Convergence in Measure in Abstract Spaces

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January 2007

Since the advent of the Lebesgue integral, many attempts have been made to generalize it for vector-valued functions and measures. We note two particularly interesting efforts. The first is from Musielak and Orlitz [3]. They give the following definition of an L^1 space: $f \in L^1(\mu)$ if f is measurable with respect to some σ-algebra $\mathcal{M}$ on a space H and

$$||f||_1 = \sup_{||y^\star|| \leq 1} \int_H ||f||_X d|y^\star \mu| < \infty.$$

Here f is a function from H to some normed linear space X and μ takes its values in some normed linear space Y, with $y^\star \in Y^\star$ the dual of Y. There need be no relation whatever between X and Y. $y^\star\mu(E)$ is then a complex measure on H with $|y^\star\mu|$ the total variation measure derived from $y^\star\mu$. The usual Lebesgue spaces are defined without any reference to an "integral, " $\int_H f d\mu$. Many of the usual convergence theorems translate over to this case, such as $f_n \to f$ in L^1 implies $f_n \to f$ in measure, etc.

Another integral for a case where $f(x) \in X$, $\mu(E) \in L(X, Y)$, the bounded linear operators from X to Y, was defined by Easton and Tucker [1]. For this integral the above convergence theorem fails.

We now offer a general integral formulation of the Easton-Tucker type but for which the above convergence result holds and which has many useful applications. One such application is the theory of differential operators in general spaces using "spectral measures."[1]

[1]See *General Spectral Measures*, vol. 2, this journal.

In the standard Lebesgue theory, the convergence theorem cited above holds because of the "maximal" inequality:

$$\epsilon\mu(\{t : |f(t)| > \epsilon\}) \leq \int_E |f| d\mu$$

where $E = (\{t : |f(t)| > \epsilon\}$. Naturally, no such inequality is useful without a function such as "absolute-value." Hence a deeper understanding of the integration process is needed.[2]

1 Preliminaries

The letters X and Y will represent separated topological vector spaces and the letters V and U denote balanced open neighborhoods of the origin in X and Y respectively. $L(X, Y)$ will denote the vector space of linear mappings from X to Y. Moreover, we suppose Y is complete. In many applications, sequential completeness is sufficient.

Let H be a set and D a ring of subsets of H on which a mapping μ is defined with range in $L(X, Y)$, where if $E_1, E_2 \in D$ with $E_1 \cap E_2 = \emptyset$ and $x \in X$ we have

(1) $x\mu(E_1 \cup E_2) = x\mu(E_1) + x\mu(E_2)$ (μ is thought of acting on the right of x.)

(2) A function f from H into X of the form

$$f(t) = \sum_{i=1}^{n} x_i \chi_{E_i}(t)$$

where $x_i \in X, E_i \in D$ will be called a D-simple function, or briefly, a simple function. Here $\chi_{E_i}(t)$ is the usual characteristic function of E_i.

(3) Let $A(D)$ be the algebra of sets generated by D. For $K \in A(D)$ and f simple, define

$$\int_K f \ d\mu = \sum_{i=1}^{n} x_i \mu(E_i \cap K)$$

[2] An earlier version of this paper appeared in *Revue Romain de Mathematiques Pures et Appliquees,* 1981.

(4) A function $g : H \to X$ such that for an open set $S \subset X$, $f^{-1}(S) \in A(D)$ is called "Borel"-measurable.

(5) We say μ has bounded semivariation if for each U there exists a V such that for all $E \in D$ there exists $t_E \in \mathbb{C}$ such that

$$\hat{\mu}_V(E) = \{\sum_{i=1}^{n} x_i \mu(E_i \cap E) : \cup E_i = E; x_i \in V\} \subseteq t_E U.$$

We say such a V "corresponds" to U. (4) is the weakest assumption that will guarantee "continuity" of the integral of simple functions. (4) is nevertheless a strong assumption. It's effect is to introduce some convexity as the following theorem shows. Integration is essentially a convex process. Note that without any difference in the statement of the definition of bounded semivariation, one could instead change the setting to where μ takes values in a topological vector space Z with an underlying map $\bullet : (X \times Z) \to Y$.

Theorem 1 *Let $0 < p < 1$ and let $X = Y = L^p(0,1)$ be the metrizable topological vector space with translation-invariant metric*

$$d(f,g) = \int_0^1 |f(t) - g(t)|^p \ dt.$$

Moreover let μ be Lebesgue measure on $(0,1)$. Then (4) fails to hold.

Proof. In this case, Y is not locally convex. Moreover, $\hat{\mu}_V(E)$ is the convex hull of V. This implies every U contains a convex neighborhood of zero, a contradiction.

2 Integration

Definition 1 A net $\{f_\alpha\}$ of functions from H into X is said to be Cauchy in measure on $E \in A(D)$ when for each U and corresponding V and each $K \in D$, $V_1 \subseteq \lambda V$, $(\lambda \in \mathbb{C} - \{0\})$, there exists α_1 such that if $\alpha_2, \alpha_3 \geq \alpha_1$, then

$$K \cap \{t \in E : f_{\alpha_2}(t) - f_{\alpha_3}(t) \notin V_1\} \subseteq F_{\alpha_2,\alpha_3} \in D$$

where $\hat{\mu}_V(F_{\alpha_2,\alpha_3}) \subseteq U$.

Convergence in measure is defined in a similar fashion. If the space X is locally convex, we may weaken the definition somewhat by restricting V_1 to be a scalar multiple of V. While we employ nets in this definition, sequences are sufficient for many purposes.

Theorem 2 *If $\{f_\alpha\}$ converges in measure to f, then $\{f_\alpha\}$ is Cauchy in measure.*

Proof. Choose $V_2 \subseteq V_1$ with $V_2 + V_2 \subseteq V_1$. Then

$$\{t \in E : f_{\alpha_2}(t) - f_{\alpha_3}(t) \notin V_1\}$$
$$\subseteq \{t \in E : f_{\alpha_2}(t) - f(t) \notin V_2\} \cup \{t \in E : f_{\alpha_3}(t) - f(t) \notin V_2\}.$$

Definition 2 A function f is said to be integrable on a set $E \in A(D)$ provided there exists a net f_α of simple functions converging to f in measure on E such that for each U there exists $U_1 = \lambda U$ for some $\lambda \in \mathbb{C} - \{0\}$ and corresponding V_1 such that for $E_1 \in D$ if $\hat{\mu}_{V_1}(E_1) \subseteq U_1$ then

$$\int_{E_1} f_\alpha \, d\mu \in U$$

and there exists $F \in D$ such that

$$\int_{K-F} f_\alpha \in U$$

for all α and $K \subseteq E, K \in A(D)$.

Theorem 3 *If f is an integrable function on a set $E \in A(D)$ and $\{f_\alpha\}$ is as in the previous definition, then for each $K \subseteq E, K \in A(D)$ we have*

$$\lim_\alpha \int_K f_\alpha \, d\mu = \eta_f(K)$$

exists and the convergence is uniform in K. Furthermore the limit does not depend on $\{f_\alpha\}$.

Proof. Fix U and choose $U_2 \subseteq U$ such that $U_2 + U_2 + U_2 + U_2 + U_2 \subseteq U$. Now choose $U_1 \subseteq U_2$ and a corresponding V_1 such that when $\hat{\mu}_{V_1}(E_1) \subseteq U_1$ then

$$\int_{E_1} f_\alpha \, d\mu \in U_2$$

by definition 2. Choose also a set $F \in D$ such that

$$\int_{E-F} f_\alpha \; d\mu \in U_2$$

as in definition 2. Now pick t_F such that $\hat{\mu}_{t_F V_1}(F) \subseteq U_2$. We can pick such a t_F since V_1 corresponds to $U_1 \subseteq U_2$ so in fact V_1 corresponds to U_2 as well. Now choose α_1 such that $\alpha_2, \alpha_3 \geq \alpha_1$ implies that $\hat{\mu}_{V_1}(E_2) \subseteq U_1$ where

$$\{t \in F : f_{\alpha_2}(t) - f_{\alpha_3}(t) \notin t_F V_1\} = E_2$$

by Theorem 1. Thus

$$\int_{E_2} f_{\alpha_2} \; d\mu - \int_{E_2} f_{\alpha_3} \; d\mu \in U_2 - U_2 = U_2 + U_2.$$

Let $E_3 = F - E_2$. Then

$$\int_{E_3} f_{\alpha_2} \; d\mu - \int_{E_3} f_{\alpha_3} \; d\mu \in U_2$$

since

$$f_{\alpha_2}(t) - f_{\alpha_3}(t) \in t_F V_1$$

on $F - E_2 = E_3$. Therefore for any $K \in D, K \subseteq E$ we have

$$\int_K f_{\alpha_2} - f_{\alpha_3} \; d\mu = \int_{K \cap F} f_{\alpha_3} + \int_{K-F} f_{\alpha_2} - f_{\alpha_3} \; d\mu$$

$$= \int_{E_2 \cap K} f_{\alpha_2} - f_{\alpha_3} \; d\mu + \int_{E_3 \cap K} f_{\alpha_2} - f_{\alpha_3} \; d\mu + \int_{K-F} f_{\alpha_2} - f_{\alpha_3} d\mu$$
$$\in U_2 + U_2 + U_2 + U_2 + U_2 \subseteq U.$$

This shows that the limit exists uniformly in K. Uniqueness is established in a similar way.

Definition 3 Let f be an integrable function. For $K \in A(D)$, define

$$\int_E f \; d\mu = \eta_f(E).$$

The integrals of integrable functions have the equicontinuity properties of definition 2. This is

Theorem 4 *Let f be integrable and $E \in A(D)$. Then for each U there exists $U_1 = \lambda U$ and a corresponding V_1 such that for $E_1 \in D$ if $\hat{\mu}_{V_1}(E_1) \subseteq U_1$ then*

$$\int_{E_1} f \, d\mu \in U$$

and there exists $F \in D$ such that

$$\int_{K-F} f \, d\mu \in U$$

for all $K \subseteq E, K \in A(D)$.

Proof. This holds by the uniform convergence of theorem 3.

Theorem 5 *Let f, g be integrable, β_1, β_2 be scalars. Then $\beta_1 f + \beta_2 g$ is integrable and moreover for $E \in A(D)$,*

$$\int_E \beta_1 f + \beta_2 g \, d\mu = \beta_1 \int_E f \, d\mu + \beta_2 \int_E g \, d\mu$$

Proof. For integrable f and g, the sum of the corresponding nets is a net for $f+g$. Since the theorem holds for simple functions, it holds for integrable functions.

Theorem 6 *Let $\{f_\alpha\}$ be a net of integrable functions converging in measure to f and suppose further that this net satisfies the conditions of definition 2. Then f is integrable and*

$$\lim_\alpha \int_E f_\alpha \, d\mu = \int_E f \, d\mu$$

for all $E \in A(D)$.

Proof. Given the uniform convergence in theorem 3, we may select a net from those defining the integrals $\int_E f_\alpha \, d\mu$ whose integrals converge to $\int_E f \, d\mu$.

Definition 4 Let $\{f_\alpha\}$ be a net of functions on H with values in X which converges to f. We say it converges *uniformly* to f if for every V, there exists α_1 such that if $\alpha_2 \geq \alpha_1$ then $f_{\alpha_2}(t) - f(t) \in V$ for all $t \in H$.

Theorem 7 *Let $\{f_\alpha\}$ be a net of measurable simple functions which converge uniformly to f. Then f is integrable and for $E \in A(D)$,*

$$\lim_\alpha \int_E f_\alpha \, d\mu = \int_E f \, d\mu.$$

3 Convergence in Measure

The theorem of the introduction requires some preliminary definitions mainly required by the nature of operator-valued measures (see example 1 below).

Definition 5 a set $K \in A(D)$ has *zero measure* if $\mu(M)$ is the zero operator for all $M \in D, M \subseteq K$.

Definition 6 The essential range of a function f on $F \in A(D)$ is defined as $\{x \in X : \{t \in F : f(t) - x \in V\}$ does not have zero measure for all $V\}$. We denote this set by $R_f(F)$. A function f is called an *essential range function* if for each $F \in A(D)$ which does not have zero measure, $R_f(F) \cap f(F) \neq \emptyset$.

When the range space is not a normed space, or the measure is not countably additive, various pathologies may arise, compared to the classical theory. For example

(1) There may exist functions which are not essential range functions.[2]

(2) A sequence of functions may converge in measure to one function but converge point-wise to another.[5]

(3) A sequence of functions may converge in measure, but no subsequence converges point-wise.[4]

Lemma 1 *Let f and g be essential range functions where for some V and $E \in D$ we have $f(t) - g(t) \in V$ for all $t \in E$. If $b \in R_f(E)$ then for each V_1, there exists $a \in R_g(E)$ such that $b - a \in V + V_1$.*

We now state the general form of a sufficient condition for the convergence theorem of the introduction to hold.

Definition 7 A net of functions $\{f_\alpha\}$ is said to dominate sets $\{B_\alpha\}$ (subsets of Y) on sets $\{E_\alpha\} \subseteq A(D)$ relative to $\{U_\beta\}$ when there exist sets $\{S_\alpha\}$ in D and collectons $\{U_\alpha^{S_\alpha}\}$ and corresponding $\{V_\alpha^{S_\alpha}\}$ such that if $S_\alpha^{'} \in D$ and $\hat{\mu}_{V_\alpha}(S_\alpha - S_\alpha^{'}) \subseteq U_\alpha^{S_\alpha}$ then there exist sets $S_\alpha^{"} \in D, S_\alpha^{"} \subseteq S_\alpha^{'}$ and $\hat{\mu}_\alpha(S_\alpha - S_\alpha^{"}) \subseteq U_\alpha^{S_\alpha}$ and when any sum of the form

$$\sum_{i=1}^{n} x_i \mu(E_i \cap S_\alpha^{"}) \in U_\beta,$$

where $x_i \in R_{f_\alpha}(E_i \cap S_\alpha^{"})$, then $B_\alpha \subseteq U_\beta$.

Theorem 8 *Let $\{f_\alpha\}$ be a net of Borel measurable essential range functions, integrable on some set $E \in D$. Let $E_{\alpha_V} = \{t \in E : f_\alpha(t) \notin V\}$ and suppose for each V_1 where V_1 corresponds to some U_1 and $V \subseteq \lambda V_1$ for some λ, f_α dominates $\hat{\mu}_{V_1}(E_{\alpha_V})_\alpha$ with respect to U_{β_V} on E_{α_V}. If $\int_K f_\alpha \, d\mu \to 0$ uniformly in $K \subseteq E, K \in D$, then $f_\alpha \to 0$ in measure.*

Proof. Fix V. Let $\{g_\beta^\alpha\}$ determine $\int f_\alpha \, d\mu$ on E. Choose sets U^α and $U_1^\alpha \subseteq U_2^\alpha$ and V_1^α corresponding to U_1^α. Choose $V_2 \subseteq t_E V_1^\alpha$ such that $V_2 + V_2 \subseteq t_E V_1^\alpha$ and $\hat{\mu}_{t_E V_1^\alpha}(E) \subseteq U_2^\alpha$, where $\hat{\mu}_{V_1^\alpha}(S_\alpha - S_\alpha^{'}) \subseteq U_1^\alpha$. Then $\int_{S_\alpha - S_\alpha^{'}} f_\alpha \, d\mu \in U_2^\alpha$. Since $g_\beta^\alpha \to f_\alpha$ in measure, we can choose $Q_\beta^\alpha \in D$ such that (for "large" β) $Q_\beta^\alpha \supseteq \{t \in E_{\alpha_V} : g_\beta^\alpha(t) - f_\alpha(t) \notin V_2\}$ where $\hat{\mu}_{V_1^\alpha}(Q_\beta^\alpha) \subseteq U_2^\alpha$ satisfies the conditions on $S_\alpha^{'}$ in definition 7. So choose $S_\alpha^{"}$ as in definition 7.

Now $g_\beta^\alpha(t) = \sum x_i \chi_{E_i}(t)$ and since simple functions are essential range functions, we can choose $b_i \in R_{g_\beta^\alpha}(E_i \cap S_\alpha^{"})$ when $E_i \cap S_\alpha^{"}$ is not of zero measure. By lemma 2 we choose $a_i \in R_{f_\alpha}(S_\alpha^{"} \cap E_i)$ with $b_i - a_i \in V_2 + V_2$. Therefore $\int_{S_\alpha^{"} g_\beta^\alpha} - \sum a_i \chi_{E_i} \, d\mu \in U_2^\alpha$. Since $\int f_\alpha \, d\mu$ is the uniform (on $D \cap E_{\alpha_V}$) limit of $\int g_\beta^\alpha \, d\mu$ we have for all large β, $\int f_\alpha \, d\mu - \int g_\beta^\alpha \, d\mu \in U_2^\alpha$ on $D \cap E_{\alpha_V}$. Since $\int_{S_\alpha} f_\alpha \, d\mu - \int_{S_\alpha^{"}} f_\alpha \, d\mu \in U_2^\alpha$, we have

$$\int_{S_\alpha} f_\alpha \, d\mu - \int_{S_\alpha^{"}} f_\alpha \, d\mu + \int_{S_\alpha^{"}} f_\alpha \, d\mu - \int_{S_\alpha^{"}} g_\beta^\alpha \, d\mu$$

$$+ \int_{S_\alpha^{"}} g_\beta^\alpha \, d\mu - \int_{S_\alpha^{"}} \sum a_i \chi_{E_i} \, d\mu \in U_2^\alpha + U_2^\alpha + U_2^\alpha \subseteq U^\alpha.$$

Therefore $\int_{S_\alpha} f_\alpha \, d\mu - \sum a_i \mu(E_i \cap S_\alpha^{"}) \in U^\alpha$. Since U^α was arbitrary the conclusion now follows.

The hypotheses of theorem 8 are satisfied of course when μ is a positive measure and the functions take values in the complex numbers, etc. The rather complicated conditions of definition 7 and theorem 8 take a somewhat different (and more familiar form) in Banach spaces or in locally convex spaces, but are still required there in general. A simple example shows this is true.

Example 1 *Let $X = Y = \mathbb{R}^2$ and $(x, y)\mu(E) = (x\mu(E), 0)$ where E is a Borel subset of $[0, 1]$ and μ is Lebesgue measure on $[0, 1]$. Let $f_n(t) = (0, n)$, then $\int_0^1 f_n d\mu = (0, 0)$ for all n but f_n does not converge in measure.*

References

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