A Note on Operator-Valued Measures And the Easton-Tucker Integral

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1 Introduction

In this short note we examine a question about the Easton Tucker integral raised by Tucker [6] together with a version of the mean value theorem for operator measure-vector function integrals. We will employ the integral introduction by Bartle [1] in our exposition.

Below, X and Y are Banach spaces $L(X,Y)$ the space of continuous linear operators from X to Y. If μ is a finitely additive set function on an algebra of subsets A of some set H, with values in $L(X,Y)$, we define as usual for $E \in A$,

$$\hat{\mu}(E) = \sup\{\|\Sigma x_i \mu(E_i \cap E)\|\}$$

where the supremmum is taken over all $x_i \in X_i$ (the unit ball of X) and all finite disjoint sequences $\{E_i\}$. $\hat{\mu}$ is called the semivariation of μ. We assume $\hat{\mu}(H) < \infty$. If f is X valued and strongly measurable, then the essential range of f on subset F of H, $\mathrm{er}_F(f) \neq \emptyset$ is defined in the usual way (relative to $\hat{\mu}$). In [4] the measure μ is $L(X, Y^{**})$ valued and H is compact Hausdorff space while A may be taken as the Borel field (see [5]). We shall assume these conditions unless otherwise noted. The lemmas immediately following are due to Rieffel [2] in the vector case.

Lemma 1 *If μ is c.a. in the weak* operator topology, then $\hat{\mu}(F) > 0$ implies that $f(F) \cap \mathrm{er}_F(f) \neq \emptyset$.*

Proof. If $f(F) \cap \mathrm{er}_F(f)$ is empty, then for each $x \in F$ there exists $\epsilon_x > 0$, s.t. $V(y^*\mu)\{y \in \ F \mid \|f(y) - f(x)\| < \epsilon_x\} = 0$ for all y^* s.t. $\|y^*\| \leq 1$ (because $\hat{\mu}(F) = \sup_{\|y^*\| \leq 1} V(y^*\mu)$. The sets

$$f(\{y \in f \mid \|f(y) - f(x)\| < \epsilon_x\}),$$

cover $f(F)$. We may assume that $f(F)$ is a separable metric space so there exists a sequence $\{\epsilon_{x_n}\}$ such that the balls $B(f(x_n), \epsilon_{x_n})$ cover $f(F)$. $F \subseteq \bigcap_{n=1}^{\infty}\{y \mid \|f(y) - f(x_n)\| < \epsilon_{x_n}\}$ which together with the c.a. of $y^*\mu$ implies $V(y^*\mu)F = 0$ and since $\sup_{y^* \in Y_1^*} V(y^*\mu) = \hat{\mu}$ we have $\hat{\mu}(F) = 0$, a contradiction.

Lemma 2 *If f is measurable and $\hat{\mu}(F) > 0$ and $N = \{x \in F \mid \ \notin f(x)\mathrm{er}_F(f)\}$, then $\hat{\mu}(N) = 0$.*

Lemma 3 *If f, g are measurable and $\hat{\mu}(F) > 0$ with $\|f(x) - g(x)\| < M$, for all $x \in F$, then $b \in \mathrm{er}_F(f)$ implies that the distance from b to $\mathrm{er}_F(g)$ is less than of equal to M.*

Proof. If $E = \{x \in F \,|\, \|f(x) - b\| < \epsilon\}$, then $\hat{\mu}(E) > 0$. There exists $y \in E$ s. t. $g(y) \in \mathrm{er}_E(g) \cap \mathrm{er}_E(g)$, which implies $\|b - g(y)\| < M + \epsilon$. Since ϵ was arbitrary, the proof is complete.

Definition 1 Let $M \subseteq X$. Define

$$C0_{\mu,s}(M) = \bigcup_{E \subseteq S} \mu(E)^{-1} \left\{ \sum_{1}^{n} \mu\left(E_i \cap E\right) x_i \right\}$$

where $x_i \in M$ and $\{E_i\}$ is a disjoint finite sequence.

If f is integrable in the Bartle Sense, define

$$A_{E,\mu}(f) = \bigcup_{F} \left\{ \mu(F)^{-1} \left(\int_F f d\mu \right) \mid F \subseteq E \right\}$$

call $A_{E,\mu}(f)$ the μ average value of f on E and $C0_{\mu,s}(M)$ the μ convex hull of M. For the scalar case motivation of the next result see [3] p. 77, for example.

Theorem 1 (Mean value theorem.) *If f is integrable, then*

$$A_{E,\mu}(f) \subseteq \overline{C0}_{\mu,E}\,(\mathrm{er}_E f). \qquad (E \in A)$$

Proof. Fix $\in > 0$. Since f is integrable there exists a sequence of simple functions f_n such that $\hat{\mu}(\{t \mid \| f_n(t) - f(t)\| > \delta\}) \to 0$ as $n \to \infty$. $\int_{(\cdot)} f d\mu$ is absolutely continuous with respect to $\hat{\mu}$ so we can choose δ_1 such that $\| \int_S f d\mu - \int_E f d\mu\| < \epsilon/3$ for $\hat{\mu}(S^c) < \delta_1$ and, furthermore, by choosing n large we may let S^c be $\{t| \ \| f_n(t) - f(t)\| > \frac{e}{4\hat{\mu}(H)}\}$. Now choose $b_j \in \mathrm{er}_{E_i(m)}(f_m)$ such that

$$f'_m(t) = \Sigma\, b_j \chi_{E_i(m) \cap S}$$

and $\hat{\mu}(E_i(m) \cap S) \neq 0$ with $f'_m(t) = f_m(t)$ a.e. $[\mu]$. Then by a previous lemma choose $a_i \in \mathrm{er}_{E_i(m)}(f)$ such that $\|a_i - b_i\| < \frac{\epsilon}{3\mu(E)}$. Then $\| \int_S f'_m d\mu - \Sigma a_i \mu(E_i(m) \cap S)\| < \epsilon/3$ so $\| \int_E f d\mu - \Sigma\mu(E_i(m) \cap S) a_i\| < \epsilon$. Thus, $\overline{\bigcup_{s \subseteq E} \{\Sigma\mu(E_i \cap S) x_i : x_i \in \underset{E_i \cap S}{\mathrm{er}(f)}\}}$ and $\mu(E)^{-1} \int_E f d\mu \subseteq CO_{\mu,E}(\mathrm{er}(f))$.

Q.E.D.

Tucker [6] asked the question as to when the equivalence classes defining the Easton-Tucker integral [3] actually contain a "function." To answer this question we need another definition. Let $f : H \to X$ and $E \in A$ and suppose there exists $S \subseteq E$ such that $\ker \mu(S) \neq X$ (ker stands kernel i.e., the null space of the operator). Suppose also there exists $\delta_S > 0$ s.t. if $K_S = \mathrm{span}\{\Sigma x_i \mu(E_i \cap S)\}$, then for all $x_i \in \underset{E_i \cap S}{\mathrm{er}(f)}$ we have

$$\Sigma x_i \mu(E_i \cap S) \in K_S \backslash B(0, \delta_S).$$

Definition 2 If f and E are as above we say that if "f is not μ-equivalent to zero on E."

We shall denote the Easton-Tucker norm of an integrable function restricted to $E \subset H$ by $\|f\|_1(E)$.

Theorem 2 *Let μ be countably additive (c.a.) in the strong operator topology, f integrable, $\hat{\mu}(E) > 0$. Suppose that if $E_1 \subset E$ and $\hat{\mu}(E_1) > 0$, then f is not μ-equivalent to zero on E_1. Then $\|f\|_1(S) = 0$ implies that $\hat{\mu}(S) = 0$ if $S \subset E$.*

Proof. Let the f_n be a sequence of simple functions determining $\int f d\mu$. Choose $S \subset E$ s.t. $\hat{\mu}(S) > 0$ and $f_n \to f$ uniformly on S. (The existence of S is an easy consequence of Egoroff's theorem and the fact that $x\mu(\cdot)$ is countably additive). Since f is not μ-equivalent to zero on S, there is $S_1 \subset S$ s.t. $\ker\mu(S_1) \neq X$ and $\delta_{S_1} > 0$ with $\Sigma x_i\mu(E_1 \cap S_1) \in B(0, \delta_{S_1})^c, x_i \in \mathrm{er}_{E_i \cap S}(f)$. Now choose a positive integer N s.t. $\|f_n(t) - f(t)\| \leq \frac{\delta_{S_1}}{2\hat{\mu}(E)+1}$, when $n \geq N$. Let $f_n(t) = \Sigma x_i^{(n)}\chi_{F_{n,i}}(t)$ where $F_{n,i} = E_i^n \cap S_1$ (we can assume $\{E_i^{(n)}\}$ partitions S_1) and let $b_i \in \mathrm{er}_{F_{n,i}}(f_n)$, then $f_n(t) = \Sigma b_i\chi_{F_{n,i}}(t)$ a.e. on S_1 and choose $a_i \in \mathrm{er}_{F_{n,i}}(f)$ s.t. $\|b_i - a_i\| \leq \frac{\delta_{S_1}}{2\hat{\mu}(E)+1}$. We have $\|\int_{S_1} f_n d\mu - \Sigma a_i\mu(E_i \cap S_1)\| < \frac{\delta_{S_1}}{2}$, since $\|\int_{S_1} f_n - f d\mu\| < \frac{\delta_{S_1}}{2}$ we have $\|\int_{S_1} f d\mu - \Sigma a_i\mu(E_i \cap S_1)\| < \delta_{S_1}$ so $\|\int_{S_1} f d\mu\| > 0$ and this gives the result.

We require one more definition which will apply in case μ is only c.a. in the weak* topology. But first we note the following corollary.

Corollary 1 *If μ is a c.a. in the weak* topology and the other hypotheses of the theorem above hold and in addition Y^{**} contains no copy of ℓ_∞ then $\|f\|_1(S) = 0$ implies $\hat{\mu}(S) = 0$. ($S \subset E$).*

Let $f : H \to X, E \in A$. Suppose there exists $S \subset E$ such that $\hat{\mu}(S) > 0$ and a positive number δ such that for each $S_1 \subset S$ with $\hat{\mu}(S \backslash S_1) < \delta$ we have that there is $S_1'' \subset S_1$ s.t. $\Sigma x_i\mu(E_i \cap S_1'') \in B(0, \delta\hat{\mu}(E))^c$ for all $x_i \in \mathrm{er}_{E_i \cap S_1''}(f)$ where $\{E_i\}$ partitions S_1''.

Definition 3 If f, E and δ are as above we say that f is not δ equivalent to zero in E.

Theorem 3 *If f is (Bartle) integrable and $\hat{\mu}(E) > 0$ then if f is not δ-equivalent to zero on of F whenever $F \subset E$, ($\hat{\mu}(F) \neq 0$) then*

$$\|f\|_1(S) = 0 \text{ implies } \hat{\mu}(S) = 0. \qquad (S \subset E)$$

The proof is similar to the preceding, and we omit it.

We now give our (partial) answer to Tucker's question in the form of

Theorem 4 *Let f_n be a sequence of integrable functions and suppose that $\|f_n\|_1(E) \to 0$. If f_n is not $\delta_{n,k}$-equivalent to zero on $E_{n,k}$ where $E_{n,k} = \{t \,|\, \|f_n(t)\| > 1/k\}$ and $\inf \delta_{n,k} > 0$, then there is a sequence $\{f_{n_\ell}\}$ with $f_{n_\ell} \to 0$ a.e. $[\mu]$.*

Proof. The proof is an easy application of the previous results.

We leave it to the reader to formulate the corresponding results under the stronger hypotheses of countable additivity used previously. The previous results are of course related to the Radon-Nikodym theorem. It would be interesting to see the form of the Radon-Nikodym theorem for finitely additive operator measures and the Bartle integral.

References

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[4] Tucker, D. H. and R. J. Easton, A generalized Lebesque-type integral, *Math. Ann* **181** (1969), 311–324.

[5] Tucker, D. H. and R. J. Easton, Absolute continuity and the Randon-Nikodym therorem, *J. Reine Angew. Math.* **244** (1970), 1–9.

[6] Tucker, D. H. (Personal Communication).