = \tilde{D}_1(x, y) + \tilde{D}_2(x, y, \lambda) \tag{1.20}$$

where

$$\tilde{D}_1(x, y) = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta(x - y) & 0 & 0 & 0 \\ 0 & \delta(x - y) & 0 & 0 \end{bmatrix} \tag{1.21}$$

and

$$\tilde{D}_2^{\pm}(x, y, \lambda) = i\sqrt{\frac{\pi}{2}}\frac{\exp(\pm i\lambda\,|x-y|)}{|x-y|}\begin{bmatrix} \lambda & 0 & i & 0 \\ 0 & & 0 & i \\ -\lambda^2 & 0 & \lambda & 0 \\ 0 & -\lambda^2 & 0 & \lambda \end{bmatrix}. \tag{1.22}$$

As in [3] we define

$$s(\lambda) = (I + \mathscr{K} + K_1 D_2(\lambda)\, K_2)^{-1}, \tag{1.23}$$

then $s(\lambda)$ is a bounded operator on Hilbert space for all $\lambda \in \mathbb{C}\setminus S$ (see Lemma 2.10 [3]) with values $s^+(\lambda)$, $s^-(\lambda)$ for $\lambda \in \bar{\mathbb{C}}_+\setminus S$ and $\lambda \in \bar{\mathbb{C}}_-\setminus S$, respectively. Set

$$\Psi'_{\pm}(\cdot, p, \lambda) = s^{\pm}(\lambda)\, \Psi'_0(\cdot, p, \lambda) \tag{1.24}$$

where

$$\Psi'_0(x, p, \lambda) = K_1(x)\, \hat{R}'_0(x, p, \lambda)\, \hat{\sigma}^*(p) \tag{1.25}$$

and

$$\hat{R}'_0(x, p, \lambda) = (|p|^2 - \lambda^2)\, \Phi_p^* R_0(x, \cdot, \lambda). \tag{1.26}$$

Then for im $\lambda \neq 0$,

$$\psi'(x, p, \lambda) = \Psi'_0(x, p, \lambda) - K_1 R_0(\lambda)\, K_2 \Psi'(\cdot, p, \lambda). \tag{1.27}$$

It is easily seen that $\hat{R}'_0$ is smooth and bounded in (x, p) for all λ and is smooth (analytic) in λ.

LEMMA 1.1. *The columns $\Psi'_{\pm}(\cdot, p, \lambda)_l$, $1 \leqslant l \leqslant 6$, of $\Psi'_{\pm}(\cdot, p, \lambda)$ are continuous $L^2 \oplus L^2$-valued functions of $(p, \lambda) \in (\mathbb{R}^3\setminus\{0\} \times A_{\pm}$, where $A_{\pm} \subset\subset \bar{\mathbb{C}}\setminus S$. (That is, $A_+ \subset\subset \bar{\mathbb{C}}_+\setminus S$, $A_- \subset\subset \bar{\mathbb{C}}_-\setminus S$.)*

Proof.

$$\|\Psi'_{\pm}(\cdot, p_1, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_2)\|^2 \leqslant 2\,\|\Psi'_{\pm}(\cdot, p_1, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_1)\|^2 + 2\,\|\Psi'_{\pm}(\cdot, p_2, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_2)\|^2$$

since

$$\Psi'_{0_{\pm}}(x, p, \lambda) = (2\pi)^{3/2}\, M(p, \lambda)\, K_1(x)\, \hat{\sigma}^*(p). \tag{1.28}$$

Equation (1.13) implies that

$$(Pf)(x) = \sum_{j \neq 0} \operatorname*{l.i.m.}_{M \to \infty} \int_{|p| \leqslant M} \Psi_j^0(x, p)(Pf)_j(p)\, dp. \tag{1.14}$$

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leading to the "kernel" equation

$$R(x, y, \lambda) = R_0(x, y, \lambda) - \int_{\mathbb{R}^3} R_0(x, y, \lambda)\, \sigma^* PB\sigma R(z, y, \lambda)\, dz. \tag{1.19}$$

Here $R_0(x, y, \lambda)$ is given by $\Phi^* \hat{R}(\cdot, \lambda)$ (see (0.23) above), and (1.18) holds by an appeal to the proof of (2.7) of [3], and assumption (4).

$R_0(x, y, \lambda)$ may be decomposed as

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$$s(\lambda) = (I + \mathscr{K} + K_1 D_2(\lambda)\, K_2)^{-1}, \tag{1.23}$$

then $s(\lambda)$ is a bounded operator on Hilbert space for all $\lambda \in \mathbb{C}\backslash S$ (see Lemma 2.10 [3]) with values $s^+(\lambda)$, $s^-(\lambda)$ for $\lambda \in \bar{\mathbb{C}}_+\backslash S$ and $\lambda \in \bar{\mathbb{C}}_-\backslash S$, respectively. Set

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Proof.

$$\begin{aligned}\|\Psi'_{\pm}(\cdot, p_1, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_2)\|^2 \leqslant 2\, &\|\Psi'_{\pm}(\cdot, p_1, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_1)\|^2 \\ &+ 2\, \|\Psi'_{\pm}(\cdot, p_2, \lambda_1) - \Psi'_{\pm}(\cdot, p_2, \lambda_2)\|^2\end{aligned}$$

since

$$\Psi'_{0\pm}(x, p, \lambda) = (2\pi)^{3/2}\, M(p, \lambda)\, K_1(x)\, \hat{\sigma}^*(p). \tag{1.28}$$

We have

$$\|\Psi'_{0\pm}(\cdot, P_1, \lambda_1) - \Psi'_{0\pm}(\cdot, p_1, \lambda_2)\|^2$$

$$\leqslant C \int_{\mathbb{R}^3} |K_1(x)|^2 \, dx \, M(p_1, \lambda_1) - M(p_1, \lambda_2)|^2 \tag{1.29}$$

$$\leqslant C \|K_1\|_2^2 \left(3 + \frac{2}{|p_1|^2}\right) |\lambda_1 - \lambda_2|^2 \tag{1.30}$$

and

$$\|\Psi'_{0\pm}(\cdot, p_1, \lambda_2) - \Psi'_{0\pm}(\cdot, p_2, \lambda_2)\|^2$$

$$\leqslant 2 |\lambda_2|^2 \|K_1(\cdot)[\exp(i(\cdot \circ p) - \exp i(\cdot \circ p_2)] f(p_1)\|^2$$

$$+ 2 |\lambda_2|^2 \|K_1(\cdot) \exp i(\cdot \circ p_2) f(p_2)$$

$$- K_1(\cdot) \exp i(\cdot \circ p_2) f(p_1)\|^2$$

$$= 2 |\lambda_2|^2 |f(p_1)|^2 \|K_1(\cdot)[\exp i(\cdot \circ p_1) - \exp i(\cdot \circ p_2)\|^2$$

$$+ 2 |\lambda_2|^2 \|K_1\|_2^2 |f(p_2) - f(p_1)|^2 \tag{1.31}$$

where $f(\eta)\, n = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \setminus \{0\}$ is equal to

$$\sqrt{2}\left(\left(\frac{\eta_1}{|\eta|}\right)^2 + \left(\frac{\eta_2}{|\eta|}\right)^2\right)^{1/2} \tag{1.32}$$

$$\begin{bmatrix} -\frac{\eta_1\eta_3 i}{|\eta|^3} & -\frac{\eta_2\eta_3 i}{|\eta|^3} & \frac{i\eta_1^2}{|\eta|^3} + \frac{i\eta_2^2}{|\eta|^3} & \frac{i\eta_2}{|\eta|} & -\frac{i\eta_1}{|\eta|} & 0 \\ -\frac{i\eta_2}{|\eta|^2} & \frac{i\eta i}{|\eta|^2} & 0 & -\frac{i\eta_1\eta_3}{|\eta|^2} & -\frac{\eta_2\eta_3}{|\eta|^2} & \frac{i\eta_1 + i\eta_2}{|\eta|^2} \\ \frac{\eta_1\eta_2}{|\eta|^2} & -\frac{\eta_2\eta_3}{|\eta|} & \frac{\eta_1^2 + \eta_2^2}{|\eta|} & \frac{i\eta_2}{|\eta|} & -\frac{i\eta_1}{|\eta|} & 0 \\ -\eta_2 & \eta_1 & 0 & -\frac{i\eta_1\eta_3}{|\eta|^2} & -\frac{i\eta_2\eta_3}{|\eta|^2} & \frac{i\eta_1^2 + i\eta_2^2}{|\eta|^2} \end{bmatrix}.$$

Thus $f(p_1) \to f(p_2)$ as $p_1 \to p_2$ if $|p_1|$, $|p_2| > \varepsilon > 0$ for all $\varepsilon > 0$. Since

$$\|K_1(\cdot)[\exp i(\cdot \circ p_1) - \exp i(\cdot \circ p_2\|^2$$

$$\leqslant C_1 \int_{|x| > R} |K_1(x)|^2 \, dx$$

$$+ C_2 |p_1 - p_2|^2 \int_{|x| < R} |K_1(x)|^2 \, |x|^2 dx, \tag{1.33}$$

we see that

$$(p, \lambda) \to \Psi'_{0\pm}(\cdot, p, \lambda)$$

is uniformly continuous on compact subsets of $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{C}_{\pm}$. Now the fact that $s(\lambda)$ is continuous on $\bar{\mathbb{C}}_{+} \setminus S$ and $\mathbb{C}_{-} \setminus S$ in the uniform norm shows that

$$(p, \lambda) \to \Psi'_{\pm}(\cdot, p, \lambda) \tag{1.34}$$

are uniformly continuous on compact subsets of $(\mathbb{R}^3 \setminus \{0\}) \times (\bar{\mathbb{C}}_{-} \setminus S)$.

DEFINITION 1.2. Define the 6×6 matrix-valued functions $\Psi_{\pm}(x, p, \lambda)$ and $V_{\pm}(x, p, \lambda)$ as

(1)
$$V_{\pm}(x, p, \lambda) = \hat{\sigma}(x) \int_{\mathbb{R}^3} \{\tilde{D}_1(x, y) + \tilde{D}_2^{\pm}(x, y)\} \times K_2(y)\, \Psi'_{\pm}(y, p, \lambda)\, dy \tag{1.35}$$

(2)
$$\Psi_{\pm}(x, p, \lambda) = \hat{\sigma}(x)\, \hat{R}'(x, p, \lambda) - V_{\pm}(x, p, \lambda). \tag{1.36}$$

THEOREM 1.3. *$V_{\pm}(x, p, \lambda)$ are bounded continuous functions in compact subsets of*

$$(\mathbb{R}^3 \setminus \{0\} \times (\mathbb{R}^3 \setminus \{0\}) \times (\bar{\mathbb{C}}_{\pm} \setminus S)$$

$\Psi_{\pm}$ is the unique solution to

$$\Psi_{\pm}(x, p, \lambda) = \hat{\sigma}(x) \exp(ix \circ p)\, \hat{R}'(x, p, \lambda) - \hat{\sigma}(x) \int_{\mathbb{R}^3} (\tilde{D}_1(x, y) + \tilde{D}_2^{\pm}(x, y, \lambda)\, \overset{*}{\hat{K}}(y)\, \hat{\sigma}^*(y)\, \Psi_{\pm}(y, p, \lambda)\, dy \tag{1.37}$$

for $\lambda \notin S$.

Proof. Multiplying $\Psi_{\pm}(x, y, \lambda)$ by $K_1(x)\, \hat{\sigma}^*(x)$ we have $\Psi'_{\pm}(x, p, \lambda) = K_1(x)\, \hat{\sigma}^*(x)\, \Psi_{\pm}(x, p, \lambda)$. By (1.24), substitution into (1.35) yields (1.37). The required continuity of $V_{\pm}$ follows from Lemma (1.1) and by the fact that the first two rows of $\overset{*}{\hat{K}}$ are continuous together with

$$\int_{\mathbb{R}^3} |\tilde{D}_2(x_1, y, \lambda) - \tilde{D}_2(x_2, y, \lambda)|^2\, |K_2(y)|^2\, dy$$
$$\leqslant 2C_1(\lambda)\, |x_1 - x_2|^2 \int_{\mathbb{R}^3} \frac{|K_2(y)|^2}{|x - y|^2}\, dx$$
$$+ 2C_2(\lambda)\, |x_1 - x_2|\, \pi \left(\int_0^1 \ln\left(\frac{u+1}{u-1}\right) \frac{du}{u} \right) M + \delta \tag{1.38}$$

where

$$\delta = \int_{|y|>R} \frac{|K_2(y)|^2}{|x_1-y|^2\,|x_2-y|^2}\,dy \tag{1.39}$$

and x_1, $x_2 \in \{x \mid |x|<R\}$. $M=2$ ess.$_{|x|<R}$ sup $|K_2|^2$. Since $\int_0^1 \ln((u+1)/(u-1))\,du/u$ converges as an elementary improper integral, the theorem follows.

THEOREM 1.4. *The function* $\Psi_\pm(x, p, \lambda_1(p))$, $\Psi_\pm(x, p, \lambda_{-1}(p))$ *satisfy Eq.* (1.15). *The functions*

$$\Psi_j(x, p) = \Psi_\pm(x, p, \lambda_j(p))(\lambda_j(p) \notin S) \tag{1.40}$$

are thus the generalized eigenfunctions sought. Suppose (1.42) *holds.*

There exist adjoint eigenfunctions $\Psi_j^*(x, p)$ *for which Eqs.* (1.12) *and* (1.13) *are satisfied in the family of subspaces* Y_δ (*see Theorem* 1.8).

Proof. Note that the first term in the RHS of (1.15) is equal to the first term of the RHS of (1.37). The $\Psi_j(x, p)$ satisfy the "Lippman–Schwinger" equation. The adjoint functions exist, where the adjoint of

$$\sigma^* A(D)\,\sigma + \sigma^* PH\sigma$$

is defined on the space $L^2(\mathbb{R}^3, \mathbb{C}^2) \oplus BL(\mathbb{R}^3, \mathbb{C}^2)$, and the analogous arguments to those above hold for this system.

Here the realization of σ and σ^* are the obverse mappings $\hat\sigma_0 \hat\sigma_0^*$ with

$$\begin{aligned}\hat\sigma_0(p)\hat f &= \sqrt{2}\,(2(w_1^2+w_2^2))^{-1/2}\,(ib(w)\hat f^1,\ |p|\,a(w)\hat f^2)\\ \hat\sigma_0^*(p)\hat f &= \sqrt{2}\,(2(w_1^2+w_2^2))^{-1/2}\,(-ib^*(w)\hat f^1,\ |p|^{-1}\,a^*(w)\hat f^2)\end{aligned} \tag{1.41}$$

or, with our assumptions on $\overset{*}{\hat K}$

$$\begin{aligned}(\sigma^* A(D)\,\sigma)^* &= \sigma_0^* A(D)\,\sigma_0\\ (\sigma \overset{*}{\hat K} \sigma^*)^* &= \sigma_0 \overset{*}{\hat K}{}^* \sigma_0^*\end{aligned} \tag{1.42}$$

and $\overset{*}{\hat K}{}^*$ has properties identical to those of $\overset{*}{\hat K}$ on $BL \oplus L^2$. (Again we assume $1 \notin \sigma(\mathscr{K})$.)

That the expansion converges on certain subspaces of H_1 follows from Theorem (1.8) below.

THEOREM 1.5. *The limit*

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\mathscr{E}_\delta} (R(\lambda+i\varepsilon) - R(\lambda - i\varepsilon)[f, g)\,d\lambda \tag{1.43}$$

exists in $L^2 \oplus L^2$ *for* $\mathscr{E}_\delta \subseteq \mathbb{R}^1 \backslash S$, *where for* $\delta > 0$, $\mathscr{E}_\delta = \{t \in \mathbb{R} \mid |t| \leqslant \delta^{-1}$ *and* $|t - s| \geqslant \delta$ *for all* $s \in S\}$. *We have* $\bigcup_{\delta > 0} \mathscr{E}_\delta = \mathbb{R}^1 \backslash S$.

Proof. We know that for $\lambda \notin S$ and ε small,

$$\begin{aligned} R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon) \\ &= R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon) \\ &\quad - R_0(\lambda + i\varepsilon)\, \overset{*}{K} R(\lambda + i\varepsilon) + R_0(\lambda - i\varepsilon)\, \overset{*}{K} R(\lambda - i\varepsilon) \end{aligned} \tag{1.44}$$

and therefore

$$\begin{aligned} ([R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]\, f, g) \\ &= ([R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)]\, f, g) \\ &\quad - (K_1 R(\lambda + i\varepsilon) f, K_2^* R_0(\lambda - i\varepsilon)\, g) \\ &\quad + (K_1 R(\lambda - i\varepsilon) f, K_2^* R_0(\lambda + i\varepsilon)\, g). \end{aligned} \tag{1.45}$$

Integration and application of the Schwartz inequality and Lemma 3.1 and Lemma 2.10 of [3] give the result.

THEOREM 1.6. *For each* $\mathscr{E}_\delta$ *there exists a bounded projection* $\mu(\mathscr{E}_\delta)$ *defined on* $L^2 \oplus L^2$ *and fixing* $BL \oplus L^2 \cap L^2 \oplus L^2$ *given by the limit* (1.43). $\mu(\mathscr{E}_\delta)$ *commutes with* $R(\lambda)$ *and* $R(\lambda)^{-1}$. *Furthermore,* $\mu(\cdot)\, x$ *can be extended as a countably additive vector measure on the Borel sets of* $\mathbb{R}^1$ *for any* x *in the range of* $\mu(\mathscr{E}_\delta)$, *for some* δ *satisfactory,* $0 < \delta < 1$. *The integral*

$$\int f\, d\mu \tag{1.46}$$

defines operators $f(R(0)^{-1})$ *on* $\mathscr{D}(R(0)^{-1})$ *or some subspace thereof* (*see* [5]).

Proof. It is clear that the limit (1.43) defines a bounded bilinear form $\mu(f, g)(\mathscr{E}_\delta)$ with

$$|\mu(f, g)(\mathscr{E}_\delta)| \leqslant C(\delta)\, \|f\|\, \|g\|. \tag{1.47}$$

The Riesz theorem shows that

$$\mu(f, g)(\mathscr{E}_\delta) = (\mu(\mathscr{E}_\delta) f, g) \tag{1.48}$$

for some (unique) bounded operator $\mu(\mathscr{E}_\delta)$. That $\mu(\mathscr{E}_\delta)$ commutes with $R(\lambda)$ and $R(\lambda)^{-1}$ follows by standard arguments. Since for a Borel set in $\mathscr{E}_\delta$ we have

$$\mu(N)\, R(\lambda)^{-1} = R(\lambda)^{-1}\, \mu(N) \tag{1.49}$$

and

$$R(\lambda): BL \oplus L^2 \cap L^2 \oplus L^2 \to BL \oplus L^2 \cap L^2 \oplus L^2 \tag{1.50}$$

we see that the range of $\mu(\cdot)$ has the required properties. Now for any polynomial f, $(R(0)^{-1} = L)$

$$f(R(0)^{-1})\, \mu(\mathscr{E}_\delta) = \lim_{\varepsilon \to 0} \int_{\mathscr{E}_\delta} [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]\, f(\lambda)\, d\lambda \tag{1.51}$$

with the inequality

$$\| f(R(0)^{-1})\, \mu(\mathscr{E}_\delta)\, x \| \leqslant \sup_{\lambda \in \mathscr{E}_\delta} |f(\lambda)|\, \|x\|\, \|\mu(\mathscr{E}_\delta)\|,$$

for with $\lambda \in \mathscr{E}_\delta$,

$$\begin{aligned} (f(\lambda)\, R(\lambda \pm i\varepsilon)| &\leqslant \sup_{\lambda \in \mathscr{E}_\delta} |f(\lambda)|\, |R(\lambda \pm i\varepsilon)\, x, g)| \\ &\leqslant \sup_{\lambda \in \mathscr{E}_\delta} |f(\lambda)|\, \|K_1 R(\lambda \pm i\varepsilon)\, x\|\, \|K_2^* R_0(\lambda \mp i\varepsilon)\, g\| \end{aligned} \tag{1.52}$$

integration leads to the required inequality.

The Weierstrass theorem, together with a routine approximation argument, shows that

$$f \to \int_{\mathscr{E}_\delta} f(\lambda)\, d\mu \tag{1.53}$$

defines an algebra map on the collection of bounded Borel functions. Since $\mu(\cdot)\, x \equiv \mu_x(\cdot)$ is clearly weakly countably additive for $x \in \bigcup_{\delta > 0}$ range $(\mu(\mathscr{E}_\delta))$ it is strongly countably additive by the Pettis theorem. Hence (1.53) may be extended to unbounded functions, see Theorem (2.11) of [5]. Equation (1.53) shows that $\mu(N)$ is a projection.

THEOREM 1.7. *$R(\lambda)$ defines a family of strongly differentiable Fourier semigroups $\mathscr{L}_\delta(t)$ on the range of μ $(= \bigcup_{\delta > 0}$ range $\mu(\mathscr{E}_\delta))$.*

Proof. By the spectral representation (1.53) we have

$$R(\lambda)\, x = \int_{\mathbb{R}^1} \frac{d\mu_x(z)}{\lambda - z}, \quad \lambda \in \rho(R(0)^{-1}) \tag{1.54}$$

where $x \in$ range μ, and that

$$\mathscr{L}(t)\, x = \int_{\mathrm{R}^1} e^{itz}\, d\mu_x(z) \tag{1.55}$$

is strongly continuous by the bounded convergence theorem (see (2.11) of [5]).

The same result implies that the difference quotients

$$\frac{\mathscr{Z}(t+h)\,x-\mathscr{Z}(t)\,x}{h}.$$

Converge to $i\mathscr{Z}(t)\,R(0)^{-1}\,x$. $(x\in\mathscr{D}(L))$.

Remark. If e is a Borel subset of some $\mathscr{E}_\delta$, then $\mu(e)$ is continuous on $\mathscr{D}(R(0)^{-1})$ to itself with $L^2\oplus L^2$-topology on its domain and the graph norm topology of $R(0)^{-1}$ on its range. That is, let $x_n\to x$ in $\mathscr{D}(R(0)^{-1})$ then

$$\mu(e)\,x_n\to\mu(e)\,x$$

and

$$R(0)^{-1}\,\mu(e)\,x_n\to R(0)^{-1}\,\mu(e)\,x$$

by the operational calculus. Thus $\mu(e)$ is continuous on $L^2\oplus L^2$ with the gradient Sobolev norm. Finally, since the graph norm topology is stronger than the Sobolev gradient topology, we obtain the continuity of $\mu(e)$ on $BL\oplus L^2\to BL\oplus L^2$. We have the following result.

THEOREM 1.8. *$(-i)[A(D)+PB]$ defines a family of strongly differentiable semigroups $\mathscr{Z}_\delta(t)$ on the subspaces*

$$(BL\oplus L^2\cap\text{range}(\mu(\mathscr{E}_\delta)))=Y_\delta. \tag{1.56}$$

Remark 1.9. It is easily checked that $A(D)+PB$ is bounded on the spaces Y_δ via computation like that preceding Theorem 1.8. Hence, $A(D)+PB$ is closed and invariant on each Y, a fact we shall need in the sequel. Bu, PBu, P_0Bu are well defined as tempered distributions with $Bu=PBu+P_0Bu$. If we restrict each of these so that the (maximal) domain and range lie in H, and P_0B is everywhere defined, then the domains of B and PB are the same. Furthermore, if we take P_0B to be compact (condition (0.24)), then since B is bounded as a mapping from H into L^∞ ($\mathbb{R}^3$, $\mathbb{C}^6$), both it and PB are closed. It is easily seen then that (i) the (again maximal) domains of $A(D)+PB$ and $A(D)+B$ coincide, and (ii) these operators are closed on H.

2. The Cauchy Problem

In this section, we pose and solve first a restricted Cauchy problem for $A(D)+B$, and then a more relaxed form solved by the strong limit of solutions to problems of the restricted type. We conclude with some remarks on the stationary problem.

By the term *restricted Cauchy problem* we shall mean the following:

$$u'(t)=(-i)[A(D)+PB]\,u(t)+(-i)\,P_0Bu(t), \qquad t>0 \tag{2.1}$$
$$H-\lim_{t\to 0^+} u(t)=f,$$

with

$$f_0=P_0f\in\mathscr{D}(B)\ \&\ f_1=Pf\in D(A(D)+PB) \tag{2.2}$$

$$PBf_0, f_1\in Y_\delta \qquad \text{for some } \delta>0. \tag{2.3}$$

In addition, P_0B is assumed to be *nuclear* (that is, of the form given in (0.24)) and satisfying the conditions

(i) $\phi_j\in\mathscr{D}(B)$ with $\mathscr{V}_j=PB\phi_j\in Y_\delta$ (same δ and all j), and

(ii) the convergence hypothesis $\sum \mu_j\,\|\mathscr{V}_j\|<\infty$.

By a *strict solution* of (2.1)–(2.3) we mean a strongly continuous function u: $[0,\infty)\to\mathscr{D}(A(D)+PB)$ having a strong derivative and satisfying (2.1)–(2.3).

If we decompose a solution $u(t)$ into its H_0 and H_1 projections, then the system (2.1)–(2.2) may be written in the coupled form

$$u_1'(t)=(-i)[A(D)+PB](u_1(t)+u_0(t)), \qquad t>0 \tag{2.4}$$

$$u_0'(t)=(-i)\,P_0Bu_0(t)+(-i)\,P_0Bu_1(t), \qquad t>0 \tag{2.5}$$

$$f_1=H-\lim_{t\to 0^+} u_1(t), \tag{2.6}$$

$$f_0=H-\lim_{t\to 0^+} u_0(t). \tag{2.7}$$

Since P_0B is bounded, we can use the semigroup properties of $\exp(-itP_0B)$ on (2.5) to obtain

$$u_0(t)=\exp(-itP_0B)\,f_0+P_0B\left(\int_0^t \exp(-i(t-s)\,P_0B)\,u_1(s)\,ds\right), \tag{2.8}$$

where it now follows that $u_0(t)\in\mathscr{D}(B)$ for $t>0$ and hence $u_1(t)\in\mathscr{D}(A(D)+PB)$ for $t>0$. Substituting this expression into (2.4) gives us

$$u_1'(t)=(-i)[A(D)+PB[u_1(t)+(Gu_1)(t)+F(t)\,f_0 \tag{2.9}$$

where for convenience we introduce the auxiliary operators

$$(\tilde{G}u)(t)=\int_0^t \exp(-i(t-s)\,P_0B)\,u(s)\,ds, \tag{A1}$$

$$(Gu)(t)=(-i)\,PBP_0B(\tilde{G}u)(t), \tag{A2}$$

$$F(t)\,f_0=(-i)\,PB\exp(-itP_0B)\,f_0. \tag{A3}$$

Under the assumptions (i) and (ii) placed upon P_0B, (2.9) may be viewed as an evolution equation with state space Y_δ. If $A_\delta=(-i)[A(D)+PB]$ *restricted* to Y_δ, then by Theorem (1.8), A_δ generates a strongly differentiable semigroup, $\mathscr{X}_\delta(t)$, which allows us to recast (2.6) and (2.9) as the integral equation

$$u_1(t)=\mathscr{X}_\delta(t)f_1+\int_0^t \mathscr{X}_\delta(t-s)[(Gu_1)(s)+F(s)f_0]\,ds. \tag{2.10}$$

Before proceeding with the restricted problem, it shall be helpful to establish certain properties of the auxiliary operators (A1), (A2), and (A3). Toward this end, let $\tau>0$ and, for Y a subspace of H, define the space $C(\tau, Y)=C([0,\tau];Y)$ with the norm

$$\|u\|_{0,\tau}=\sup\{\|u(t)\|\mid 0\leqslant t\leqslant\tau\}.$$

LEMMA 2.1.

(i) $\tilde{G}\in\mathscr{B}(C(\tau, Y_\delta), C(\tau, H))$,

(ii) $G\in\mathscr{B}(C(\tau, Y_\delta), C(\tau, Y_\delta))$.

(iii) *for* $f_0\in H_0$, $F(t)\,f_0\in C(\tau, Y_0)$.

Proof. (A1), (A2), and the fact that $\|P_0B\|=\mu_1$ we find that for $0\leqslant t\leqslant \hat{t}\leqslant\tau$ and $u\in C(\tau, Y_\delta)$

$$\|(\tilde{G}u)(\hat{t})-(\tilde{G}u)(t)\|\leqslant e^{\mu_1\tau}[(\hat{t}-t) + \|\exp(-i\hat{t}P_0B)-\exp(-itP_0B)\|\,t]\,\|u\|_{0,\tau},$$

$$\|(Gu)(\hat{t})-(Gu)(t)\|\leqslant(\Sigma\mu_j\,\|\mathscr{V}_j\|)\,\|(\tilde{G}u)(\hat{t})-(\tilde{G}u)(t)\|.$$

Appealing to the continuity of $\exp(-itP_0B)$ then shows that $\tilde{G}u\in C(\tau, H)$ and $Gu\in C(\tau, Y_\delta)$. Further, putting $t=0$ in these inequalities gives us the operator bounds

$$\|\tilde{G}\|_\tau\leqslant\tau e^{\mu_1\tau}, \qquad \|G\|_\tau\leqslant(\Sigma\mu_j\,\|\mathscr{V}_j\|)\,\|\tilde{G}\|_\tau.$$

Finally, the strong continuity of $F(t)f_0$ follows from the properties of $\exp(-itP_0B)$ and the convergence hypothesis $\Sigma\mu_j\,\|\mathscr{V}_j\|<\infty$.

Remark. The operators $\tilde{G}$ and G are actually compact.

LEMMA 2.2. *For fixed* $f_0 \in H_0$ *and* $u \in C(\tau, Y_\delta)$, *the functions* $(\tilde{G}u)(t)$, $(Gu)(t)$, *and* $F(t) f_0$ *all have strongly continuous strong derivatives on* $(0, \tau)$.

Proof. For $(\tilde{G}u)(t)$ this follows from the properties of the semigroup $\exp(-itP_0B)$. For $(Gu)(t)$ and $F(t) f_0$, the result depends on the closedness of PB and the convergence hypothesis.

LEMMA 2.3. *If* $g\colon [0, \tau] \to Y_\delta$ *possesses a strongly continuous strong derivative, then for* $0 < t < \tau$

(i) $h(t) = \int_0^t \mathscr{Z}_\delta(t-s)\, g(s)\, ds \in \mathscr{D}(A_\delta)$,

(ii) $h'(t) = A_\delta h(t) + g(t)$.

Proof. In addition to the hypothesis on g, the argument (cf. [1, pp. 167–169]) requires principally two things: (1) the properties of $\mathscr{Z}_\delta(t)$ already stated, and (2) the closedness of A_δ as shown in Remark 1.9.

Remark. The uniform boundedness in t of $\|\mathscr{Z}_\delta(t)\|$ means that $\sup\{\|\mathscr{Z}_\delta(t)\| \mid t \geqslant 0\} < \infty$. We denote this supremum by $z(\delta)$.

THEOREM 2.4. *Assume that for some* $\delta > 0$

(i) $\phi_j \in \mathscr{D}(B)$ *and* $\mathscr{V}_j = PB\phi_j \in Y_\delta$ *all* j,

(ii) $\Sigma \mu_j \|\mathscr{V}_j\| < \infty$.

Then the initial-value problem (2.6) & (2.9) *has a unique strict solution with a strongly continuous derivative and continuous dependence on* f_1.

Proof. Suppose that $u_1(t)$ satisfies Eq. (2.10) for $0 \leqslant t \leqslant \tau$. Then it can be shows that

$$\|u_1(t)\| \leqslant k_1 + k_2 \int_0^t \|u_1(\hat{s})\|\, d\hat{s}, \tag{2.11}$$

where

$$k_1 = z(\delta)[\|f_1\| + \|PBf_0\|\, \tau + \|f_0\|\, \tau e^{\mu_1 \tau}(\Sigma \mu_j \|\mathscr{V}_j\|)]$$

and where

$$k_2 = z(\delta)\, \tau e^{\mu_1 \tau}(\Sigma \mu_j \|\mathscr{V}_j\|),$$

so that by Gronwall's inequality

$$\|u_1(t)\| \leqslant k_1 e^{k_2 t}. \tag{2.12}$$

These inequalities establish three things: (1) uniqueness, (2) continuous dependence on f_1, and (3) uniform boundedness of $\|u_1(t)\|$ for $0 \leqslant t < \tau$. Further, an inequality like (2.11), together with (3) and the continuity of $\mathscr{X}_\delta(t)$, shows that a strong left-sided limit for $u_1(t)$ exists at τ. This means that the maximal interval of existence must be closed.

Next, require that τ be small enough to make k_2 less than one. Then by the Contraction Mapping Principle, the equation (2.10) has a (local) solution (that is, one in $C(\tau, Y_\delta)$). To continue this solution, note that (2.10) may, for $\tau \leqslant t \leqslant \tau + \varepsilon$, be written as

$$u_1(t) = k(t) - i \int_0^t \mathscr{X}_\delta(t-s)$$

$$\times PBP_0B\left(\int_0^t \exp(-i(s-\hat{s})\, P_0B)\, u_1(\hat{s})\, d\hat{s}\right) ds,$$

where $k(t)$ is strongly continuous and depends only on those states $u_1(t)$ preceeding τ. Now, if $\varepsilon > 0$ is small enough, the CMP can again be applied to obtain an extension over $[\tau, \tau + \varepsilon]$. Thus, no finite interval can be maximal and $u_1(t)$ must extend globally.

By employing Lemma (2.2) and (2.3), we can establish that $u_1(t)$ has a strong derivative and satisfies (2.9). The initial condition (2.6) is built in. From the properties of $\mathscr{X}_\delta(t)$, Lemma (2.3), and the condition (2.2), we get that for all $t > 0$, $u_1(t) \in \mathscr{D}(A_\delta)$.

The strong continuity of $u'(t)$ follows from (2.9), (2.10). and Lemma (2.2) since

$$A_\delta u_1(t) + g(t) = A_\delta \mathscr{X}_\delta(t)\, f_1 + A_\delta \int_0^t \mathscr{X}_\delta(t-s)\, g(s)\, ds + g(t)$$

$$= \mathscr{X}_\delta(t)[A_\delta f_1 - iPBf_0] + \int_0^t \mathscr{X}_\delta(t-s)\, g'(s)\, ds,$$

where

$$g(t) = (Gu_1)(t) + F(t)\, f_0.$$

This completes the proof.

Remark. The solution $u_1(t)$ is not necessarily continuous with respect to f_0 since PB is (in general) unbounded.

If we now use $u_1(t)$ to obtain $u_0(t)$ via (2.8), then $u(t) = u_0(t) + u_1(t)$ is the unique strict solution of the restricted Cauchy problem (2.1)–(2.3). Further, $u'(t)$ is strongly continuous and $u(t)$ depends continuously on f_1.

Next we consider a Cauchy problem whose solution will be obtained as the strong limit of solutions to restricted problems. Accordingly, assume that P_0B is still nuclear, but satisfying now

(i) $\phi_j \in \mathscr{D}(B)$ with $\mathscr{V}_j = PB\phi_j \in Y_j = Y_{\delta_j}$ for δ_j a sequence of positive numbers decreasing to zero, and

(ii) the stronger (since, in general, $\varkappa(\delta_j) \to \infty$) convergence hypothesis $\Sigma \mu_j \varkappa(\delta_j) \|\mathscr{V}_j\| < \infty$.

THEOREM 2.5. *The initial-value problem* (2.1)–(2.3) *with* P_0B *satisfying* (i) & (ii) *has a strict solution* (*with strongly continuous derivative*) *which is the strong limit of solutions to the problems*

$$u'(t) = (-i)[A(D) + PB]\, u(t) + (-i)(P_0B)_n\, u(t), \qquad t > 0$$

$$H - \lim_{t \to 0^+} u(t) = f,$$

with

$$f_0 = P_0 f \in \mathscr{D}(B)\ \&\ f_1 = Pf \in \mathscr{D}(A(D) + PB)(CP)_n$$

$$PBf_0, f_1 \in Y_\delta \qquad \textit{for some} \quad \delta > 0$$

where $(P_0B)_n$ *is the finite rank operator given by*

$$(P_0B)_n\, u = \sum_{j \leqslant n} \mu_j(u, \Psi_j)\, \phi_j.$$

Proof. Fix $\tau > 0$ and consider the sequence $\{u_n(t),\ n \geqslant n_0\}$ in $C(\tau, Y_0 = \bigcup Y_{n \geqslant n_0})$, where n_0 is large enough so that $\delta_{n_0} \leqslant \delta$ and $u_n(t)$ is the unique strict solution of $(CP)_n$ in $C(\tau, Y_n)$. Our immediate task is to show this sequence uniformly convergent.

To this end, note that

$$\|u_n(t)\| \leqslant \varkappa(\delta)(\|f_1\| + \tau\, \|PBf_0\|) + \left(\sum_{j \leqslant n} \mu_j \varkappa(\delta_j)\, \|\mathscr{V}_j\|\right) \tau e^{\mu_1 \tau} \left(\|f_0\| + \int_0^t \|u_n(\hat{s})\|\, d\hat{s}\right),$$

since $\mathscr{Z}_n(t)$ is an extension of $\mathscr{Z}_j(t)$, for $j \leqslant n$, and $\|\mathscr{Z}_j(t-s)\, \mathscr{V}_j\| \leqslant z(\delta_j)\, \|\mathscr{V}_j\|$. Consequently, from Gronwall's inequality $\|u_n(t)\| \leqslant [\varkappa(\delta)(\|f_1\| + \tau\, \|PBf_0\|) + b_n\, \|f_0\|]\, e^{b_n t}$, for $b_n = (\sum_{j \leqslant n} \mu_j \varkappa(\delta_j)\, \|\mathscr{V}_j\|)\, \tau e^{\mu_1 \tau}$, and the sequence is uniformly bounded.

Now, for $n > m$ and $0 \leqslant t \leqslant \tau$, it can be shown that

$$\|u_n(t) - u_m(t)\| \leqslant \varepsilon(n, m) + k \int_0^t \|u_n(\hat{s}) - u_m(\hat{s})\|\, d\hat{s},$$

where k is a positive constant and $\varepsilon(n, m)$ is the expression

$$\left[\left(\sum_{m<j\leqslant n} \mu_j \varkappa(\delta_j)\, \|\mathscr{V}_j\|\right) + \mu_m \mu_1^{-1}\left(\sum_{j\leqslant m} \mu_j \varkappa(\delta_j)\, \|\mathscr{V}_j\|\right)\right] \times \tau^2 e^{\mu_1 \tau}(\|u_n\|_{0,\tau} + \|f_0\|).$$

The convergence of the sequence now follows from the convergence hypothesis (ii), the fact that $\mu_m \to 0$ as $m \to \infty$, and the completeness of $C(\tau, Y_0)$.

Set $\tilde{u}(t) = H - \lim_{n\to\infty} u_n(t)$. Then $\tilde{u}(t) \in C(\tau, Y_0)$ for all $\tau > 0$ and satisfies the integral equation

$$\begin{aligned}\tilde{u}(t) = {} & \mathscr{Z}_\delta(t) f_1 - i \int_0^t \mathscr{Z}_\delta(t-s)\, PBf_0\, ds \\ & - i \int_0^t \sum_{j\geqslant 1} \mu_j \left(\int_0^s \exp(-(i(s-\hat{s})\, P_0 B)\, \tilde{u}(\hat{s})\, d\hat{s}, \Psi_j\right) \mathscr{Z}_j(t-s)\, \mathscr{V}_j\, ds \\ & - i \int_0^t \sum_{j\geqslant 1} \mu_j \left(\left[\sum_{k\geqslant 1} \frac{(-is)^k\, (P_0 B^{k-1}}{k!}\right] f_0, \Psi_j\right) \mathscr{Z}_j(t-s)\, \mathscr{V}_j\, ds.\end{aligned}$$

This can be used to show that $\tilde{u}(t)$ has a strongly continuous strong derivative; instead, we demonstrate this in a way better suited to our other needs.

If $\tau > 0$, $\tilde{f} = A_\delta f_1 - iPBf_0$, and $g_n(t) = (G_n u_n)(t) + F_n(t) f_0$ $(G_n + F_n$ defined using $(P_0 B)_n)$, then

$$\begin{aligned}u_n'(t) & = A_{\delta_n} u_n(t) + g_n(t) \\ & = \mathscr{Z}_\delta(t)\tilde{f} + \int_0^t \mathscr{Z}_n(t-s)\, g_n'(s)\, ds \\ & = \mathscr{Z}_\delta(t)\tilde{f} - i \int_0^t \sum_{j\leqslant n} \mu_j(u_n(s) + (P_0 B)_n\, (g_n(s) + f_0), \Psi_j) \\ & \quad \times \mathscr{Z}_j(t-s)\, \mathscr{V}_j\, ds,\end{aligned}$$

and it follows that $u_n'(t)$ converges in $C(\tau, Y_0)$, say to $v(t)$. So that, by the properties of the strong Riemann integral, $\tilde{u}(t)$ has a strong derivative and $\tilde{u}'(t) = v(t)$. Further, since $u_n(t) \in \mathscr{D}(A_{\delta_n}) \subseteq \mathscr{D}(A(D) + PB)$ for all $n \geqslant n_0$, and because $A_{\delta_n} u_n(t)$ has a strong limit for each $t > 0$, we have (1) $\tilde{u}(t) \in \mathscr{D}(A(D) + PB)$ (all $t > 0$), and (2) $\tilde{u}(t)$ satisfies (2.6) & (2.9).

To complete the proof, obtain $u_0(t)$ from (2.8) and set $u(t) = u_0(t) + \tilde{u}(t)$.

Our final remarks pertain to the *stationary* problem

$$(A(D) + B)\, u = 0, \qquad (S)$$

where P_0B is still assumed to be nuclear, but with no attendant domain or convergence conditions.

This problem enjoys considerable richness which we try to illustrate with some simple examples.

The most trivial example occurs when $B=0$ (of course, here H_0 constitutes the set of stationary solutions). This in turn suggests the more general cases $P_0B=0$ and $PB=0$. In the first, (S) becomes

$$[A(D)+PB]\,u_1=-PBu_0,$$

where we have assumed that $u_0\in\mathscr{D}(B)$. This is essentially the steady-state problem of [2] and may for u_0 in $\mathscr{D}(B)$ be solved using the limiting absorption principle up to (perhaps) a multiple of the identity (that is, the frequency may have to be nonzero). If $PB=0$, then $u_1=0$ and the stationary solutions are exactly those u_0 orthogonal to $M=\overline{\operatorname{span}\Psi_j}$. Finally, consider the possibility that (i) $M=H_0$ or (ii) $M=H_1$. In case (i), $u_0=0$ and $u_1\in\mathscr{D}(A(D)+PB)$ with $[A(D)+PB]\,u_1=0$. Whether this equation has nontrivial solutions will depend again on whether "0" is in the singular spectrum. On the other hand, (ii) means that $u_1=0$ and $Bu_0=0$ so that the intersection of H_0 with the null space of B determines precisely the set of stationary solutions.

Conclusion

The convergence hypothesis (ii) is a measure of how badly B scatters stationary data to data which can propagate in the unperturbed medium. The "mode mixing" of B is the key to the analysis of this nonelliptic nonselfadjoint operator. This limited mixing condition we impose on B has an analog in the case where perturbation is imposed by a boundary. We refer the reader to Schulenberger [4]. Limited mixing can be applied to elastic waves in $\mathbb{R}^3$ with results similar to those given here. This work will be reported on elsewhere.

If anisotropy is introduced into the medium the treatment of the problem becomes considerably more complex since the number of propagation speeds may increase etc. (see Gilliam [2]).

An interesting special case of our problem is given under the condition that S is bounded and $S\cap\mathbb{R}^1=\varnothing$. It can be shown in this case that $A(D)+B$ is a spectral operator on H, when the conditions (1.42) are satisfied. Here the space Y_0 is simply H, and the solution to the Cauchy problem takes a more classical form.

What happens in case the modes are allowed to mix in a more arbitrary way is unknown. However, the problem is badly behaved and is known to be ill posed in general.

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