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DIFFERENTIAL OPERATORS IN BANACH SPACES WITH DENSELY DEFINED SPECTRAL MEASURES

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ABSTRACT. We prove in this paper that certain second order differential operators in B-spaces have locally defined spectral measures associated with them. This is the "best possible" result in general.

1. Introduction. In [10] we established the existence of densely defined spectral measures for rather large classes of operators in some nonlocally convex spaces. It is our intention to deal with the locally convex case here.

Specifically we shall be concerned with differential operators. Rather than trying to prove a general theorem we shall take some specific (and important) examples to work with. We shall try at the same time to push the known theory as far as possible in these particular cases. Our most important example shall be the operator determined by the formal differential operator

$$\text{(1)}\qquad \begin{aligned} &\ell = -\frac{d^2}{dt^2} + q(t) \qquad t \in (-\infty,\infty) \\ &\int_{-\infty}^{\infty} |q(t)|dt < \infty \end{aligned}$$

where q(t) is allowed to be complex valued. We show that in $L^p(-\infty,\infty)$, $(1 \leqslant p \leqslant \infty)$ ℓ determines a unique closed densely defined operator L. We shall develop the spectral theory of L and give some conditions under which L is spectral. Although important work has been done on related operators in $L^2(0,\infty)$ (see Folland [3], Kemp [5] for examples) almost all of the basic facts are contained in Stone [11]. We shall pursue a different course than Stone however, since he assumes q to be very rapidly decreasing (and real valued).

We rely on [10] and Rota [8] for many of our results. Our reason for attacking the B-space problem as opposed to Hilbert space is that it presents genuine examples of locally defined measures and hence at most dense eigenfunction expansions.

Necessary preliminaries are taken care of in Section 2. Section 3 contains the main results on ℓ and some other examples for comparison and contrast.

The author would like to acknowledge the very helpful comments of the referee. The referee also very kindly added the pertinent reference [12] of which the author was not aware and pointed out that the proof of Theorem 3 could be omitted.

2. **Preliminaries.** The Lagrange adjoint of ℓ is denoted as ℓ^* and is

$$-\frac{d^2}{dt^2}+\overline{q(t)}.$$

The main result on the solutions of $\ell y = \lambda y$ is contained in the following, which is proved differently in Stone [11], and elsewhere.

THEOREM 1. *There exist two solutions of* $Ly = \lambda y$, $f_1(\cdot,\sqrt{\lambda})$, $f_2(\cdot,\sqrt{\lambda})$ *which are continuously differentiable such that (assume* $\mathrm{im}\sqrt{\lambda} \geq 0$ *and* $s = \sqrt{\lambda} \neq 0$*)*

(1) $f_1(x,s) = e^{isx} + \int_x^\infty \frac{\sin s(y-x)}{s} q(y) f_1(y,s)\,dy$

(2) $|f_1(x,s)| \leq e^{-\mathrm{im}\, sx}\exp(|s|^{-1}\sigma_1(x))$, $\sigma_1(x) = \int_x^\infty |q(y)|\,dy$

(3) $f_2(x,s) = e^{-isx} + \int_{-\infty}^x \frac{\sin s(x-y)}{s} q(y) f_2(y,s)\,dy$

(4) $|f_2(x,s)| \leq e^{\mathrm{im}\, sx}\exp(|s|^{-1}\sigma_2(x))$, $\sigma_2(x) = \int_{-\infty}^x |q(y)|\,dy$

(5) *For fixed* x, f_1 *and* f_2 *are analytic in* $\mathbf{R}^2_+$ *and continuous in* $\overline{\mathbf{R}^2_+}\backslash\{0\}$ $(\mathbf{R}^2_+ = \{z \in \mathbf{C} | \mathrm{im}\, z > 0\}$.

(6) $f_1^*(\cdot,-\bar{s}) = f_2(\cdot,s)$; $f_2^*(\cdot,-\bar{s}) = \overline{f_1(\cdot,s)}$.

PROOF. Solving (1) by successive approximation leads to $f_1(x,s) = \Sigma f_1^{(n)}(x,s)$, where

$$f_1^{(0)}(x,s) = e^{isx}$$

$$f_1^{(n)}(x,s) = \int_x^\infty \frac{\sin s(y-x)}{s} q(y) f_1^{(n-1)}(y,s)\,dy.$$

We must go through the details to make the remainder of the paper intelligible.

$$\left|\frac{\sin xt}{s}\right| \leq |x|^{-1} e^{\mathrm{im}\, st},$$

so

$$f_1^{(1)}(x,s) = \int_x^\infty \frac{\sin s(t-x)}{s} q(t) e^{ist}\,dt$$

$$|f_1^{(1)}(x,s)| \leq \int_x^\infty |s|^{-1} e^{\mathrm{im}\, s(t-x)} |q(t)| e^{-\mathrm{im}\, st}\,dt = \frac{e^{-\mathrm{im}\, sx}}{|s|} \int_x^\infty |q(t)|\,dt$$

continuing

$$|f_1(x,s)| \leqslant \Sigma_0^\infty e^{-imsx} \frac{\{\int_x^\infty |q(t)|dt|s|^{-1}\}^n}{n!} = e^{-imsx}\exp\{\sigma_1(x)|s|^{-1}\}.$$

The proof of (3), (4) is analogous.

To prove (5), we consider $q_a(t) = q(t) \underset{(-a,a)}{\chi(t)}$ and the iterates $f_{1,a}^{(n)}$ for q_a. Since q_a satisfies

$$\int_{-\infty}^{\infty} |t|\ |q_a(t)|dt < \infty,$$

we can apply the following reasoning.

Set

$$F(s) = \int_S f(x,s)dx;\ s_1,s_2 \in K,$$

$\overline{K}$ some compact subset of $\mathbf{R}_+^2$.

$$\frac{F(s_1)-F(s_2)}{s_1-s_2} = \frac{2}{s_1-s_2}\int_S \int_{s_2}^{s_1} \partial_2 f(x,z)dzdx$$

$\sup_z |\partial_2 f(x,z)| \leqslant g(x) \in L_1(S)$ implies (by the Fubini theorem) that there exists

(*) $F'(s) = \int_S \partial_2 f(x,s)dx.$

Set

$$|f_{1,a}^{(n)}(x,s)| \leqslant c(s)$$

and set

$$\partial_z \frac{\sin z(t-z)}{z} f_{1,a}^{(n)}(t,z) = g_n(t,z),$$

we have

$$|g_n(t,z)| \leqslant c(z)e^{imz(t-x)}\{(t-x)|f_{1,a}^{(n)}(t,z)| + |\partial_2 f_{1,a}^{(n)}(t,z)|\}.$$

Now induction:

$$n = 0:\ f_{1,a}^{(0)}(t,z) = e^{itz}$$

$$|g_0(t,z)| \leqslant c(z)e^{-imzx}\{|x| + |t|\}$$

$$|g_0(t,z)|\ |q_a(t)| \in L_1$$

so by (*), $f_{1,a}(x,\cdot)$ is analytic in K and

$$|\partial_2 f_{1,a}^{(1)}(x,z)| \leqslant c(z)e^{-imzx}\int_x^\infty (|t| + |x|)|q_a(t)|dt$$

$$\equiv c(z)e^{-imzx}(|x| + 1).$$

Hence

$$|g_2^*(t,z)| \leqslant c(z)e^{-imzx}\{|t| + |x|\}$$

and $\partial_2 f_{1,a}^{(2)}$ exists with

$$|\partial_2 f_{1,a}^{(2)}(x,z)| \leqslant c(z)e^{-imzx}(1 + |x|).$$

Induction hypothesis:

$$\partial_2 f_{1,a}^{(n-1)}(x,z)$$

exists and

$$|\partial_2 f_{1,a}^{(n-1)}(x,z)| \leqslant c(z)e^{-imzx}(1 + |x|)$$

and the induction step is clear. Thus

$$f_{1,a}^{(n)}(x,s) = \int_x^\infty \frac{\sin s(y-x)}{s} q_a(y) f_{1,a}^{(n-1)}(y,s)dy$$

is analytic.

Now consider $x_0 = x$,

$$|f_{1,a}^{(n)}(x,s) - f_1^{(n)}(x,s)| \leqslant c(\lambda)|\int_{R^n} \Pi_1^n \sin s(x_i - x_{i-1})[q_a(x_i)]$$

$$- \Pi_1^n \sin s(x_i - x_{i-1})[q(x_i)]\, dx_1 dx_2 \cdots dx_n|$$

$$\leqslant \frac{c(\lambda)}{n!|s|^n}\left(\int_{|x|>a} |q(x)|dx\right)^n \to 0$$

for s in $\overline{K}$ uniformly as $a \to \infty$, and therefore $f_1^{(n)}(x,s)$ is analytic in R_+^2 (and continuous in $\overline{R_+^2}\setminus\{0\}$). Taking $S_N(x,s) = \Sigma_1^N f_1^{(n)}(x,s) S_N \to f_1$ uniformly and thus f_1 is analytic in R_+^2, continuous in $\overline{R_+^2}\setminus\{0\}$. (6) follows from the obvious check and Theorem 1 is established.

We now define an operator in B-space from ℓ.

DEFINITION 1. (Rota) (a). Let $1 \leqslant p \leqslant \infty$ and let L be an operator in $L^p(-\infty,\infty)$ such that D(L) (the domain of L) is dense in the $L^q(-\infty,\infty)$ topology of $L^p(-\infty,\infty) p^{-1} + q^{-1} = 1$. By the adjoint of L we mean the operator L^* in $L^q(-\infty,\infty)$ given by

$$D(L^*) = \{g|\text{ there exists an } h \text{ in } L^q(-\infty,\infty) \text{ such that } (Lf,g) = (f,h)\ f \in D(L)\};$$

$$L^*g = h,\ g \in D(L^*).$$

(b) A^n is the set of functions with $n-1$ continuous derivatives and with absolutely continuous $n-1$st derivative that is, all f such $f^{(n)}(t)$ exists a.e. and is in $L^1_{\ell oc}(-\infty,\infty)$.

(c) K^n is the set of all functions f on $(-\infty,\infty)$ with n everywhere defined derivatives which vanish outside a compact set (the set may vary with f).

(d) Let $1 \leqslant p \leqslant \infty$ and let $L(\ell,p)$ be the operator in $L^p(-\infty,\infty)$ defined by

$$D(L(\ell,p)) = \{ f \in L^p | f \in A^2, \ell f \in L^p \}$$

$$L(\ell,p) = \ell f \text{ if } f \in D(L(\ell,p)).$$

$$L(\ell,p)^* \text{ is denoted by } L_0(\ell^*,q).$$

$$p^{-1} + q^{-1} = 1.$$

THEOREM 2. (Rota) *Suppose* L *is the restriction of* $L(\ell^*,q)$ *to* K^2. *Then*

$$L \subset L_0(\ell^*,q) \subseteq L(\ell^*,q)$$

$$L^* = L_0(\ell^*,q) = L(\ell,p).$$

If $1 \leqslant q < \infty$ $(p^{-1} + q^{-1} = 1)$, *then the closure of* L, $\bar{L}$ *is* $L_0(\ell^*,q)$. *(The two different uses of "*" should be noted.)*

NOTE. Rota assumes the coefficient q is locally bounded. This is not necessary to the proof in our case, however.

THEOREM 3. $L(\ell,p)$ *is a closed densely defined operator.*

PROOF. (May be found in Goldberg [4].)

We now desire to obtain the resolvent kernel for L in a useful form so that we may investigate the spectrum of L. First we note that for ims = 0, $s \neq 0$ the Wronskian of $f_1(x,s)$, $f_1(x,-s)$ may be computed via (1), (2), (3), (4), as $x \to \infty$ to be 2is and by Abel's formula, the Wronskian

$$W_x[f_1(\cdot,s),f_1(\cdot,-s)]$$

is independent of x. The same holds as well for $f_2(\cdot,s),f_2(\cdot,-s)$ for which the Wronskian is -2is. It therefore follows that

(7) $f_1(x,s) = c_{11}(s)f_2(x,s) + c_{12}(s)f_2(x,-s)$

(8) $f_2(x,s) = c_{22}(s)f_1(x,s) + c_{21}(s)f_1(x,-s)$.

Differentiation of (7) and (8) yields another set of equations which may be solved simultaneously with (7), (8) to obtain

$$a(s) = c_{21}(s) = c_{12}(s) = -(2is)^{-1} W[f_1 f_2](s)$$

$$b(s) = c_{22}(s) = -c_{11}(-s) = (2is)^{-1} W[f_1(\cdot,-s), f_2(\cdot,-s)].$$

THEOREM 4. ($ims \geqslant 0$).

(9) $a(s) = 1 - (2is)^{-1} \int_{-\infty}^{\infty} e^{ist} q(t) f_2(t,s)dt$

(10) $b(s) = (2is)^{-1} \int_{-\infty}^{\infty} e^{+ist} q(t) f_2(t,s)dt$

(11) $a(s)a(-s) = 1 + b(s)b(-s) \quad (ims = 0)$

(12) $f_1(x,s) = -b(-s)f_2(x,s) + a(s)f_1(x,-s)$

(13) $f_2(x,s) = b(s)f_1(x,s) + a(s)f_1(x,-s)$

(14) $b(s) = 0(|s|^{-1}), \ |s| \to \infty$

(15) $a(s) = 1 - (2is)\int_{-\infty}^{\infty} q(t)dt + 0(|s|^{-1}), \ |s| \to \infty.$

PROOF. (12), (13) follow from the discussion above. We then proceed as follows: (initially we assume $s \neq 0$ and $ims = 0$)

$$f_2(x,s) = e^{-isx} + \int_{-\infty}^{x} \frac{\sin s(x-t)}{s} q(t) f_2(t,s)dt$$

$$= e^{-isx} + \frac{e^{isx}}{2is} \int_{-\infty}^{x} e^{-ist} q(t) f_2(t,s)dt - \frac{e^{isx}}{2is} \int_{-\infty}^{x} e^{ist} q(t) f_2(t,s)dt$$

$$= e^{-isx} + \frac{e^{isx}}{2is} \int_{-\infty}^{\infty} e^{-ist} q(t) f_2(t,s)dt - \frac{e^{-isx}}{2is} \int_{-\infty}^{\infty} e^{ist} q(t) f_2(t,s)dt$$

$$- \frac{e^{-isx}}{2is} \int_{x}^{\infty} e^{-ist} q(t) f_2(t,s)dt + \frac{e^{-isx}}{2is} \int_{x}^{\infty} e^{-ist} q(t) f_2(t,s)dt$$

$$= e^{-isx} 1 - \frac{1}{2is} \int_{-\infty}^{\infty} e^{ist} q(t) q(t) f_2(t,s)dt + \frac{e^{isx}}{2is} \int_{-\infty}^{\infty} e^{-ist} q(t) f_2(t,s)dt$$

$$- \frac{e^{isx}}{2is} \int_{x}^{\infty} e^{-ist} q(t) f_2(t,s)dt + \frac{e^{-isx}}{2is} \int_{x}^{\infty} e^{ist} q(t) f_2(t,s)dt$$

$$= b(s) f_1(x,s) + a(s) f_1(x,-s)$$

$$= b(s) e^{isx} + b(s) \int_{x}^{\infty} \frac{\sin s(t-x)}{s} q(t) f_1(t,s)dt$$

$$+ a(s) e^{-isx} + a(s) \int_{x}^{\infty} \frac{\sin s(t-x)}{s} q(t) f_1(t,-s)dt$$

therefore

$$e^{-isx} \{a(s) - [1 - \frac{1}{2is} \int_{-\infty}^{\infty} e^{ist} q(t) f_2(t,s)dt]\}$$

$$+ e^{isx}\{b(s) - \frac{1}{2is}\int_{-\infty}^{\infty} e^{ist}q(t)f_2(t,s)dt\}$$

$$= \int_x^{\infty} \frac{\sin s(t-x)}{s} q(t)[f_2(t,s) - b(s)f_1(t,s) - a(s)f_1(t,s)]\,dt$$

$$= o(1)$$

as $x \to \infty$ since $[\cdots]$ is bounded, ($\operatorname{im} s = 0$), and thus the coefficients are $o(1)$ as $x \to \infty$. Since they are independent of x we have (9) and (10). For (11), we see that by substitution of (12) into (13) we obtain (11). Utilizing (4) we have

$$\int_{-\infty}^{\infty} |e^{ist}q(t)f_2(t,s)dt \leq \int_{-\infty}^{\infty} e^{-imst}|q(t)|e^{imst}\exp(|s|^{-1}\sigma_2(t))dt$$

$$\leq c(s)\int_{-\infty}^{\infty}|q(t)|dt < \infty$$

and thus a(s) (and so b(s)) extends analytically to $\mathbf{R}^2_+$; i.e., $\operatorname{im} s > 0$.

Now we have by substitution of (1) into (2)

$$a(s) = 1 - (2is)^{-1}\int_{-\infty}^{\infty} q(t)dt - (2is)^{-1}\int_{-\infty}^{\infty} e^{ist}q(t)g(t,s)dt$$

where

$$g(t,s) = \int_{-\infty}^{t} \frac{\sin s(t-y)}{s} q(y)f_2(y,s)dy,$$

we have

$$|g(t,s)| \leq \int_{-\infty}^{t} |s|^{-1} e^{ims(t-y)}|q(y)|\exp(|s|^{-1}\sigma_2(y)e^{imsy}dy$$

$$= e^{imst}\int_{-\infty}^{t} \partial_y \exp(|s|^{-1}\sigma_2(y))dy$$

$$= e^{imst}[\exp(|s|^{-1}\sigma_2(t) - 1)],$$

$$\int_{-\infty}^{\infty}|e^{ist}q(t)g(t,s)|dt \leq \int_{-\infty}^{\infty} e^{imst}|q(t)|e^{-imst}[\exp(|s|^{-1}\sigma_2(t)) - 1]\,dt$$

$$= \int_{-\infty}^{\infty}|q(t)|[\exp(|s|^{-1}\sigma_2(t)) - 1]\,dt$$

$$\leq [\exp(|s|^{-1}\sigma_2(\infty) - 1]\,\sigma_2(\infty) = 0(1),\ |s| \to \infty.$$

This gives (15)

$$2|s|\ |b(s)| \leq \int_{-\infty}^{\infty}|q(t)|\exp(|s|^{-1}\sigma_2(t))dt$$

$$\leq \exp(|s|^{-1}\sigma_2(\infty))\int_{-\infty}^{\infty}|q(t)|dt = 0(1)$$

and this gives (14). This completes the proof of the theorem

We remark here that (11) and (14) imply that $a(\cdot)$ has no zeros outside some bounded set. Further it is clear that for positive real λ, no combination of f_1, f_2 can be in $L^p(-\infty,\infty)$ $(p < \infty)$, and therefore, as we shall see, $(0,\infty) \supseteq \sigma_e(L)$. σ_e denotes the essential spectrum (essential spectrum is defined as in [8]).

We define the "resolvent kernel" as (χ is a characteristic function)

$$R(x,y,\lambda) = [(W[f_1,f_2])(\lambda)]^{-1}\{f_1(x,s)f_2(y,s)\chi_{(-\infty,x)}(y) + f_1(y,s)f_2(x,s)\chi_{(x,\infty)}(y)\}.$$

We proceed now to the next result where it is shown that the zeros of $a(\cdot)$ have a close relation to the oscillatory properties of the imaginary part of the potential.

THEOREM 5.

(17) *The zeros of* $a(\cdot)$ *are nowhere dense in* $\overline{R^2_+}$, *and if a neighborhood of the positive real axis is removed from the* λ*-plane,* $a(\sqrt{\lambda})$ *has only finitely many zeros.*

(18) *If* $a(\pm s) \neq 0$, $(\operatorname{im} s = 0)$

$$R(x,y,\lambda+i0) - F(x,y,\lambda-i0) = -[2is\, a(s)a(-s)]^{-1}\{f_2(x,-s)f_2(y,s) + f_1(x,-s)f_1(y,s)\}$$

$$= -[2is\, a(s)a(-s)]^{-1}\{f_1(x,s)f_1(y,-s) + f_2(x,s)f_2(y,-s)\}.$$

(19) *The eigenvalues* λ *of* L *are exactly those* $\lambda \notin R_+$ *with* $a(s) = 0$.

(20) *If* $\operatorname{im} q(t)$ *has constant sign, then the zeros of* $a(\cdot)$ *can accumulate only at zero.*

PROOF. (17) follows from the previous theorem and the usual facts about analytic functions.

For (18)

$$R(x,y,+i0) - R(x,y,-i0) = -(2is)^{-1}\{\frac{f_1(x,s)f_2(y,s)}{a(s)}\chi_{(-\infty,x)}(y) + \frac{f_1(y,s)f_2(x,s)}{a(s)}\chi_{(x,\infty)}(y)\}$$

$$- (2is)^{-1}\{\frac{f_1(x,-s)f_2(y,-s)}{a(-s)}\chi_{(-\infty,x)}(y) + \frac{f_1(y,-s)f_2(x,s)}{a(s)}\chi_{(x,\infty)}(y)\}$$

$$= - (2is\, a(s)a(-s))^{-1}\{[a(-s)f_1(x,s)f_2(y,s)$$

$$+ a(s)f_1(x,-s)f_2(y,s)]\chi_{(-\infty,x)}(y)$$

$$+ [a(-s)f_1(y,s)f_2(x,s) + a(s)f_1(y,-s)f_2(x,-s)]\chi_{(x,\infty)}(y)\}$$

using (12) and (13), we have

$$- (2is\, a(s)a(-s))^{-1}\{(f_1(x,s)[f_1(y,-s) + b(s)f_2(x,-s)$$

$$+ [f_2(x,s) - b(s)f_1(x,s)]\, f_2(y,-s))\chi_{(-\infty,x)}(y)$$

$$+ (f_2(y,-s)[f_2(x,-s) - b(s)f_1(x,-s)]$$

$$+ f_1(y,-s)[f_1(x,s) + b(-s)f_2(x,s)]\,\chi_{(x,\infty)}(y)\}$$

$$= (- 2is\, a(s)a(-s))^{-1}\{ f_1(x,s)f_1(y,-s) + f_2(x,s)f_2(y,-s)\}.$$

For (19) we note that if $a(s) = 0$, then

$$f_1(x,s) = c(s)f_2(x,s)$$

and since $\lambda \notin \mathbf{R}_+$

$$f_1 = 0(e^{-imsx})\quad x \to \infty$$

$$f_2 = 0(e^{imsx})\quad x \to \infty.$$

$Lf_{1,2} = \lambda f_{1,2}$ then implies λ is an eigenvalue of L. Since $R(x,y,z)$ is analytic elsewhere, this completes the proof of (19).

Note that

$$Ly = \lambda y$$

$$\int_{-\infty}^{\infty} Ly\bar{y} - y\overline{Ly} = (\lambda - \bar{\lambda})\int_{-\infty}^{\infty} |y|^2,$$

but

$$\int_{-\beta}^{\beta} Ly\bar{y} = - y'\bar{y}\Big|_{-\beta}^{\beta} + \int_{-\beta}^{\beta} |y'|^2 + \int_{-\beta}^{\beta} q|y|^2$$

$$\int_{-\beta}^{\beta} \overline{Ly}y = - \bar{y}'y\Big|_{-\beta}^{\beta} + \int_{-\beta}^{\beta} |y'|^2 + \int_{-\beta}^{\beta} \bar{q}|y|^2.$$

Thus, from (1) and (3),

$$2i \int_{-\infty}^{\infty} |y|^2 im(q)dx = 2i\, im\lambda \int_{-\infty}^{\infty} |y|^2$$

and so $imq = 0$, if it has constant sign and $\lambda \in \mathbf{R}_-$.

For (20), it follows from the argument immediately preceding that $\lambda > 0$ implies

$$\lambda \int_{-\beta}^{\beta} |y|^2 = - is[1 + 0(1)] + \int_{-\beta}^{\beta} |y'|^2 + \int_{-\beta}^{\beta} q|y|^2$$

$$\lambda \int_{-\beta}^{\beta} |y|^2 = is[1 + 0(1)] + \int_{-\beta}^{\beta} |y'|^2 + \int_{-\beta}^{\beta} \bar{q}|y|^2.$$

Thus

$$- s[1 + 0(1)] + \int_{-\beta}^{\beta} imq|y|^2 = 0.$$

Assume without loss of generality $\mathrm{im}\, q \leqslant 0$. This gives a contradiction.

Now we line up necessary facts about the resolvent.

THEOREM 6. *Suppose* $g \in D(L(\ell^*,q)) \cap K^2$. *Then*

$$\int_{-\infty}^{\infty} R(x,y,\lambda)(\ell_\lambda^* g)(x)dx = g(y),$$

where $a(\sqrt{\lambda}) \neq 0$, $\mathrm{im}\sqrt{\lambda} > 0$.

PROOF. Green's formula may be applied since g has compact support.

THEOREM 7. *Define for* $f \in L^p(-\infty,\infty)$, $1 \leqslant p \leqslant \infty$,

$$R(\lambda,L)f(t) = \int_{-\infty}^{\infty} R(x,t,\lambda)f(x)dx$$

for all λ s.t. $a(\sqrt{\lambda}) \neq 0$, $\mathrm{im}\sqrt{\lambda} > 0$. *Then the range of* $R(\lambda,L)$ *is the domain of* $L(\ell_\lambda,p)$ *and* $L(\ell_\lambda,p)R(\lambda,L)$ *is the identity on* $L^p(-\infty,\infty)$.

PROOF. We have for g as in Theorem 6 and $f \in L^p(-\infty,\infty)$, (note $R(x,y,\lambda) = R(y,x,\lambda)$)

$$\int_{-\infty}^{\infty} f(y)g(y)dy = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} R(x,y,\lambda)(\ell^* g)(x)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\, R(x,y,\lambda)(\ell^* g)(x)dydx$$

$$= \int_{-\infty}^{\infty} R(\lambda,T)(\ell^* g)(x)dx.$$

(The Fubini theorem is justifiably used here. See below.)

Now consider $\int_{-t}^{t} R(x,y,\lambda)dy$. From Theorem 1 we have

$$\sup_{-t<x<t} |f_1(x,s)| \int_{-t}^{t} |f_2(y,s)|dy \leqslant K(\lambda)/\mathrm{im}\, s.$$

It follows then that if $s = \sqrt{\lambda}$ is bounded away from the origin, the zeros of $a(\cdot)$ and the real axis we have,

$$\sup_{-t<x<t} \int_{-t}^{t} |R(x,y,\lambda)|dy < c(\lambda) < \infty.$$

It is an easy matter to verify that

$$\|R(\lambda,L)f\|_{L^1} \leqslant \|f\|_{L^1} \sup_x \int_{-\infty}^{\infty} |R(x,y,\lambda)|dy < c(\lambda)\|f\|_{L^1}$$

and

$$\|R(\lambda,L)f\|_{L^\infty} \leqslant \|f\|_\infty c(\lambda),$$

and therefore $R(\lambda)$ is a bounded operator on L^1 and L^∞ with norm at most $c(\lambda)$. The

Riesz convexity theorem now implies that $R(\lambda)$ is bounded on L^p $(1 \leqslant p \leqslant \infty)$. Suppose f is continuous with compact support. Using the definition of $R(x,y,\lambda)$, it is easy to see that

$$(\ell_\lambda R(\lambda,L)f)(x) = \int_{-\infty}^{\infty} \ell_\lambda R(x,y,\lambda)f(y)dy + f(x)$$

but this implies (since $\ell_\lambda R(x,y,\lambda) = (\lambda - \ell)\, \Sigma\, f_1 f_2 = 0)$ $R(\lambda,L)f \in D(L)$ and so by Lemma 2.1 of Rota [8] we have

$$\int_{-\infty}^{\infty} \ell_\lambda (R_\lambda f)(t)g(t)dt = \int_{-\infty}^{\infty} f(t)g(t)dt.$$

K^2 is dense in L^q, $q < \infty$ implies $\ell_\lambda(R(\lambda,L)f)(t) = f(t)$ for almost all t. Since $L(\ell,p)$ is closed the same equation is valid in L^p $(1 < p \leqslant \infty)$. Now, however, K^2 is L^1-dense in L^∞ and this implies the result for $p = 1$ by the dominated convergence theorem. This completes the proof.

NOTE. The properties of $a(\cdot)$ now imply that $\sigma_e(L) \subseteq (0,\infty)$.

THEOREM 8. $L_0(\ell,p) = L(\ell,p)$, $1 < p \leqslant \infty$.

PROOF. For $1 < p \leqslant \infty$ we have

(A) $L_0(\ell^*,p) - \bar{\lambda} = (L(\ell,q) - \lambda)^*$

(B) $L_0(\ell,p) - \lambda = (L(\ell^*,q) - \bar{\lambda})^*$

which for $\sqrt{\lambda}$ restricted as in the previous theorem implies that each of these operators has a bounded everywhere defined inverse, since the unstarred right-hand sides of (A) and (B) do. However,

$$L_0(\ell,p) - \lambda \subseteq L(\ell,p) - \lambda$$

and for $p > 1$ both have everywhere defined inverses. Hence they are equal and therefore

(C) $L_0(\ell,p) = L(\ell,p)$.

For $p = 1$, $L_0(\ell,1) = L(\ell^*,\infty)$ and since by (C)

$$L(\ell^*,\infty)^* = L_0(\ell^*,\infty)^*,$$

we have

$$L_0(\ell,1) = L_0(\ell^*,\infty)^*$$

and applying Theorem 2, the proof is complete.

REMARK. We have shown now that $L(\ell,p)$ is a uniquely defined operator in all the Banach spaces $L^p(-\infty,\infty)$, $1 \leqslant p \leqslant \infty$. From now on we simply denote it as "L."

We now identify the spectrum of L.

THEOREM 9. *The spectrum of* L, $\sigma(L)$ *consists of all the points* s^2 *in* C ($\mathrm{im}\, s \geqslant 0$), *such that* s *is a zero of* $a(\cdot)$, *or* $s^2 \geqslant 0$.

REMARK. For $\mathrm{im}\, s = 0$, $f_{1,2}$ are both in $L^\infty(-\infty,\infty)$, $(s \neq 0)$. $f_{1,2}$ appear to be eigenfunctions of L. However, our duality theory (weak * convergence on $L^\infty \oplus L^\infty$) does not admit them. (See 3.1 of Rota [8].)

Much the same result holds for n^{th} order operators under the assumption of regularity and sufficient smoothness and boundedness conditions on the coefficients. Kemp [6] gives a theory for singularities interior to the interval of definition in the general n^{th} order case.

3. **Spectral theory of L.** It is apparent that the zeros of $a(\cdot)$ on the real axis play a large role in the spectral theory of L.

THEOREM 10. *If* $a(\cdot)$ *has no zeros on* $\mathbf{R}^1$, *then* L *is a spectral operator in* $L^2(-\infty,\infty)$.

THEOREM 11. L *has a locally defined spectral measure.*

THEOREM 12. *If* $a(\cdot)$ *has no zeros on* $\mathbf{R}^1$, *then* L *has* L^p *eigenfunction expansions for* $1 < p \leqslant 2$. *In general, no complete expansion exists for* $p > 2$.

REMARK. We will discuss part of the proof of Theorem 11 and also the "spectral representations" associated with it. We then indicate how to derive the rest of Theorem 11.

The other two results will follow from the discussion of the mappings "Φ" defined below.

The nature of the zeros of $a(\cdot)$ must be discussed in more detail now.

LEMMA. *Considering* $a(\cdot)$ *as a function of* $\sqrt{\lambda}$, $\mathrm{im}\sqrt{\lambda} \geqslant 0$. *The zeros of* $a(\cdot)$ *on* $\mathrm{im}\sqrt{\lambda} = 0$ *form a set of* $\mathbf{R}^1$*-measure zero.*

PROOF. Excluding $s = 0$ by a semicircle of radius "ϵ" in the upper half plane we can map to this the unit disc preserving the boundary. Then $a(\cdot)$ may be thought of as a function bounded and analytic in the disc. $a(0) \neq 0$, so Jensen's formula implies

$$|a(0)|\Pi_{n=1}^{N} \frac{r}{|a_n|} = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|a(re^{it})|dt\}$$

$\{a_n\}$ being the zeros of $a(\cdot)$ in the closed disc of radius r. Let

$$K_r = \int_{-\pi}^{\pi} \log|a(re^{it})|dt.$$

We can choose r so that $K_r > -\infty$ since $a(\cdot)$ is not identically zero. Since K_r is monotone increasing in r by Jensen's formula, Fatou's lemma implies $K_r \leqslant K_1$. This implies the result, since ϵ was arbitrary and Lebesgue measure is countably additive.

We now give the formula which generates the measure on the essential spectrum. Let $\delta_\epsilon \subseteq \mathbf{R}^1_+$ be a bounded set whose closure is a distance ϵ away from the zeros of $a(\cdot)$. Define

$$\mu(\delta) = \frac{1}{2\pi} \int_{\lambda\in\delta} \frac{\{ f_1(\cdot,s) \otimes f_1(\cdot,s) + f_2(\cdot,s) \otimes f_2(\cdot,s)\} ds}{a(s)a(-s)}$$

where

$$f \otimes g(\phi(t)) = f(t) \int g(y,-s)\phi(y)dy.$$

The verification that μ is multiplicative can proceed directly though it is rather tedius. For $g \in \boldsymbol{D}$ or $\boldsymbol{S}$ (test functions, etc.) $\mu(\delta_\epsilon)g$ is certainly in $L^p(-\infty,\infty)$. The domain of μ therefore contains a dense subset of $L^p(-\infty,\infty)$. Define three terms as follows:

(i) $u^\pm_{1,2}(\cdot,s) = f_{1,2}(\cdot,\pm s)/a(\pm s)$

(ii) $\Phi^\pm_{1,2} g(s) = (2\pi)^{-1/2} \int_{\mathbf{R}} u^\pm_{1,2}(x,s)g(x)dx \qquad g \in \boldsymbol{D}$ or $\boldsymbol{S}$

(iii) $\Phi^\pm g(s) = \langle \Phi^\pm_1 g(s), \Phi^\pm_2 g(s)\rangle \qquad \Phi^\pm g(s) \in L^p(\mathbf{R}_+, \mathbf{C}^2, ds)$.

Now if $\psi \in L^q(\mathbf{R}_+,\mathbf{C}^2,ds) \cap \boldsymbol{S}$, then

$$\begin{aligned}(\Phi^\pm g, \psi) &= (\Phi^\pm_1 g, \psi_1) + (\Phi^\pm_2 g, \psi_2)\\ &= (2\pi)^{-1/2} \int_{\mathbf{R}\times\mathbf{R}_+} [u^\pm_1(x,s)g(x)\psi_1(s)\\ &\quad + u^\pm_2(x,s)g(x)\psi_2(s)]\, dxds\\ &= (g, (\Phi^\pm)^*\psi)\end{aligned}$$

hence

$$\begin{aligned}(\Phi^\pm)^*\psi(x) &= (2\pi)^{-1/2} \int_{\mathbf{R}^+} [\bar{u}^\pm_1(x,s)\psi_1(s) + \bar{u}_2(x,s)\psi_2(s)]\, ds\\ &= (2\pi)^{-1/2} \int_{\mathbf{R}_+} [u^{\pm *}_1(x,s)\psi_1(s) + u^{\pm *}_2(x,s)\psi_2(s)]\, ds.\end{aligned}$$

It is a useful guide to consider the selfadjoint case, $\operatorname{im} q(t) \equiv 0$. In that case it is easily seen that (cf. Theorem 1)

$$a(-s) = \overline{a(s)}$$

$$\overline{f}_{1,2}(x,s) = f_{1,2}(x,-s)$$

and $a(s) \neq 0$, $s \in \mathbf{R}$, $(\lambda \in \mathbf{R}_+)$. Assume $p = 2$. Then

$$\|\mu(\delta)g\|_2^2 = \int_{-\infty}^{\infty} |\int_{\delta \ni s^2} \frac{f_1(x,s)}{a(s)} \Phi_1 \overline{g}(s) + \frac{f_2(x,s)}{a(s)} \Phi_2 \overline{g}(s) ds|^2 dx$$

$$= \int_{-\infty}^{\infty} |\int_\delta u_1^+ \Phi_1^- g(s) + u_2^+ \Phi_2^- g(s) ds|^2 dx$$

$$= \int_{-\infty}^{\infty} \int_{\delta \ni s^2} u_1^+ \Phi_1^- g(s) + u_2 \Phi_2^- g(s) ds \int_{\delta \ni s^2} \overline{u_1^+ \Phi_1 \overline{g}(s)} + \overline{u_2^+ \phi_2 \overline{g}(s)} ds dx$$

$$= (\Phi^{*-} \chi_\delta \Phi \overline{g}, \Phi^{*-} \chi_\delta \Phi \overline{g}).$$

Since $a(\cdot) \neq 0$, we can let $\delta = (0,\infty)$ and $\Phi^\pm, \Phi^{*\pm}$ are seen to be isometries. This illustrates the difficulty for $p \neq 2$, $\mathrm{im}q(t) \neq 0$. If we assume $\mathrm{im}q(t) \neq 0$ and $p = 2$, then we obtain the representation

$$\|\mu(\delta)g\|_2^2 = (\overline{\Phi^{+*}\chi_\delta \Phi \overline{g}}, \overline{\Phi^{+*}\chi_\delta \Phi \overline{g}})$$

or

$$\mu(\delta)g = \overline{\Phi^{+*}\chi_\delta \Phi \overline{g}}$$

which is bounded as long as δ is some positive distance from the zeros of $a(\cdot)$. The appearance of $\overline{\Phi^{+*}}$ above indicates that the solutions of L^* must also be considered if the expansion is to be analogous to the Fourier theory. This is illustrated by the consideration of an appropriate physical example, such as the vibrating finite string with one flexible support.

Now let us consider the general case where $p \neq 2$, $\mathrm{im}q(t) \neq 0$. When $\chi_\delta \Phi \overline{g}$ is bounded (or at least in $L^q(-\infty,\infty)$), then the representation given above is valid. However, little is to be gained by this course. This follows from the fact that

$$(\overline{\Phi^{*+}\chi_\delta \Phi \overline{g}}, \overline{\Phi^{*+}\chi_\delta \Phi \overline{g}})^{1/p} = \sup_{\|\psi\|_q = 1} \int_{-\infty}^{\infty} (\int_\delta u_1^+ \Phi_1 \overline{g} + u_2^+ \Phi_2 \overline{g} ds) \psi(x) dx$$

which in general is unbounded (for $p > 2$) by virtue of the same fact relative to the Fourier transform $(q(t) = 0)$. Therefore the most that can be said is $\mu(\delta)$ is not a bounded operator in $L^p(-\infty,\infty)$ $p \neq 2$ (whether or not $\mathrm{im}q(t) = 0$), in general. We can generate the locally defined measure for L by intersection of the ranges $\mu(\delta_\epsilon)\boldsymbol{D}$ over $\epsilon > 0$ and defining $\mu(\{\lambda\}) = 0$ if $a(s) = 0$, $s = \sqrt{\lambda}$. Then μ clearly satisfies the hypotheses of Section 2 of [9].

In general, μ "vanishes" on the subspace consisting of limits of generalized eigenfunctions (see the discussion on pages 571-577 of [2]). The hypotheses of Theorem 20 of [1] may now be verified to show Theorem 10 is valid. Some of the "damage" can be repaired by the following theorem whose proof is exactly analogous to that given in Rutovitz [9].

THEOREM. *For* $xq(x) \in L^1(-\infty,\infty)$ *and of bounded variation, we have*

$$(\lim \chi_\delta \Phi_{1,2}^{-} f = f)$$

in L^p *when* $1 < p \leqslant 2$ *and* $a(\cdot)$ *has no zeros on the real axis. (The limit is taken as* δ *expands to* $\mathbf{R}^1$.*)*

The "complex" square well provides an example. Set

$$L = -\frac{dt}{dt^2} - (c + id)\,\chi(t)_{(-a,a)} \qquad \left\{ \begin{matrix} c > 0 \\ d < 0 \end{matrix} \right.$$

$$\rho = \sqrt{s^2 + (c + id)} \qquad (\mathrm{im}\rho \geqslant 0)$$

then

$$f_1(t,s) = e^{ist}\,\chi(t)_{(a,\infty)} + e^{isa}[\cos \rho(t - a) + is\,\rho^{-1} \sin \rho(t - a)]\,\chi(t)_{(-a,a)}$$

$$+ [a(s)e^{ist} + b(s)e^{-ist}]\,\chi(t)_{(-\infty,a)}$$

where

$$a(s) = e^{2isa}[\cos (2a\rho) - ia\{s\}^{-1}(2s^2 + c + id)\frac{\sin 2a\rho}{2a\rho}].$$

Theorem 5 shows $a(\cdot)$ has no zeros on the real axis except possibly at $s = 0$. A computation shows this to be impossible.

The theorem above takes on added importance for computational purposes if we consider the following result.

THEOREM. $\overline{\Phi^{+*}\chi_\delta\Phi^{-}} = \Phi_2^{*-}a(-p)\chi_\delta(p^2)\Phi_1^{-}$.

PROOF. $\mu(\delta)g(s) = \int_{\lambda\in\delta}[u_1^{+}(x,s)\Phi_1^{-}g(s) + u_2^{+}(x,s)\Phi_2^{-}g(s)ds$ using the fact that

$$f_2(x,s) = b(s)f_1(x,s) + a(s)f_1(x,-s)$$

$$\Phi_2^{-}g(p) = (2\pi)^{-1/2} \int \frac{b(-p)f_1(x,-p) + a(-p)f_1(x,p)}{a(-p)} g(x)dx$$

$$(*) \quad u_2^{+}(x,p)\Phi_2^{-}g(p) = \frac{b(-p)}{a(p)} f_2(x,p)\Phi_1^{-}g(p) + f_2(x,p)\Phi_1^{-}g(-p)$$

$$(**)\quad u_1^+(x,p)\Phi_1^- g(p) = \frac{f_1(x,p)}{a(p)}\Phi_1 g(p) = [-\frac{b(-p)}{a(p)}f_2(x,p) + f_2(x,-p)]\,\Phi_1^- g(p)$$

hence

$$(*) + (**) = f_2(x,p)\Phi_1^- g(-p) + f_2(x,p)\Phi_1^- g(p)$$

hence

$$\mu(\delta)g(x) = \int_{\sqrt{\delta}}[f_2(x,p)\Phi_1^- g(-p) + f_2(x,-p)\Phi_1^- g(p)dp$$

$$= \int f_2(x,-p)\Phi_1^- g(p)\chi_\delta(p^2)dp.$$

This gives us the "single term" spectral representation above and concludes the proof.

For the formally selfadjoint case the measure is

$$\bar{\Phi}_2^+\overline{a(p)}\chi_\delta(p^2)\Phi_1^-.$$

Verification is left to the reader. The reader is referred to (cf. page 1205 [2]) for contrast.

We have not as yet considered the point spectrum of L. We do this in the following

THEOREM 13. *Let* $\{\xi_i\}_1^\infty$ *be the eigenvalues of* L. *Then the measure* μ *of* $\delta = \{\xi_i\}_{i=1}^N$ *is given by*

$$\mu(\delta) = \Sigma_\delta \operatorname{Res}.R(\lambda,L)$$

$$= \Sigma_{\xi\in\delta}\left(\frac{d}{dz}\right)^{K_\xi-1}\left[\frac{(z-\xi)^{K_\xi-1}m(s)}{(K_\xi-1)!a(s)}f_2(\cdot,\xi)\otimes f_2(\cdot,\xi)\right]_{z=\xi}$$

where K_ξ *is the order of* ξ *as a zero of* $a(\cdot)$. *If* $K_\xi = 1$

$$\mu(\xi) = \frac{f_2(\cdot,s)\otimes f_2(\cdot,s)}{f_2(\cdot,s)f_2(\cdot,s)}\qquad (s=\sqrt{\xi}).$$

PROOF. It suffices to show the $K_\xi = 1$ version.

$$f_1(x,s)f_2(y,s)\chi_{(-\infty,x)}(y) + f_1(y,s)f_2(x,s)\chi_{(x,\infty)}(y) = c(s)f_2(x,s)f_2(y,s).$$

We claim

$$\dot{a}(s) = -ic(s)\int_{-\infty}^{\infty}(f_2(x,s))^2dx.$$

Proceed as follows: (D means x derivative; $(\cdot)$ means $\sqrt{\lambda}$ derivative).

$$-2isa(s) = W[f_1(\cdot,s),f_2(\cdot,s)]$$

$$[-2isa(s)]^{\bullet} = -2ia(s) - 2is\dot{a}(s) \qquad (a(s) = 0)$$

$$= W[\dot{f}_1,f_2] + W[\dot{f}_1,f_2]$$

$$L\dot{f}_{1,2} = (s^2 f_{1,2})^{\bullet} = 2sf_{1,2} + s^2\dot{f}_{1,2}.$$

Then

$$DW[\dot{f}_1,f_2] = W[D\dot{f}_1,f_2] + W[\dot{f}_1,Df_2]$$

$$= \det \begin{bmatrix} D\dot{f}_1 & f_2 \\ D^2\dot{f}_1 & Df_2 \end{bmatrix} + \det \begin{bmatrix} \dot{f}_1 & Df_2 \\ D\dot{f}_1 & Df_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} D\dot{f}_1 & f_2 \\ q\dot{f}_1 - 2sf_1 - s^2\dot{f}_1 & Df_2 \end{bmatrix} + \begin{bmatrix} \dot{f}_1 & Df_2 \\ D\dot{f}_1 & qf_2 - s^2 f_2 \end{bmatrix}$$

$$= 2sf_1 f_2.$$

We know that

$$\dot{f}_1(x,s) = 0(e^{-imsx}|x|)$$
$$D\dot{f}_1(x,s) = 0(|x|e^{-imsx})$$
$$f_2(x,s) = c(s)f_1(x,s) = 0(e^{-imsx})$$
$$Df_2(x,s) = c(s)Df_1(x,s_0) = 0(e^{-imsx}),\ x \to \infty.$$

Hence

$$2s \int_x^\infty f_1(t,s)f_2(t,s)dt = 3s \lim_{\beta\to\infty} \int_x^\beta f_1(t,s)f_2(t,s)dt$$

$$= \lim_{\beta\to\infty} W[\dot{f}_1,f_2](\beta) - W[\dot{f}_1,f_2](x)$$

$$= - W[\dot{f}_1,f_2](x).$$

In the same way

$$DW[f_1,\dot{f}_2] = - 2sf_1 f_2$$

and

$$W[f_1,\dot{f}_2] = -2s \int_{-\infty}^x f_1(x,s)f_2(x,s)dx.$$

This gives the claim above. Now, since

$$\lim_{z\to\lambda} \frac{a(\sqrt{\lambda})}{z-\lambda} = \frac{1}{2\sqrt{\lambda}} \lim_{\sqrt{z}\to\sqrt{\lambda}} \frac{a(\sqrt{z})}{\sqrt{z}-\sqrt{\lambda}} = \frac{1}{2s}\dot{a}(s)$$

we have

$$\operatorname*{Res}_{z=\lambda} R(z) = - \frac{c(s)}{\dot{a}(s)} f_2(\cdot,s) \otimes f_2(\cdot,s)$$

which then gives (21) and this completes the proof.

Unfortunately, the series of eigenfunctions diverges in general, as might be expected. We simply remark that for p = 2, the situation is no better.

REMARK. For q more rapidly decreasing, for example

$$\int_{-\infty}^{\infty} e^{|x|}\,|q(x)|dx < \infty$$

or

$$\int_{-\infty}^{\infty}(1 + x^2)|q(x)|dx < \infty.$$

Then $f_{1,2}(\cdot,s)$ is continuous at zero. The first condition comes from Stone [11]. That the second implies the required continuity we note that

$$\left|\frac{d}{ds}\left(\frac{\sin s(t-y)}{s}\right)\right| \leqslant |t - y|^2$$

we have, $((\ )^{\bullet}$ indicates $\frac{d}{ds})$,

$$\left|\int_{\infty}^{y}\left(\frac{\sin s(t-y)}{s}\right)^{\bullet} q(t)f_2(t,s)dt\right| \leqslant \int_{-\infty}^{y}|t - y|^2|q(t)|\,|f_2(t,s)|dt < c(s)|1 + y^2|$$

where the last inequality follows by

$$|f_2(x,z)| \leqslant e^{imzs}\exp\left(\int_{-\infty}^{x}(x - y)|q(y)|dy\right),$$

which is proved exactly as (4). We obtain by another similar calculation that

$$|\dot{f}_2(x,z)| \leqslant c(z)(1 + x^2)\exp(\sigma_2(x))$$

and using the corresponding equation for $\dot{f}_2$ we see $\dot{f}_2(x,0)$ exists and this shows the required continuity. We can therefore state the following

THEOREM 14. *Let* q *be decreasing as in the above remark. Suppose* imq *has constant sign. Then* L *is a spectral operator in* $L^2(-\infty,\infty)$.

THEOREM 15. *If* $\text{im}q(t) \leqslant 0$ *for* $|t| > a \geqslant 0$ *and there exists* $N > 0$ *such that*

$$\left|\int_{-a}^{a}\text{im}q\left[\exp\int_{-\infty}^{\infty}|x - y|\,|q(y)|dy\right]dx\right| \leqslant \left|\int_{[-a,a]^c\cap(\beta,B)}\text{im}q\,dx\right|$$

whenever $N < \beta$, *then* L *is a spectral operator in* $L^2(-\infty,\infty)$.

PROOF. The proof follows once we establish

$$f_1(x,s)| \leqslant \exp\left\{\int_{x}^{\infty}(y - x)|q(y)|dy\right\} < \infty$$

(ims = 0) and apply the computation of Theorem 5. This inequality follows from the second rapid decrease condition as in Theorem 1 using $|\frac{\sin sx}{s}| \leqslant |x|$.

CONJECTURE. For $1 < p < 2$, L has L^p-convergent eigenfunction expansions if q(t) satisfies **the first** condition (is exponentially decreasing) above.

REMARK. There is a relation between the case $a(\cdot) \neq 0$ on the real axis and the finite expansions of G. L. Krabbe in L^p $(1 < p < \infty)$ for $-d^2/dt^2$ [7]. As Krabbe notes in [7], his finitely additive measures do not admit countably additive extensions in general, although they are bounded, everywhere defined projection valued measures. It therefore appears that one must give up either boundedness or countable additivity in the general case.

Finally we note one further example of the form we have been studying:

$$\frac{d}{dt}p(t)\frac{d}{dt} + q(t) \qquad t \in (-\infty,\infty).$$

The variety of behavior here is significantly greater than for L. For example take $p(t) = t^2$, $q(t) = 0$. Then the solutions of the corresponding equation may fail to be in L^p, depending on what value of p we select. Thus the essential spectrum may vary with p. The solutions are of the form $t^{-(1-u/2)}$, $t^{-(u+1/2)}$, $u = \sqrt{1-\lambda}$, or $\lambda = 1 - u^2$ for the eigenequation. If $\mathrm{re}(1 \pm u/2) = 1/p$, then the corresponding λ must lie in the essential spectrum (i.e., $\lambda = 1 - (2(1/p + is) - 1)^2$ where s is a real parameter) if we restrict ourselves to $(-\infty,-1)$, say. This, of course, makes for a much less specific result in the Sturm-Liouville case.

For Banach spaces then, densely defined measures exist (or even "local" measures) where there can be no expansion in the ordinary sense.

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