Spectral Measures
In Abstract Spaces

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Abstract

We study the theory of spectral measures in topological vector spaces. We extend the Hilbert space theory to this setting and generalize the notion of spectral measures in some useful ways to provide a framework for operator theory in this setting. The Riesz representation theorem is proved without assuming local convexity. This theorem is applied to give sufficient conditions for an operator (continuous or otherwise) to be "spectral". A uniqueness problem is pointed out and the function calculus is extended to the case of several variables. A Radon-Nikodym theorem is proved. We then extend the theory of spectral measures to the case where values are assumed in the set of discontinuous (in normed spaces "un-bounded") operators. Examples of operators in nonlocally convex spaces are given which have densely defined measures.

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1 General Theory

1.1 Introduction

Research in functional analysis has been relatively meager in the area of general topological vector spaces without any local convexity hypothesis. This paper is an attempt to extend the spectral theory of linear operators to such a setting and, in particular, to generalize the notion of "spectral measure". This will be carried out in various ways while at the same time preserving as much as possible of the Hilbert space–Banach space results. "Regularity" of the measures involved turns out to be a basic assumption if we are to have sufficient structure to work with. We note here that two types of integrals are required in order to state the theory consistently, a Riemann theory [14] and a Lebesgue-type theory [40]. We shall see below that the properties of a spectral measure make it possible to work in certain Banach (normed) subspaces of the topological vector spaces we consider below. This fact can be used to shorten a number of arguments considerably.

Some previous work may be compared with the results below:

1. Colojoară and Folas [2]. Their approach is to assume an operational calculus with some family of functions, defined (at least) on the spectrum of the operator, somewhere in scope between the analytic functions and continuous functions.

   Maeda's work [24] is in the same direction but in certain locally convex spaces. Our work includes much of this theory but does not subsume it without some juggling of topologies.

2. Schaefer [35]. Schaefer's work treats spectral theory from an order-theoretic point of view reminiscent of the Riesz–Sz.-Nagy approach to the spectral theorem. His work is in the locally convex case for weakly complete spaces. Our work includes the results of this work in the comparable situations.

3. Other contributions have been made by Naǐmark [27], Marcenko [25], Ljance [19] and Folland [12]. The work of these authors is very closely related to the comments in Section 6.5 below.

Section 1 is divided into two parts by the consideration of continuous and discontinuous operators and, at the same time, spectral measures that are countably additive and those that may fail to be countably additive. In every case, the values of our measures are continuous projections. This hypothesis will be relaxed in sections 6 and 7 below.

1.2 Preliminaries

In this paper we shall usually denote by "A" a sequentially complete Hausdorff topological algebra, i.e., an algebra whose underlying vector space is a topological vector space with the property that the multiplication of A is separately continuous in that topology. All topological spaces are assumed to be Hausdorff.

A *spectral measure in* A is a set of function μ defined on a ring R of subsets of some set H, which takes values in A and has the following three properties:

(1) $\mu(C \cap B) = \mu(C) + \mu(B) \quad (C \cap B = \emptyset)$

(2) $\mu(C \cap B) = \mu(C)\mu(B)$ (the product in A will be denoted by juxtaposition)

(3) μ is of *bounded semivariation* when considered as having its range in $B(\mathbb{C}, A)$ via scalar multiplication in A. That is ([40]) for $F \in R$, ($\uplus$ means disjoint union),

$$\hat{\mu}(F) = \left\{ \sum_{i=1}^{n} \alpha_i \mu(E_i \cap F) \,\middle|\, \uplus E_i = F,\ \alpha_i \in \mathbb{C} \text{ and } |\alpha_i| \leq 1 \right\}$$

is a topologically bounded subset of A.

From time to time we may require other properties. They will be stated explicitly when needed.

We need the following lemmata:

Lemma 1.1 *Suppose $E_1, \ldots, E_l$ are sets in R and $R(E_i, \ldots, E_l)$ is the algebra in R generated by $E_1, \ldots, E_l$. We can conclude that $R(E_1, \ldots, E_l)$ is a finite subset of R and there is a finite collection of mutually disjoint sets $\{K_m\} \subseteq R$ such that each element of $R(E_1, \ldots, E_l)$ is uniquely representable as a finite union of sets in $\{K_m\}$.*

Proof. The proof follows by the taking the $\{K_i\}$ to be all possible sets of the form $\bigcap_{i=1}^{l} M_i$ where $M_i = E_i$ or $M_i = \Omega \backslash E_i$, where $\Omega = \bigcup_{i=1}^{l} M_i$.

Lemma 1.2 *Suppose A is a topological vector space and z is a closed, bounded, convex, balanced subset of A. Denote by A_z the subspace of A generate by* *z. If A is sequentially complete, then A_z with the Minkowski functional z as norm is a Banach space.*

Proof. The proof may be found in [10].

Corollary 1.1 *Suppose z is a closed, absolutely convex, bounded subset of A, which is a subsemigroup of A. If A is sequentially complete, then A_z (See Lemma 1.2 for notation), with the Minkowski functional of z as its norm, is a Banach algebra.*

Proof. If $a, b \in z$, then $ab \in z$ so $\|a\| \leq 1,\ \|b\| \leq 1$ implies $\|ab\| \leq 1$. The result follows immediately.

If X is topological vector space, $B(X)$ denotes the set of continuous linear mappings of X into itself. $L(X)$ denotes the set of linear mappings of X into itself, while $L(A, X)$ is the set of linear mappings of space A into X and similarly for $B(A, X)$.

Remark 1.1 If A_z is as in Corollary 1.1, we shall usually assume it has its induced supspace topology (from A) unless otherwise noted.

1.3 Integration Theory

We will need a theory of integration for functions with values in a general topological vector space with respect to a scalar measure and conversely a theory to handle scalar functions and vector-valued measures. To carry out our program we will use two integrals, the integration theory of [14] (a Riemann integral) for the vector function scalar measure case and the theory developed in [40] for the scalar function vector measure case which allows a unified treatment of the finitely additive cases which we need here (see Section 6 below). We give an outline of the Riemann theory here in our setting, since [14] deals only with the Banach space case.

Definition 1.1 Let λ be a complex Borel measure on $\mathbf{R}^n$ (or $\mathbf{C}^n$). If J is a measurable set in $\mathbf{R}^n$, with finite measure a partition of J is a finite sequence of closed measurable sets $J_1, \ldots, J_n$ such that $J = \bigcup_{i=1}^{n} J_i$ and $\lambda\left(J_i \bigcap J_j\right) = 0$, if $i \neq j$. The mesh of a partition of J is the maximum of the metric diameters of sets J_i. We shall use the symbol $N\{J_i\}$ to denote the mesh of the partition

$\{J_i\}$. If $P_n = \{J_i^n\}$ defines a sequence of partitions of J with $NP_n \to 0$ as $n \to \infty$ and f is a function defined on J with values in a topological vector space M and

$$\lim \Sigma f(t_i^n)\, \lambda(J_i^n)$$

(t_i^n is any point in J_i^n) exists for any such sequence $\{P_n\}$, then we say that f is integrable on J.

Proposition 1.1 *Let f be integrable on J as in Definition 1.1. The value of*

$$\lim_{n\to\infty} \Sigma f(t_i^n)\, \lambda\, (J_i^n)$$

is independent of which sequence of partitions occurs in the limit as well as the choice of the points t_i^n.

The proof consists in using the standard technique of "meshing" sequences. See [29] for example.

Definition 1.2 We define $\int_J f\, d\lambda$ to be

$$\lim_{n\to\infty} \Sigma f(t_i^n)\, \lambda\, (J_i^n)$$

as in Proposition 1.1 for any f integrable on J.

Proposition 1.2 *Let f and g be integrable on J. Then $f+g$ is integrable on J with*

$$\int_J f + g\, d\lambda = \int_J f d\lambda + \int_J g d\lambda.$$

If k is a constant, then kf is integrable on J and

$$\int_J k\, f d\lambda = k \int_J f\, d\lambda.$$

If f is integrable on J' and $\lambda(J' \cap J) = \Phi$, then

$$\int_J f\, d\lambda + \int_{J'} f\, d\lambda = \int_{J \cup J'} f\, d\lambda.$$

The proofs are elementary.

We must also consider integration of vector-valued functions defined in the complex plane over some piecewise smooth curve.

Definition 1.3 We say that $f : \mathbb{C} \to M$ is *integrable over* γ, a continuous piecewise C^1 curve in $\mathbf{C}$ if

$$\int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

exists (where $[\alpha, \beta]$ is the parameter domain of γ). The usual relationships hold for reparameterizations.

Proposition 1.3 *Let f be integrable over γ. If γ_1 and γ_2 are parameterizations of γ, then*

$$\int_{\alpha_1}^{\beta_1} f(\gamma_1)\gamma_1'\,dt = \int_{\alpha_2}^{\beta_2} f(\gamma_2)\gamma_2'\,dt,$$

where γ_1 and γ_2 are related as γ_1 composed with $h = \gamma_2$ with

$$h(t) = \frac{\alpha_1(\beta_2 - t)}{\beta_2 - \alpha_2} + \frac{\beta_1(t - \alpha_2)}{\beta_2 - \alpha_2}.$$

Existence of one integral is assumed. Existence of the other is part of the conclusion.

Proof. The proof consists in checking the Riemann sums with appropriate use of h as in elementary complex analysis.

The usual relations for path integrals hold as a consequence of propostions 1.2 and 1.3. For example, let f be integrable over two paths γ_1 and γ_2 which meet at their "end points". If γ denotes the union of these paths in the usual sense, then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

Remark 1.2 We use our Riemann integral to integrate functions with vector values. It is possible to define other integrals for vector functions and scalar measures as in [11], for example. Etter uses the concept of "ultra convergence" which is equivalent to uniform convergence in locally convex spaces, see [49]. We shall not describe this integral here, but we shall have occasion to return to [11] in a later section of this paper.

When A is metrizable and "locally pseudoconvex", it has been shown ([29]) that the integral exists for certain smooth functions, but that simple continuity of the integrand does not imply the existence of the integral. It is possible to prove a number of existence theorems for the Riemann integral in our case. We shall not go into this now, however.

1.4 Products of measures

Definition 1.4 Suppose μ_1 and μ_2 are two spectral measure on rings R_1 and R_2, respectively, with values in A. If $\mu_1(c_1)\mu_2(c_2) = \mu_2(c_2)\mu_1(c_1)$ for all $c_1 \in R_1$, $c_2 \in R_2$, then we define the product measure $\mu_3 = \mu_1 \times \mu_2$ on $R(R_1 \times R_2)$ in the obvious way where $R(R_1 \times R_2)$ denotes the ring of "elementary" sets (see Rudin [34]) generated by R_1 and R_2.

Using the notation of Definition 1.4, μ_3 clearly satisfies (1) and (2) of Lemma 1.1, but there is no assurance that it is of bounded semivariation. We shall return to this problem below.

Definition 1.5 Suppose $\mu : R \to A$ is a spectral measure and that $F \in R$. Denote by $R_F(\mu)$ the range of μ on $\{F \cap K \mid K \in R\}$ and by $\text{cob}(R_F(\mu))$ the absolutely convex hull of $R_F(\mu)$. $\overline{\text{cob}}(R_F(\mu))$ is the closure of $\text{cob}(R_F(\mu))$.

Lemma 1.3 *$\overline{\text{cob}}(R_F(\mu))$ is bounded commutative subsemigroup of A.*

Proof. $\overline{\text{cob}}(R_F(\mu))$ is clearly a semigroup. Consider
(*) $$\Sigma\alpha_i\mu(E_i \cap F) \in \text{cob}(R_F(\mu))(\Sigma|\alpha_i| \leq 1).$$

By Lemma 1.1, we may rewrite (*) as

$$\sum_m (\sum_{k \leq n} \alpha_k)\mu(L_m)$$

with $L_m \cap L_n = \emptyset$ when $n \neq m$. Thus, $\text{cob}(R_F(\mu)) \subseteq \hat{\mu}(F)$ and hence the lemma follows.

Corollary 1.2 *$A_{\overline{\text{cob}}(R_F(\mu))}$ is a commutative Banach algebra, with the Minkowski functional of $\overline{\text{cob}}(R_F(\mu))$ as a norm (here, $\overline{\text{cob}}(R_F(\mu))$ is the "z" of Lemma 1.2).*

Remark 1.3 In general, the problem of determining when the "product" of the two vector measures extends (in some continuous, or countably additive (c.a.) fashion) to a vector measure on a larger collection than the "elementary" sets is difficult [30]. We note the following result. (Assume that A is complete.)

Theorem 1.1 *Let R_1 and R_2 be σ-algebras in sets H_1 and H_2 with μ_1 and μ_2 spectral measures on R_1 and R_2. Let R be the ring of elementary sets generated by R_1 and R_2, and let $S(R)$ denote the σ-algebra generated by R. Suppose μ_1 and μ_2 are c.a. The following hold:*

(1) If $B \in S(R)$, $h_2 \in H_2$, the h_2 section of B, B^{h^2}, is R_1 measurable.

(2) The integral

$$\int_{H_2} \mu_1(B^{h^2}) d\mu_2(h_2)$$

is well defined [26]. Suppose that for each neighborhood U of zero in A and finite partition P of H_2 in R_2 there exists a finite set $\Delta(P) \subseteq P$ such that if F is a refinement of $P \subseteq \Delta(P)$, then

$$\sum_{\alpha \in F} \{\textit{range of } \mu\} \mu_2(\alpha) \subseteq U$$

and further that μ_2 has the property that for each neighborhood of U of zero in A, there exists a neighborhood V of zero in A, if K is a finite partition of H_2 in R_2, then

$$\sum_{\alpha \in F} (V) \mu_2(\alpha) \subseteq U.$$

Then $\mu_1 \times \mu_2$ extends to a unique c.a. measure on $S(R)$ defined by the integral 2 above.

Proof. The proof is essentially like the standard scalar case [26].

Definition 1.6 If H_1 and H_2 are sets, and R_1 and R_2 are rings of subsets of H_1 and H_2, then a mapping $f : H_1 \to H_2$ is called *measurable* (relative to the pair (R_1, R_2)) if, for any $F \in R_2$, $f^{-1}(F) \in R_1$. If H_2 is a linear space and $N(f) = \{t \mid f(t) \neq 0\}$, then we only require $f^{-1}(F) \cap N(f) \in R_1$.

Lemma 1.4 *Suppose μ is a spectral measure on a ring R of subsets of a set H and H is the product $\bigtimes_{i=1}^{n} H_i$ with R_i a ring in H_i.*

Let the projection map f_i of H onto H_i be measurable for each i. Then there exist spectral measures μ_i on R_i such that for each measurable rectangle, $\bigtimes_{i=1}^{n} F_i$,

$$\mu(\bigtimes_{i=1}^{n} F_i) = \bigtimes_{i=1}^{n} \mu_i(F_i).$$

If R and R_i are δ-rings (or σ-rings, etc.), then the same result holds. Furthermore, if

$$F \in \bigcap_{i=1}^{n} \{ {f_i}^{-1}(K_i) | K_i \in R_i \}),$$

then the decomposition $\bigtimes_{i=1}^{n} \mu_i$ *of μ is unique on*

$$F \cap R = \{ F \cap K | K \in R \} ,$$

relative to $A_{\overline{cob}(R_F(\mu))}$. ($\mu(F)$ is the "identity" in $A_{\overline{cob}(R_F(\mu))}$).

Proof. Define $\mu_i(F_i) = \mu({f_i}^{-1}(E_i))$. Suppose $n = 2$, then

$$\mu(F_1 \times F_2) = \mu({f_i}^{-1}(E_1) \cap {f_2}^{-1}(E_2)) = \mu_1(F_1)\mu_2(F_2).$$

The proof proceeds by induction. Uniqueness is proved as follows: suppose $\mu_1', \ \mu_2'$ are such that $\mu(F_1 \times F_2) = \mu_1'(F_1)\mu_2'(F_2)$. Then, since

$$\mu(f_1(E) \times f_2(F)) = \mu_1'(E_1)\mu_2'(F_1) = \mu_1(E_1),$$

and $\mu_2'(F_1)$ is the identity by hypothesis, the conclusion follows.

Remark 1.4 Each spectral measure on the Borel subsets of $\mathbb{C}$ is thus the product of a pair of spectral measures on the real line.

Proposition 1.4 *Let μ_1 and μ_2 be two spectral measures on ring R_1 and R_2 with $\mu_1 \times \mu_2$ defined on $R(R_1 \times R_2)$ as above, then (1) through (4) below are equivalent. (*) implies (1) as well.*

(1) $\mu_1 \times \mu_2$ is a spectral measure.

(2) $\mu_1 \times \mu_2$ is of bounded semivariation.

(3) $\overline{\mathrm{cob}}(R_F(\mu_1 \times \mu_2))$ is bounded in A for each $F \in R$.

(*) A is locally convex and $R_F(\mu_1 \times \mu_2)$ is bounded.

(4) μ_1 and μ_2 are bounded with respect to each other's range–i.e.,

$$\left\{\sum_{i=1}^{l}\alpha_i\sum_{j=1}^{k}\beta_j\mu_1(E_i\cap F^1)\mu_2(F_j\cap F^2)\mid F^1\in R_1,\quad F^2\in R_2,\ |\alpha_i|\le 1,\ |\beta_j|\le 1\right\}$$

is bounded. (Here, $\{E_i\}$ and $\{F_j\}$ are mutually disjoint finite collections from R_1 and R_2, respectively.)

Proof. $1\to 2\to 3\to 4$ are obvious and $4\to 1$ simply by the definition of R.

Remark 1.5 In case R_1 and R_2 are σ-algebras and μ_1 and μ_2 are c.a., one wishes to extend $\mu_1\times\mu_2$ (if it is spectral) to the the σ-algebra generated by R.

If μ has a c.a. extension, then it will have bounded semivariation and, hence, is a spectral measure. (It is easy to check that μ is multiplicative.) Generally countable additivity is not a desirable property for a spectral measure. However a weak form of countable additivity is necessary for further development. We shall return to this in section 1.6.

Suppose μ is a spectral measure with values in A, defined on a subring R of the Borel sets in the complex plane, with the property that for each $F\in R$ and Borel set C with $C\subseteq F$ we have $C\in R$. Let S be a compact subset of $\mathbb{C}$, $f(t_1,t_2)$ as continuous function from $\mathbb{C}\times\mathbb{C}$ to $\mathbb{C}$. Suppose F is a bounded set in R. Then $f(t_1,t_2)$ is the uniform limit (on $S\times F$) of simple functions which are of the form $\sum_{i=1}^{n}\alpha_i\chi_{E_i}$ where E_i is an elementary set. We shall refer to a sequence of simple functions of this type as an "elementary sequence".

Proposition 1.5 *Using the notation indicated above and letting $F'\subseteq F$, $S'\subseteq S$, the iterated integrals,*

$$\int_{S'}\int_{F'}f(t_1,t_2)d\mu(t_1)d\lambda(t_2)$$

and

$$\int_{F'}\int_{S'}f(t_1,t_2)d\lambda(t_2)d\mu(t_1)$$

make sense and are equal. (λ is Lebesque measure.)

Proof.

$$\int_{S'}f(t_1,t_2)d\lambda(t_2)$$

is bounded μ-measurable function and so its μ integral exists. Let $\{f_n\}$ be an elementary sequence with $f_n \to f$ uniformly on $S \times F$ and suppose

$$f_n(t_1, t_2) = \sum_{j=1}^{K_n} \alpha_j^{(n)} \chi_{E_j}(t_1, t_2), \quad E_j = E_{jn}^1 \times E_{jn}^2.$$

Then,

$$\int_{S'} \int_{F'} f_n(t_1, t_2)\, d\mu\,(t_1)\, d\lambda\,(t_2) = \int_{F'} \int_{S'} f_n(t_1, t_2) d\lambda(t_2)\, d\mu\,(t_1)$$

Meanwhile,

$$\lim_n \int_{F'} \int_{S'} f_n(t_1, t_2)\ d\lambda\,(t_2) d\mu\,(t_1)$$

exists and equals

$$\int_{F'} \int_{S'} f(t_1, t_2)\ d\lambda\,(t_2) d\mu\,(t_1)$$

Moreover,

$$\lim_n \int f_n(t_1, t_2) d\mu(t_1)$$

converges uniformly in t_2, $(NP_n \to 0)$ for given $\epsilon > 0$,

$$\int f_n(t_1, t_2) d\mu(t_1) - \int f_n(t_1, t_2') d\mu(t_1) =$$

$$\int f_n(t_1, t_2) - f_n(t_1, t_2') d\mu(t_1) \in \epsilon \hat{\mu}(F')$$

for sufficiently large n. By corollary 1.2 and the Moore–Smith convergence theorem we may interchange limits to get

$$\lim_n \int_{S'} \int_{F'} f_n(t_1, t_2) d\mu(t_1) d\lambda(t_2) = \int_{S'} \lim_n \int_{F'} f_n(t_1, t_2) d\mu(t_1) d\lambda(t_2)$$

$$= \int_{S'} \int_{F'} f(t_1, t_2) d\mu(t_1) d\lambda(t_2). \qquad (**)$$

But (**) equals

$$\int_{F'} \int_{S'} f(t_1, t_2) d\lambda(t_2) d\mu(t_1).$$

This completes the proof.

1.5 Calculation in a topological algebra

Proposition 1.6 *Let μ be a spectral measure on R with values in A, and suppose $F \in R$. Denote by $Bd(F)$ the collection of all bounded R-measurable complex-valued functions on F.*

The integral is a continuous homomorphism of $Bd(F)$ into A and into $A_{\overline{\mathrm{cob}}(R_f(\mu))}$ with its norm topology, where $Bd(F)$ has the topology of uniform convergence.

Proof. It is clear that for simple functions f and g,

$$\int_F fg\, d\mu = \int_F f d\, \mu \int_F g \; d\mu = \int_F g\, d\mu \int_F f\, d\mu.$$

Since μ has bounded semivariation in A, the rest of the proposition follows by taking uniform limits of simple functions. This completes the proof.

Proposition 1.7 *Let μ be a spectral measure on R with values in A, and let f be complex-valued and R-measurable. If f is bounded, then, for all $F \in R$ and for all bounded Borel functions g,*

(*) $$\textstyle\int_F gf\, d\mu = \int_{f(F)} g(z) dm_f(z)$$

where $g \quad f(z) = g(f(z))$ and

$$m_f(E) = \mu(f^{-1}(E) \cap F)$$

with E a Borel set in $\mathbb{C}$.

Proof. First note that (*) makes sense because m_f has bounded semivariation. Let $g = \chi_K$, K a Borel subset of $\mathbb{C}$. Then

$$\int_F \chi_K \circ f d\mu = \mu(f^{-1}(K) \cap F) m_f(f(F) \cap K) = \int_{f(F)} \chi_K(z) dm_f(z)$$

and, therefore, the equality holds for all simple functions g and so for all bounded Borel functions g. This completes the proof.

Proposition 1.8 *Any spectral measure in A takes its values in a commutative subalgebra of A, which allows a stronger locally convex topology than the topology induced by A.*

Proof. The inductive limit

$$\lim A_{\overline{\text{cob}}(R_F(\mu))},$$

where $F \in R$ and R is partially ordered by inclusion, works. That is, we take the Banach space topology on each member of this increasing family of subalgebras. It remains to check the continuity of the inclusion maps. This is trivial from the definition. (Norms are increasing.) This completes the proof.

Remark 1.6 At this point it is necessary to define what we shall mean by the "spectrum" of an element in A. The usual definition of the spectrum of an element $a \in A$ is the set-theoretic complement of the set of all complex numbers λ, such that $(\lambda e - a)^{-1}$ exists as an element of A (we assume A has an identity e). This definition requires further assumptions we wish to postpone. See [35] for the locally convex case. The idea that seems most sanguine to the situation at hand is that the resolvent is locally holomorphic on the resolvent set. However, the theory of holomorphic A-valued functions has its own special problems. For example in A, power series are not well-behaved in general (see [51]). They may converge at any finite number of points and diverge elsewhere–in contrast to the case where A is locally convex or pseudoconvex.

At this point, we do not wish to limit ourselves to the various possible special cases.

This leads us to an indirect definition of holomorphicity, which will reduce to the usual one in less general cases.

Definition 1.7 Suppose $a \in A$ and assume that A has an identity e. The "Morera" resolvent set of a is the collection of all complex numbers λ for which there is a disc N_λ about λ such that if $z \in N_\lambda$, then $(ze - a)^{-1}$ exists in A and for each piecewise smooth simple closed curve c in N_λ the integral

$$\int_C (\zeta - a)^{-1} d\zeta$$

exists (in the sense of §1.2 above) and is equal to the additive identity in A. (We mean $(ze - a)^{-1}$ when writing $(z - a)^{-1}$.)

The Morera resolvent set, or briefly, the resolvent set of a, which we denote as $\rho(a)$, is open and in the classical case Remark 4.2 implies that $(\zeta - a)^{-1}$ is locally analytic on $\rho(a)$.

As usual we use $\sigma(a)$ to mean the "spectrum of a" and define it to be $\rho(a)^c = (\mathbb{C}\backslash\rho(a))$.

Remark 1.7 $\lambda \to (\lambda - a)^{-1}$ may of course be continuous and yet it may fail to be integrable on compact sets, contrary to the condition in locally convex spaces (see [32] and also [11] and [49].) Since we do not assume our measures are defined on an "algebra" we must define some notion of invertiblity, useful in this context. We therefore define "invertibility" not with respect to the whole of A, but with respect to the measure involved. To be precise we have

Definition 1.8 Suppose $a \in A$ and

$$a = \int_F \lambda d\mu(\lambda)$$

where μ is a spectral measure on R, a subring of the Borel sets of $\mathbb{C}$, $F \in R$. Then $\lambda - a$ is taken to mean $\lambda\mu(F) - a$ and inverses are computed relative to $A_{\overline{\mathrm{cob}}R_F(\mu)}$, of which a is certainly a member and in which $\mu(F)$ acts as the identity. (Whether we have the norm topology or not makes no difference here. This is a useful ambiguity.)

The notion "regular measure" appears in the proposition below. As applied to a measure μ with values in A, we do not need the *multiplicative structure* of A. This allows us to consider A as a metric space (actually as a subspace of a product of metric spaces–this is sufficient). The notion of regularity is therefore just the same as in the *scalar* case. The support of a *regular measure* (on a Borel field) is the complement of the largest open set on which the measure vanishes. (We occasionally use the term "support" in a generalized sense elsewhere; here it is a single-valued notion.) The existence of the support is then proved by the same method as in the scalar case, i.e., let V be the union of all open sets V_i on which μ vanishes. Then by regularity $\mu(V)$ must be "zero" or there is a compact set $K \subseteq V$ with $\mu(K) \neq 0$. K is covered by finitely many of the sets making up V and this leads to a contradiction since

$$\mu(K) = \mu\left(\bigcup_{i=1}^{n} K \cap V_i\right) = \text{ sum of products of terms}$$

$$\text{of the form } \mu(K \cap V_i) = \mu(K)\mu(V_i) = 0.$$

Proposition 1.9 *Let μ be a spectral measure of the type discussed in the remark preceding Proposition 1.5 or immediately above, restricted to a Borel field. In addition, suppose μ is regular in the topology of A. Identify the support of μ as "$\text{supp}(\mu)$" and assume this set is compact. If*

$$a = \int \lambda d\mu(\lambda),$$

then $\sigma(a)$ (computed in $A_{\overline{\text{cob}R_S(\mu)}}$, where $S = \text{supp}(\mu)$) is equal to the set $\text{supp}(\mu)$.

Proof. We may extend μ to the Borel sets of $\mathbb{C}$ in the obvious way, and we suppose this is done. $\text{supp}(\mu)$ is certainly a measurable set; it is a compact set. Let $z \in [\text{supp}(\mu)]^c$. It will then be the case that

$$\int \frac{d\mu(\zeta)}{\zeta - z}$$

exists on appropriate sets, indeed,

$$\int_{\text{supp}(\mu)} \frac{d\mu(\zeta)}{\zeta - z} = \int_F \frac{d\mu(\zeta)}{\zeta - z} \qquad (F \supseteq \text{supp}(\mu)).$$

(The equality holds because it holds for the appropriate simple functions.) Moreover, $\mu(F)$ (or $\mu(\text{supp}(\mu))$ is the natural identity for the subalgebra generated by the range of μ and we take this to be the identity, as noted above. Thus,

$$(ze - a)^{-1} = \int_F \frac{d\mu(\zeta)}{z - \zeta}$$

by Proposition 1.6. Therefore, $(ze - a)^{-1}$ exists, and if we take a disc D small enough about z so that ∂D(=the boundary of D) does not intersect $\text{supp}(\mu)$ (this is possible because $\text{supp}(\mu)$ is closed), then the integral

$$\int_{\partial D} (\zeta e - \alpha)^{-1} d\zeta = \int_{\partial D} \int_{\text{supp}(\mu)} \frac{d\mu(\zeta)}{\zeta - \lambda} d\zeta$$

exists by Proposition 1.5 and is equal to the additive identity in A, again by Proposition 1.5 and the Cauchy theorem. This shows that $\sigma(a) \subseteq \text{supp}(\mu)$. Now suppose $z \notin \sigma(a)$. Choose an *open* disc D_1, whose closure does not intersect $\sigma(a)$ (again this is possible since $\sigma(a)$ is closed) so that D_1 is centered at z and then choose D_2 with $\overline{D}_2 \subset D_1$ with D_2 again being centered at z. We wish to show that $\mu(D_2) = 0$. This will show that $z \notin \text{supp}(\mu)$, and, therefore, that $\text{supp}(\mu) \subseteq \sigma(a)$. Let μ_1 be the restriction of μ to D_1 and μ_2 be the restriction of μ to D_1. Then $\mu = \mu_1 + \mu_2$. Let $e_i = \mu_i(\mathbf{C}),\ a_i = \int z d\mu_i$. It follows

$$(\lambda e - a)^{-1} - (\lambda e_1 - a_1)^{-1} = \int \frac{d\mu_2(z)}{\lambda - z}(\lambda \notin \text{supp}(\mu))$$

and

$$(\lambda e - a)^{-1} = (\lambda e_1 - a_1)^{-1} + (\lambda e_2 - \alpha_2)^{-1}$$

if $\lambda \in \rho(a) \bigcap \rho(a_1)$. We know that

$$\int_C (\lambda e - a)^{-1} d\lambda = 0$$

if $C \subseteq \overline{D}_1$. Now note that

$$\int_{C'} (\lambda e_1 - a_1)^{-1} d\lambda = 0$$

for C' any piecewise continuous closed curve in an annulus containing ∂D_1 assuming the index of C' at z is zero, for $\sigma(a_1)$ is inside $\text{supp}(\mu_1)$, which is (properly) contained in D_1 (an *open* disc). This follows from the usual complex variable argument extended directly to our case. Since $\text{supp}(\mu_2) \subseteq D_1^c$, we have

$$\int_{C_2} (\lambda e_2 - a_2)^{-1} d\lambda = 0$$

for any $C_2 \subseteq D_1$. It follows from the properties of the integral that

$$\int_{\partial D_1} (\lambda e_1 - a_1)^{-1} d\lambda = 0$$

and, thus by Proposition 1.5

$$0 = \int_{\partial D_1} \int_{D_2} \frac{d\mu_1(z)}{\lambda - z} d\lambda \int_{D_2} \int_{\partial D_1} \frac{d\lambda}{\lambda - z} d\mu_1(z) = 2\pi i \mu_1(D_2)$$

so $\mu_1(D_2) = 0$ and the multiplicative property of μ shows that $D_2 \subseteq [\text{supp}(\mu)]^c$ and, thus, $z \in [\text{supp}(\mu)]^c$ so $\text{supp}(\mu) = \sigma(a)$. This completes the proof.

Definition 1.9 If $a \in A$ is the integral $\int f d\mu$ where f and μ are as in Proposition 1.7, then we define $g(a)$ for g as in Proposition 1.7 to be

$$\int gf\, d\mu = \int g(z) dm_f(z).$$

Such elements "a" of A will be called "*scalar*" elements. At this point, it may be that $g(a)$ depends on m_f.

Proposition 1.10 *We use the notation of Proposition 1.7. If a is a scalar element of A, and g is continuous, then*

$$\operatorname{supp} m_f = \bigcap_{\mu(M)\mu(F)} \overline{f(M)} = \sigma(a)$$

and

$$g(\sigma(a)) = \sigma(g(a)).$$

(The hypothesis of regularity is in force.)

Proof. Suppose $\mu(M) = \mu(F)$, then $m_f((\overline{f(M)})^c) = 0$ and so $\operatorname{supp}(m_f) \subseteq \overline{f(M)}$. However,

$$m_f(\operatorname{supp}(m_f)) = \mu(F) = \mu(f^{-1}(\operatorname{supp}(m_f))).$$

This proves that the first equality above and $\sigma(a) = \operatorname{supp}(m_f)$ follows from Definition 1.9. To see that $g(\sigma(a)) = \sigma(g(a))$, notice that for $m_{fg}(K) = m_f(g^{-1}(K))$,

$$\operatorname{supp}(m_{fg}) = \bigcap_{m_f(K)=m_f(\operatorname{supp}(m_f))} \overline{g(K)} = \sigma(g(a)).$$

By assumption, $g(\operatorname{supp}(m_f))$ is compact since $\operatorname{supp}(m_f)$ is compact. To complete the proof it remains to show that

$$\operatorname{supp}(m_{fg}) \supseteq g(\operatorname{supp}(m_f)).$$

To show this, suppose that $\lambda \notin \operatorname{supp}(m_{fg})$ and $\lambda \in g(\operatorname{supp})m_f))$. Let V be a disc about λ not intersecting $\operatorname{supp}(m_{fg})$.

$$\int_{g^{-1}(V)} g(z) dm = \int_V z\, dm_{fg}(z) = 0$$

by Proposition 1.7 (take "g" in Proposition 1.7 to be $g(s) = s$). Now, since g is continuous, $g^{-1}(V)$ is an open set, $g^{-1}(\lambda) \in \operatorname{supp}(m_f)$ and $g^{-1}(\lambda) \in g^{-1}(V)$. Thus λ must be zero (g must vanish at $g^{-1}(\lambda)$ by definition of "support") or we have a contradiction. Suppose then that $\lambda = 0$. $g^{-1}(0)$ is either an isolated point in $\operatorname{supp}(m_f)$ or not. If it is not isolated, $g^{-1}(V)$ must contain other points in the support of m_f which would give a contradiction by repeating the argument above. Thus $g^{-1}(0)$ is an isolated point in $\operatorname{supp}(m_f)$, then $m_f(\{g^{-1}(0)\}) \neq 0$. But then

$$m_f(\{g^{-1}(0)\}) = m_{fg}(\{0\})$$

and $0 \in \operatorname{supp}(m_{fg})$ contradicting our original assumption. Therefore

$$\operatorname{supp}(m_{fg}) = g(\operatorname{supp}(m_f)) = g(\sigma(a)).$$

This completes the proof.

We close this section with two results on the generation of spectral measures. They serve as primitive versions of our work in section 1.6.

Proposition 1.11 *Let $C_0(X)$ be the space of continuous functions vanishing at "∞" on a locally compact Hausdorff space X. Suppose Λ is a continuous A-valued homomorphism on $C_0(X)$. Then there exists a measure μ on the Borel subsets of X with values in the space of linear functions on a subspace of the algebraic dual A' of A, so that*

$$\Lambda'(f) = \int_X f d\mu \qquad (f \in C_0(X))$$

where Λ' is Λ "lifted" to $F(A')$ (the space of linear self-mapping functions with no topology–hence the term "algebraic") in the way specified in the proof below.

Proof. It is easy to see that as a *Banach space* $C_0(X) \oplus \mathbb{C}$ may be identified with the space of continuous functions (with sup norm) on the one point compactification of $X, \hat{X}$, denoted by $C(\hat{X})$. If we define multiplication on $C_0(X) \oplus \mathbb{C}$ coordinate-wise, then $C_0(X) \oplus \mathbb{C}$ has a *Banach algebra* structure (different than that of $C(\hat{X})$). Denote by "P" the projection to the first coordinate $P : (f, c) \to (f, 0) \simeq f$. Define $\Lambda P(f, c) \equiv \Lambda(f)$. If $C(\hat{X})_1$ is the unit disc of the Banach space $C(\hat{X})(= \{f \mid \|f\| \leq 1\}$, then $\Lambda P(C(\hat{X}_1))$ is a bounded, convex, balanced subset of A. If $C(\hat{X})$ is given the algebraic structure of $C_0(X) \oplus \mathbb{C}$ (for multiplication), then $\Lambda P(C(\hat{X})_1)$ is also a semigroup.

Applying Corollary 1.1, we have that ΛP (call it $\hat{\Lambda}$) is a bounded (hence continuous) map of $C(\hat{X})$ into the Banach algebra A_c, where c is the A-closure of $\Lambda P(C(\hat{X})_1)$ and A_c is given its *norm* topology. The proof of Theorem XVIII. 2.4 of [9] may now be carried over wholesale, paying attention to the details concerning the multiplicative nature of μ. The only alteration being that the measure is only defined *locally* (not on $\hat{X}$, but on the Borel subsets of X). It is now a matter of considering $\hat{\Lambda}^*$ from the above cited theorem (*) has the meaning accorded it in the cited theorem),

$$\hat{\Lambda}^* : C(\hat{X}) \to L(A_c^*).$$

A_c^* being the Banach space dual of A_c, $L(A_c^*)$ denotes the continuous linear self-mappings of A_c^* (norm topology of A_c). Now A_c is a linear manifold in A and $f \in A_c^*$ may be extended "by zero" to A making $A_c^* \subset A'$ with this extension (note: A^* may be zero-dimensional). Identifying $\hat{\Lambda}^*$ with Λ' in the obvious way gives the proposition. This completes the proof.

For Proposition 1.12 we need some notation. H will be a set, R a ring of subsets of H, $B(R)$ the collection of all complex-valued R-measurable functions bounded on elements of R. Give $B(R)$ the topology of uniform convergence on members of R.

Proposition 1.12 *Suppose Λ is a continuous homomorphism of $B(R)$ into A. Then there exists a spectral measure μ on R with values in A such that $\Lambda(\chi_F \cdot f) = \int_F d\mu$ for each $F \in R$ and $f \in B(R)$.*

Proof. Fix $F \in R$. The set

$$\left\{ \sum_{i=1}^{n} \alpha_i \chi_{F \cap E_i} | E_i \in R, \cup(E_i \cap F) = F, |\alpha_i| \leq 1 \right\}$$

is bounded in $B(R)$. ($\{E_i\}$ is a disjoint finite collection.) Since Λ is continuous, the image of the above set is bounded in A. Define $\Lambda(\chi_E) = \mu(E)$ for $E \in R$. It is now easy to see that μ is a spectral measure on R. First, μ is certainly additive and multiplicative. Second, the noted boundedness above says μ has bounded semivariation. Since

$$\Lambda(\chi_F \cdot f) = \int_F f \, d\mu \tag{1.1}$$

for all simple functions f, and both sides of (1.1) are continuous for uniform convergence, the proposition follows.

1.6 Unbounded operators on a topological vector space

Additional assumptions on spectral measures

We will be looking at measures with values in $B(M)$ for a topological vector space M. In addition to the original three assumptions on a spectral measure $\mu : R \to B(M)$, we shall require two more.

(4) For each $x \in M$, $\mu(.)x$ (which we shall denote by $\mu_x(.)$) is a c.a. M-valued measure on R.

(5) The regularity hypothesis of Definition 1.8 is satisfied.

We make some assumptions on M, $B(M)$ and R.

(1) M is sequentially complete.

(2) The (vector) topology of $B(M)$ is stronger than (at least as strong as) pointwise convergence and makes $B(M)$ sequentially complete.

(3) R is a δ-ring. (R is closed under countable intersection.)

Proposition 1.13 *Suppose f_n is the sequence of functions corresponding to a measurable function f such that $f_n = \chi_{K_n} \cdot f$ where*

$$K_n = \{t|\, 0 < |f(t)| \leq n\}.$$

If $\lim_{n\to\infty} \int f_n d\mu_x$ exists in the topology of M, then f is integrable with respect to μ_x (in the sense of Smith [40]*).*

Proof. There exist simple functions $\{S_{m,n}\}$ such that $\lim_m S_{m,n} = f_n$ where convergence is uniform. It is apparent since μ_x has bounded semivariation and μ is a spectral measure (see Proposition 1.6, Lemma 1.3) that we may choose $\{S_{m_n,n}\}$ such that $S_{m_n,n} \to f$ pointwise almost everywhere and

$$\lim \int S_{m_n,n} - f_n d\mu_x = 0,$$

for $F \in R$. It follows from the Nikodym convergence theorem (e.g., [4]–[6] or [42]) that $\{\int S_{m_n,n} d\mu_x\}$ is a uniformly continuous set (as in [40]). Consider

W_1, the collection of all simple functions bounded by 1. Notice that for any k a positive integer

$$W_1 \subseteq W_1^{4k^2} + \frac{1}{k} W_1, \tag{1.2}$$

where $W_1^{4k^2}$ is the collection of all simple functions taking at most $4k^2$ different values bounded by 1. Let $E_n \searrow \emptyset$ in R and define $\mu_{x,n}(F) = \mu_x(F \cap E_n)$. It follows from the Nikodym convergence theorem again and (1.2) that $\hat{\mu}_x(E_n) \to 0$ as $n \to \infty$. This allows us to conclude that $S_{m_n,n} \to f$ in "measure" and, therefore, by [40] f is integrable. This completes the proof.

Definition 1.10 A linear mapping T acting in M (perhaps not defined on the whole of M) is a scalar type operator (briefly: "scalar operator" or "scalar"), if there is a c.a. spectral measure μ on a δ-ring R of subsets of some set H and an R-measurable function f such that if $f_n = \chi_{K_n} \cdot f$ where

$$K_n = \{t \mid 0 < |f(t)| \leq n\},$$

then

$$Tx = \lim_{n\to\infty} \int_{K_n} f_n d\mu_x.$$

The domain of $T(= D(T))$ being

$$\{x \in M \mid \lim_{n\to\infty} \int_{K_n} f_n d\mu_x \text{ exists}\}.$$

Proposition 1.14 *Let T be a scalar operator in M corresponding to the spectral measure μ and measurable function f. Then there exists a countable number of subspaces $\{M_n\}$ of M such that T restricted to $M_n(= T|_{M_n})$ is continuous. Also $\mu(N(f))M$ is the continuous sum of the M_n (i.e., if $x \in \mu(N(f))M$, then there exists a sequence $\{x_n\}$, $x_n \in M_n$ and $x = \lim_{n\to\infty} \sum_{i=1} x_i$) and for each n there is a continuous linear projection $P_n : M \to M_n$. Furthermore, for each $x \in D(T)$,*

$$Tx = \sum_{n=1}^{\infty} T_n P_n x \quad (T_n = T|_{M_n}).$$

Proof. If we let $g_n = f \cdot \chi_{K_n}$ where

$$K_n = \{t | n - 1 < |f(t)| \leq n\}$$

then by Proposition 1.13, $Tx = \sum_{n=1}^{\infty} \int g_n d\mu_x$, if $x \in D(T)$. Define $P_n = \mu(K_n)$, it is easily seen that $T_n = \int g_n d\mu$ on M_n. Thus, $T_n P_n = \int_{K_n} g_n d\mu$ and since $\int g_n d\mu_x = 0$ if $x \notin \mu(K_n)M$ we have $Tx = \sum_{n=1}^{\infty} T_n P_n x$. If $x \in \mu(N(f))M$, we know that

$$x = \mu(N(f))x = \sum_{n=1}^{\infty} \mu(K_n) \cap N(f)x = \sum_{n=1}^{\infty} \mu(K_n)(N(f))x =$$

$$\sum_{n=1}^{\infty} \mu(K_n)x = \sum_{n=1}^{\infty} x_n,$$

$$x_n \in M_n,$$

it follows by definition that $\mu(N(f))M = \sum_{n-1}^{\infty} M_n$. This completes the proof.

Proposition 1.15 *If S is a scalar operator with μ its spectral measure and $Sx = \int f d\mu_x$, then $D(S)$ is dense in $\mu(N(f))M$. If $H \in R$ and $\mu(H) = I$, the identity mapping, then $D(S)$ is dense in M. Furthermore, S is a closed operator.*

Proof. The collection of all finite sums from $\cup M_n$ is dense in $\sum M_n$ and is contained in $D(S)$. This proves the first part via Proposition 1.14. Now to show S is closed, let $\{x_\alpha\}$ be a convergent net $\{x_\alpha\} \subseteq D(S)$, $x_\alpha \to x$ and suppose $Sx_\alpha \to Y$. $Sx_\alpha = \lim_{n\to\infty} \int_{K_n} f_n d\mu_{x_\alpha}$ by definition. Now, $y = \mu(N(f))y$ since $\mu(N(f))Sx_\alpha = Sx_\alpha$ as a simple calculation shows. By the countable additivity of μ,

$$\lim_{n\to\infty} \mu(K_n)y = \mu(N(f))y;$$

$$\mu(K_n)y = \mu(K_n) \lim_{\alpha} Sx_\alpha = \lim_{\alpha} \mu(K_n)Sx_\alpha = \lim_{\alpha} \int f_n d\mu_{x_\alpha} = \int f_n d\mu_x$$

(f_n is bounded, so by sequential completeness of $L(M)$, $\int_{K_n} f_n d\mu \in L(M)$). So, $\lim_{n} \int_{K_n} f_n d\mu_x$ exists and is equal to y. Thus, S is closed. This completes the proof.

Proposition 1.16 (Dunford Calculus). *Let μ be a spectral measure in $L(M)$ on R and f and g complex-valued measurable functions. Define*

$$S(f)(x) = \int f d\mu_x = \lim_{n\to\infty} \int_{K_n} f_n d\mu_x$$

when the limit exists, with f and f_n as in Proposition 1.13. Then

(a) $D(S(f)) = D(S(|f|))$,

(b) $D(S(f)) \subseteq D(S(g))$ *if* $|f(t)| \geq |g(t)|$,

(c) $S(f) \in L(M)$ *if* f *is bounded,*

(d) $S(\alpha f) = \alpha S(f)$

(e) $S(f+g)$ *extends* $(\supseteq) S(f) + S(g)$,

(f) $D(S(f)) + S(g) = D(S(f+g)) \bigcap D(S(f))$,

(g) $S(fg) \supseteq S(f)S(g)$,

(h) $D(S(f)S(g)) = D(S(fg)) \bigcap D(S(g))$,

(i) $S(f)\,\mu\,(\delta) \supseteq \mu\,(\delta)\,S(f)$ *for* $\delta \in R$.

Proof. The proof is an application of proposition 1.15.

We indicate the proof:

For part (i): if $x \in D(S(f))$, then

$$\lim_n \mu(\delta) \int_{K_n} f_n d\mu_x = \lim_n \int_{K_n} f_n d\mu(\mu(\delta)x)$$

exists so $\mu(\delta)x \in D(S(f))$.

For (g) and (h): let

$$\delta_n = \{t| \ |f(t)| \leq n, \ |g(t)| \leq b\} \cap \ (N(f)) \cap N(g)).$$

Let

$$\tilde{S}(g)(x) = \lim_{n\to\infty} \int_{\delta_n} g\, d\mu_x,$$

$$D(\tilde{S}(g)) = \{x| \lim_{n\to\infty} \int_{\delta_n} g\, d\mu_x \text{ exists}\},$$

if $\lim_{n\to\infty} \int_{\delta_n} g\, d\mu_x$ exists, then $\lim_{n\to\infty} S(g)\mu\,(\delta_n)x$ exists and $\lim \mu(\delta_n)x$ exists by the countable additivity of $\mu(.)x$. Then $\mu(\lim_{n\to\infty} \delta_n)x \in D(S(g))$ by Proposition 1.15, but by definition

$$S(g)\mu(\lim \delta_n)x = S(g)(x)$$

and, so, $x \in D(S(g))$.

If $x \in D(S(g))$ then

$$\beta_n = \{t |\ |g(t)| \leq n\} \cap (N(g) \cap N(f))$$

then

$$\lim_{n\to\infty} \tilde{S}(g)\mu(\beta_n)x = \tilde{S}(g)(x)$$

exists again so $x \in D(\tilde{S}(g))$. Obviously $\tilde{S}(g) = \tilde{S}(g)$ on $D(S(g)) \cap D(S(g))$. The same facts hold in relation to $S(gf)$ and $S(f)$. From (i) we conclude that if

$$x \in D(S(f)S(g)) \cap D(S(g)),$$

then

$$\lim_{n\to\infty} S(gf \cdot \chi_{\delta_n})x = \lim_{n\to\infty} S(g \cdot \chi_{\delta_n})S(f)\mu(\delta_n)x = \lim_{n\to\infty} S(g)\mu(\delta_n)S(f)x =$$

$$= \lim_{n\to\infty} \mu(\delta_n)S(g)S(f)x = S(g)S(f)x$$

so $x \in D(S(gf))$. If

$$x \in D)S(gf)) \cap D(S(f)),$$

then

$$\lim S(g\chi_{\delta_n})S(f)x = \lim S(g\chi_{\delta_n})\mu(\delta_n)S(f)x = \lim S(g\chi_{\delta_n})S(f)\mu(\delta_n)x =$$

$$= \lim S(g\chi_{\delta_n})x = \lim S(gf\chi_{\delta_n})x = S(gf)x,$$

so $S(f)x \in D(S(g))$ and $S(g)S(f)x = S(gf)x$. This gives (g) and (h). For (e) and (f), the proof is similar. For (b) let $l(s) = g(s)/f(s)$ (and $l(s) = 0$ if $f(s) = 0$), then $S(l) \in L(M)$ by (c) (and so $D(S(l)) = M$). By (g)

$$D(S(f)) = D(S(l)S(f)) = D(S(f)) \cap D(S(g))$$

so $D(S(f)) \supseteq D(S(g))$. (a) follows from this, (c) is obvious by sequential completeness. This completes the proof.

Remark 1.8 In Proposition 1.7, we assumed that f was bounded. However, it is evident that for any bounded Borel measurable function $g : \mathbb{C} \to \mathbb{C}$ the formula of Proposition 1.7 holds in the more general present case, i.e., for $x \in D(S(f))$,

$$\int_F g \circ f d\mu_x = \int_{f(F)} g(z) dm_f(z)_x,$$

and, in fact, choosing f_n as in Definition 1.10, gives that (taking

$$g(\lambda) = \begin{cases} \lambda \ |\lambda| \leq n \\ 0 \ |\lambda| > n \end{cases}$$

for g in Proposition 1.7)

$$S(f_n) = \int_{G_n \cap f(F)} z dm_f(z) = \int_F f_n d\mu.$$

Here we have written G_n for the set $\{\lambda \in \mathbf{C} | \ |\lambda| \leq n\}$ and f_n as in Definition 1.10. We now conclude that

$$\lim_{n \to \infty} \int_{|\lambda| \leq n} z \, dm_f(z)_x$$

exists for each $x \in D(S(f))$. This gives (applying Proposition 1.13)

$$\int_F f d\mu_x = \int_{f(F)} z \, dm_f(z)_x.$$

In most cases we shall be interested in the m_f corresponding to $F = N(f)$ (see Proposition 1.9), for if μ finds its "support" in R, we take $F = \text{supp}(\mu)$. We use the term "support" for F here to mean that if $E \in R, \ \mu(E) = \mu(E \cap F)$, (and if G is any other set with this property, $\mu(F \cap G) = \mu(F)$) since H is not assumed to be a topological space. We have seen that a scalar operator may be interpreted as an integral of a spectral measure. The question arises as to the relation, in case the support of the measure is not compact (i.e., unbounded) between the spectrum of a scalar operator (in Definition 1.7, $(\lambda I - T)^{-1} \in B(M)$ for $\lambda \in \rho(T)$ in addition to the integration requirements) and the support of its spectral measure. We restrict ourselves to some of the classical conditions.

Proposition 1.17 *If T is a scalar operator, μ its complex spectral measure, and $\mathbb{C} \in R$ with $\mu(\mathbb{C}) = I$ the identity operator, then $\sigma(T) = \mathrm{supp}(\mu)$. ($\sigma(T)$ in general is calculated relative to $\mu(\mathrm{supp}(\mu))$ as in Proposition 1.9)*

Proof. Suppose $\mathrm{supp}(\mu) \neq \mathbb{C}$. Let D be a disc whose closure does not intersect $\mathrm{supp}(\mu)$, then for $z \in \partial D$,

$$\int_C \frac{d\mu(\xi)}{z - \xi}$$

exists. We may choose an elementary sequence f_n converging uniformly to $1/(z-\xi)$ on $\partial D \times \mathrm{supp}(\mu)$, since $1/(z-\xi)$ vanishes at "infinity". The proof of Proposition 1.5 extends trivially to show

$$0 = \int_{\partial D} \int_{\mathrm{supp}(\mu)} \frac{d\mu(\xi)}{z - \xi}\ dz \int_{\mathrm{supp}(\mu)} \int_{\partial D} \frac{dz}{z - \xi}\ d\mu(\xi).$$

Thus, $\sigma(T) \subseteq \mathrm{supp}(\mu)$. The rest of the proof is identical to Proposition 1.9. This completes the proof.

Proposition 1.18

$$\sigma(S(f)) = \bigcup_{n=1}^{\infty} \sigma(S(f_n))$$

where f and f_n are as in Proposition 1.13. $S(f)$ is defined in Proposition 1.16.

Proposition 1.19 *Let $\lambda \in \rho(T)$ where T is a closed operator with dense domain. T is a scalar operator if and only if $(\lambda I - T)^{-1} (\in B(M))$ is a scalar operator with compact spectrum. (Assume here that $\mu(\mathbb{C}) = I$ as above.)*

Proof. Let I be scalar, μ its complex spectral measure. Since $\lambda \in \rho(T)$,

$$(\lambda I - T)^{-1} = \int_{\mathbb{C}} \frac{d\mu(z)}{\lambda - z}$$

by Remark 1.8. Taking $f(\,)z = 1/(\lambda - z)$ in Definition 1.9, we see that

$$(\lambda I - T)^{-1} = \int_{\mathbb{C}} \xi dm_f(\xi).$$

(We remark that $m_f(\{0\}) = \mu(f^{-1}(\{0\})) = 0$ since $1/(\lambda - z)$ is bounded on $\mathrm{supp}(\mu)$.) Conversely if

$$(\lambda I - T)^{-1} = \int_{\mathbb{C}} \xi d\mu(\xi)$$

then

$$\int_{\mathbb{C}} \frac{d\mu_x(\xi)}{\xi}$$

exists since

$$(\lambda I - T)\mu(K_n)x = \int_{K_n} \frac{d\mu_x(\xi)}{\xi}$$

for $x \in D(T),\ K_n = \{\xi |\, |\xi \geq 1/n\}$ and if $x \in D(T)$,

$$\lim_{n\to\infty} (\xi I - T)\mu(K_n)x = \lim_{n\to\infty} \mu(K_n)(\lambda I - T)x = (\lambda I - T)x$$

by Proposition 1.16. Thus

$$Tx = \int \left(\lambda - \frac{1}{\xi}\right) d\mu_x(\xi)$$

for $x \in D(T)$. Since T is closed, the result follows from Remark 1.8 again. This completes the proof.

1.7 Relation to classical results

The classical definitions of spectrum and resolvent set are:

Definition 1.11 Let $T : M \to M$ be linear with domain $D(T)$ and range $R(T)$. Let $T(\lambda) = \lambda I - T$. Then the point spectrum $\sigma_p(T) = \{\lambda |\ T(\lambda)$ is not one to one$\}$, the continuous spectrum $\sigma_c(T) = \{\lambda |\ T(\lambda)$ is one to one, $R(T(\lambda))$ is dense in M and $T(\lambda)^{-1}$ is discontinuous from $R(T(\lambda))$ to $D(T(\lambda)\}$ and the residual spectrum $\sigma_r(T) = \{\lambda |\ T(\lambda)$ is one to one but $R(T(\lambda))$ is not dense in $M\}$. The spectrum $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ and the resolvent set $\rho(T) = \mathbb{C} - \sigma(T)$. Whence $\rho(T) = \{\lambda |\ T(\lambda)$ has dense range and its inverse is a continuous mapping of $R(T(\lambda))$ onto $D(T(\lambda))\}$.

The classical spectral operators are those which can be decomposed as the sum of a scalar operator and a nilpotent operator. We make the following formal definition.

Definition 1.12 T is called *spectral* if there is a spectral measure μ such that for all bounded sets $F \in R$ and all sets $E \in R$ we have

(1) $\mu(F)M \subseteq D(T)$, $T\mu(E)$ is continuous.

(2) $\mu(E)D(T) \subseteq D(T)$

(3) $T\mu(E)x = \mu(E)Tx$ for all $x \in D(T)$,

(4) $\sigma(T_E) \subseteq \overline{E}$ where $T_E = T$ restricted to $\mu(E)D(T)$.

(5) $\mu(\mathbb{C}) = I$.

The previous definition gives the following:

Proposition 1.20 *For any Borel set E, we have T_E is a spectral operator with spectral measure μ_E ($\mu(G)$ restricted to $\mu(E)M$ for all $G \in R$). If E is a bounded set, then T_E is continuous.*

Proof. From (1) we have T_E is bounded. (2) shows that $\mu_E(G)D(T_E) = \mu_E(D(T) \cup \mu(E)X) \subseteq D(T) \cap \mu(E \cap G)M \subseteq D(T) \cap \mu(E)X \subseteq D(T_E)$. (3) implies that $T_E\mu_E(G)x = \mu_E(G)T_Ex$, $(x \in D(T_E)$. Lastly, (4) shows that $\sigma(T_E|(D(T_E) \cap \mu(G)X)) = \sigma(T|(D(T) \cap \mu(G \cap E)X)) \subseteq \overline{E \cap G} \subseteq \overline{E}$.

Proposition 1.21 *Let T be a spectral operator. Then $D(T)$ is dense in M.*

Proof. Since μ_x is countably additive, by definition 1.12 (2) and (5) the domain of T is dense.

The nature of the point spectrum of an operator T is related to the atomic support of its spectral measure.

Proposition 1.22 *Let T be a spectral operator and assume that M is separable and that there exists U^* such that for every V there is $V^* \subseteq V$ and t where if $\mu(E)tV^* \subseteq U^*$ then $\mu(E) = 0$. Then $E_p = \{\lambda \mid \mu(\{\lambda\}) \neq 0\}$ is countable.*

Proof. Let $\lambda_1 \in E_p$, for some x, $\mu_x(\{\lambda_1\}) = x_{\lambda_1} \notin U^*$. Since $\mu(\{\lambda_1\})$ is continuous, there exists V such that $\mu(\{\lambda_1\})V \subseteq U^*$. Whence, $x_{\lambda_1} \notin V$. Let $\lambda_2 (\neq \lambda_1) \in E_p$. Then $\mu(\{\lambda_1)x_{\lambda_2} = 0$. It follows that $x_{\lambda_1} - x_{\lambda_2} \notin V$.

Let $\{x_n\}$ be a countable dense subset of M and $V_1 + V_1 \subseteq V$. Let $\{x_{n1}\} = \{x_n \mid x_n - x_{\lambda_1} \in V_1\}$ and $\{x_{n2}\} = \{x_n \mid x_n - x_{\lambda_2} \in V_1\}$. Since $x_{\lambda_1} - x_{\lambda_2} \notin V$, $\{x_{n1}\} \cap \{x_{n2}\} = \emptyset$. Hence E_p has cardinality less than or equal to that of the set of disjoint subsequences of $\{x_n\}$ which is the cardinality of $\{x_n\}$ and this completes the proof.

2 Riesz Theorem

We shall be dealing with maps of a topological vector space M into itself. All topological vector spaces will be assumed separated and sequentially complete. We make one more modification in the definition of "spectral measure". We shall assume the measure to be defined on the entire space and to be the identity there (i.e., if μ is the measure and the Borel sets of $\mathbb{C}$ is the domain of μ, then $\mu(\mathbb{C}) = I$ the identity in the range space of operators under consideration). This is not essential to the abstract development and is simply a computational convenience.

The Riesz theorem has been proved in many contexts. In Banach spaces the result is as follows:

Theorem 2.1 *Let S be a compact Hausdorff space and let T be a weakly compact operator from $C(S)$ to X. Then there exists a measure μ defined on the Borel sets in S and having values in X such that*
(a) μ is regular and countably additive.
(b) $T(f) = \int_S f d\mu$.

Conversely, if μ satisfies (a) and (b) for T, then T is weakly compact, that is, the image of some neighborhood in $C(S)$ is relatively compact in the weak topology of X (see [8] VI.7.3). The theorem has seen extension in a number of directions. For example, instead of $C(H)$ (where H is compact Hausdorff), one may consider $C(H, Z)$ where Z is a Banach space and $T : C(H, Z) \to X$ is the mapping (see [1], [3], [47] for example). The case where Z and X are locally convex and quasi-complete has also been studied (see [13]). Even weaker requirements on Z and X have been studied [45]. The requirement that H be compact may also be relaxed to local compactness, etc. We desire to extend the result above to the case where X is only assumed to be a topological vector space (without assuming local convexity). Our procedure will be to prove a form of the Riesz theorem, following the Bourbaki approach and then, translate the result into the one which we want. To carry out this program we shall need some notation and some elementary facts.

Definitions and Facts H will stand for a locally compact Hausdorff space. $K(H)$ shall be the space of continuous functions with compact support on H. M is a topological vector space. It will make the notation simpler to assume that M is metrizable with translation invariant metric $|\ .\ |$. We will be able to discard this assumption at the conclusion of this section. If Λ is a continuous linear map of $K(H)$ into M (i.e.. for each $\epsilon > 0$ and compact $K \subset H$ there is $\delta > 0$ such that $\sup_{t \in H} |\phi(t)| < \delta$ and support $(\phi) \subset K$, implies $|\Lambda(\varphi)| \leq \varepsilon$), then define $\Lambda^0(f) = \sup_{|\varphi| \leq f} |\Lambda(\varphi)|$ for nonegative lower semicontinuous functions f (abbreviate lower semicontinuous as l.s.c.)

$$\Lambda^0(f) = \inf_{f \leq g} \Lambda^0(g) \qquad (g \text{ is l.s.c.})$$

for f with compact support and, finally, for arbitrary f:

$$\Lambda^0(f) = \sup_{h \leq f} \Lambda^0(h)$$

h with compact support (but not necessarily continuous). If $A \subset H$, then $\Lambda^0(\chi_A) = 0$ shall be the criteria for determining sets of "measure zero". (This gives meaning to "almost everywhere" (a.e.) type phases.) The above definition is consistent and the following are true:

(i) if $f \leq g$, then $\Lambda^0(f) \leq \Lambda^0(g)$.

(ii) $\Lambda^0(0) = 0$.

(iii) $\Lambda^0(f_1 + f_2) \leq \Lambda^0(f_1) + \Lambda^0(f_2)$.

(iv) $\Lambda^0(\Sigma f_n) \leq \Sigma \Lambda^0(f_n)$ for any sequence $\{f_n\}$.

(v) $\Lambda^0(f) = 0$ implies $f = 0$ a.e.

(vi) If f is bounded with compact support $\Lambda^0(f) < \infty$, then $\lim_{\lambda \to 0} \Lambda^0(\lambda f) = 0$.

$L^1(\Lambda)$ shall be the collection of all functions of f such that for each $\varepsilon > 0$, there exists $\varphi \in K(H)$ for which $\Lambda^0(|f - \varphi|) \leq \varepsilon$. $L^1(\Lambda)$ is a topological vector space with metric Λ^0 and the functions equal to zero almost everywhere constitute a closed subspace of $L^1(\Lambda)$. Λ can be extended to $L^1(\Lambda)$ with this topology in the obvious way. We note that if h is bounded and continuous then $hL^1(\Lambda) \subset L^1(\Lambda)$.

Λ will be called a *Radon measure*, if $L^1(\Lambda)$ contains all the bounded Borel functions with compact support on H.

Theorem 2.2 *Let the range of Λ be bounded on the unit ball of $K(H)$. These are equivalent:*

(1) *$L^1(\Lambda)$ contains all bounded Borel functions.*

(2) *Λ is a Radon measure and $1 \in L^1(\Lambda)$.*

(3) *For every bounded l.s.c. $f \geq 0$ and $\varepsilon > 0$, there exists $\varphi \in K(H)$ with $0 \leq \varphi \leq f$, and $\Lambda^0(f - \varphi) \leq \varepsilon$.*

(4) *Λ^0 is regular, u.e., if $\chi_E \in L^1(\Lambda)$ and $\varepsilon > 0$, then there exists a compact K and open O s.t. $K \subset E \subset O$ and $\Lambda^0(O \setminus K) < \varepsilon$.*

(5) *For each $\{\varphi_n\}$, $\varphi t \in K(H)$ such that $\sum_{n=1}^{\infty} |\varphi_n(t)| \leq 1$, $\lim_{n\to\infty} \Lambda(\varphi_n) = 0$.*

(6) *Λ maps weakly compact subsets of $C_0(H)$ into relatively compact subsets of M (Λ can be extended to C_0 by the boundedness condition).*

Proof. We shall outline the proof of (6) → (5) → (3) → (2) → (1).

(6) → (5). Suppose $\Sigma|\varphi_n(t)| \leq 1$. Then, $\varphi_n(t) \to 0$ weakly. Thus $\{\Lambda(\varphi_n)\}$ is relatively compact by 6). Choose any subsequence $\{\Lambda(\varphi_{n_k})\}$ which converges. (Metrizability of M allows us to use sequences.) Let $\lim_{k\to\infty} \Lambda(\varphi_{n_k}) = m$ and assume $\Sigma|m - \Lambda(\varphi_{n_k})| < \infty$ by passing to a subsequence, if necessary. Then $\lim_{n\to\infty} \left[n(m) - \Lambda\left(\sum_{k=1}^{n} \varphi_{l_k}\right)\right] = y$ exists, so $\lim \Lambda\left(\frac{1}{n}\sum(\varphi_{lk})\right) = m$. But, $|\sum \varphi_{lk}| \leq 1$ and so $\frac{1}{n}\sum \varphi_{lk} \to 0$ uniformly in C_0; therefore, $m = 0$. This is (6) → (5).

(5) → (3). Let $f \geq 0$ be a l.s.c. bounded function and $\varepsilon > 0$. There exists $\varphi \in K(H)$ with $O \leq \varphi \leq f$ and $\Lambda^0(f - \varphi) \leq \varepsilon$. Suppose not. Then there is $\varepsilon > 0$ and f such that

$$|\,\Lambda^0(\varphi_1)| > \varepsilon,\ |\,\Lambda^0(f - \varphi_1)| > \varepsilon$$

and φ_2 such that

$$|\Lambda(\varphi_2)| > \varepsilon \text{ and } |\varphi_1| + |\varphi_2| < f$$

with $\Lambda^0(f - \varphi_1 - \varphi_2) > \varepsilon$. By induction we get a sequence $\{\varphi_n\}$,

$$\Lambda^0\left(f - \sum_1^k |\varphi_n|\right) > \varepsilon,\ (\Lambda(\varphi_{k+1})) > \varepsilon,$$

etc. But

$$0 \le \Sigma|\varphi_k| < f,$$

and this contradicts (5).

(3) $\to$ (2). From (3) it follows that we may extend Λ by continuity of Λ^0 to χ_O where O is any relatively compact open set in H. It follows also that characteristic functions of compact sets are in $L^1(\Lambda)$. If we define $B(\Lambda)$ to be all subsets E of H with the property that $\chi_{E\cap K}$ is in $L^1(\Lambda)$ for all compact K, then it can be shown that $B(\Lambda)$ is a $\sigma-$algebra containing the Borel sets of H. Let us show that if f is measurable with respect to $B(\Lambda)$, then for each compact $K \subset H$ and $\varepsilon > 0$ there exists $K' \subset K$ with $\Lambda^0(K - K') \le \varepsilon$ and $f|_{K'}$ is continuous (f is "almost" continuous on K). For suppose f is $B(\Lambda)$ measurable. Assume without loss of generality that $0 \le f \le 1$. It is enough to show that φf is almost continuous for each $\varphi \in K(H)$. Since φ is a Borel function, we can assume $0 \le f \le \chi_K$ for some compact K. Since f is a bounded $B(\Lambda)-$measurable function, there exists $\{f_n\}$, a sequence of $B(\Lambda)$ simple functions with $0 \le f_n \le \chi_K$, and $|f_n - f| \le \dfrac{1}{n}$. We have $f_n \in L^1(\Lambda)$, so since

$$\Lambda^0(f - f_n) \le \Lambda^0\left(\frac{1}{n}\chi_K\right) \to 0,$$

we have $f \in L^1(\Lambda)$. Now let $\{\varphi_n\} \subset K(H)$ be chosen so that

$$\Lambda^0(f - \varphi_n) < \frac{1}{(2^n)4}$$

and write $f(t) = \lim \varphi_n(t)$ a.e. Define $f_n = \varphi_n - \varphi_{n-1} \qquad (\varphi_0 = 0)$.

Then $\Lambda^0(|f_n|) < \dfrac{1}{2^n}$. Define

$$h(t) = \sum_{n=1}^{\infty} n|f_n(t)| < \infty$$

and $O_\alpha = \{t : h(t) > \alpha\}$. O_α is open since h is l.s.c. If $\alpha > 1$, then $\chi_{O_\alpha} \leq \dfrac{1}{\alpha} h \leq \sum \dfrac{n}{\alpha}|f_n|$ and so,

$$\Lambda^0(O_\alpha) \leq \Sigma n\Lambda^0\left(\frac{1}{\alpha}|\,f_n|\right) \leq \sum_{n=1}^{N} n\Lambda^0\left(\frac{1}{\alpha}|\,f_n|\right) + \sum_{N+1}^{\infty}\frac{n}{2^n} \leq \varepsilon$$

for large N and α. If $t \in O_\alpha^c$, then $Z^c = H \diagdown Z$

$$n\sum_{k\geq n}|\,f_k(t)| \leq h(t) \leq \alpha,$$

so $\sum f_n(t)$ converges uniformly to f on O_α^c. This shows f is almost continuous. We have seen already that $B(\Lambda)$ must contain the compact sets in H and the open sets with compact closure. Thus, Borel functions with compact support are $B(\Lambda)$ measurable and are therefore almost continuous. Let K be compact, $K = \mathrm{supp}(f)$, f a bounded Borel function. Then for some $vp \in K(H)$, $|f| \leq \varphi$, and for some $K_1 \subset K$, $\Lambda^0(K \diagdown K_1) \leq \dfrac{\varepsilon}{2^n}(\varphi \leq n)$ and $f|_{K_1}$ is continuous. Let $\overline{f}$ be a norm preserving continuous extension of f off K_1. Define $\psi = \inf(\varphi, \overline{f})$, then $\psi \in K(H)$,

$$\begin{aligned}\Lambda^0(f-\psi) &\leq \Lambda^0(\chi_{K_1}(f-\psi)) + \Lambda^0(\chi_{K\diagdown K_1}(f-\psi))\\ &\leq \Lambda^0(2n(K\diagdown K_1)) \leq 2n\Lambda^0(K\diagdown K_1) < \varepsilon\end{aligned}$$

Thus, $f \in L^1(\Lambda)$. This gives (3) $\to$ (2). (The fact that $1 \in L^1(\Lambda)$ follows from (3) for $f = 1$.)

(2) $\to$ (1). Notice now that if f is any bounded Borel function and K is a compact subset of H with $\Lambda^0(\chi_{H/K}) < \frac{\varepsilon}{M}$, $M = [\max|\,f|]$ ([.] is the greatest integer function) (we can choose such a K since, by (2), $1 \in L^1(\Lambda)$), we can choose $\varphi \in K(H)$, $f(t) \leq \varphi(t)$ on K and let

$$g = \inf(\Phi, f)\chi_{\mathrm{supp}(\varphi)}.$$

Then, $g \in L^1(\Lambda)$ by (2), so

$$\Lambda^0(f-g) \leq \Lambda^0(\chi_K(f-g)) + \Lambda^0(\chi_{H\diagdown K}(2M)) \leq 2\varepsilon,$$

so $f \in L^1(\Lambda)$ and (1) follows from (2).

The proof is complete.

Remark 2.1 It is now easy to relate to our previous development. If Λ satisfies one of the hypotheses of Theorem 2.2, then for any Borel set $E \subset H$, we may define $\mu(E) = \Lambda(E) = \Lambda(\chi_E)$. Thus μ is, we claim, a regular countably additive M-valued Borel measure. The regularity follows from Theorem 2.2, but we may give the following argument for countable additivity: let $\{E_n\}$ be a sequence of Borel sets decreasing in monotone fashion to $\emptyset$. Since each $E_n \in L^1(\Lambda)$ for any $\varepsilon\,\varphi > 0$, we can choose H_n with H_n compact in H where $\Lambda^0(E_n \diagdown H_n) \leq \frac{e}{2^n}$ for there exists $\varphi \in K(H)$ with $\Lambda^0(\chi_{E_n} - \varphi) \leq \frac{\varepsilon}{2^n}$). Let $H_n = \mathrm{supp}(\varphi)$, then

$$\Lambda^0(E_n \diagdown H_n) \leq \Lambda^0(\chi_{E_n} - \varphi) + \Lambda^0(\chi_{H_n^c} \cdot \varphi).$$

Now, choose $L_n \subset H_n$ with $\chi_{E_n}|_{L_n}$ continuous and $\Lambda^0(H_n \diagdown L_n) \leq \dfrac{\overline{\varepsilon}}{2^n}$. Then let $K_n = L_n \cup E_n$, so K_n is compact. Now let $M_n = \bigcap_{i=1}^{n} K_i$, then $E_n \diagdown M_n \subset \bigcup_{i=1}^{m} (E_i \diagdown K_i)$, so $\Lambda^0(E_n \diagdown M_n) < \varepsilon$, but $M_n \uparrow \emptyset$, so $M_n = \emptyset$ for some n. Thus, $\Lambda^0(E_n) < \varepsilon$ for some n. This implies $\mu(E_n) \to 0$ as $n \to \infty$. This implies the countable additivity of μ. It is easily seen that μ is of bounded semivariation because Λ^0 is continuous on the bounded Borel functions with the topology of uniform convergence (in particular, on the *simple functions* with this topology) because $1 \in L^1(\Lambda)$ by (2). (If f is a bounded Borel function, let $\{f_n\}$ be a sequence of simple functions converging to $f : |f_n - f| \leq \dfrac{1}{n}$. We have $|\Lambda(f - f_n)| \leq \Lambda^0\left(\dfrac{1}{n}\right) \to 0$.) It follows easily that μ must have bounded semivariation. It is now evident that (see [38])

$$\Lambda(f) = \int_H f d\mu$$

for all bounded Borel functions on H. This now gives the usual version of the Riesz theorem (see [33], p. 310, for example in the scalar case). We state this as

Proposition 2.1 *Let Λ satisfy the initial hypothesis of Theorem 2.2. If Λ maps weakly compact subsets of $C_0(H)$ into relatively compact subsets of M, then there exists a unique regular countably additive Baire measure with* *values in M which has bounded semivariation and such that*

$$\Lambda(f) = \int_H f d\mu \qquad (f \in C_0(H)). \qquad (^*)$$

Proof. The existence of such a μ has been shown above. The uniqueness may be shown as follows. Let μ_1 and μ_2 be two such measures satisfying (*). We may choose a sequence $\{f_n\}$ of functions in $K(H)$ converging monotonically to χ_K where K is a compact G_δ set in H. The dominated convergence theorem (7.2.7 of [48]) shows that $\mu_1(K) = \mu_2(K)$ and the "monotone class theorem", together with the countable additivity of μ_1 and μ_2, imply that $\mu_1 = \mu_2$ on the Baire sets of H.

Remark 2.2 For our purpose it makes no difference whether we have the Baire or Borel sets as the domain of the measures in Proposition 2.1, since we shall apply the theorem only in the case where $H = \mathbb{C}$ or $\mathbb{C}^n$.

In this section we have restricted ourselves to metrizable complete spaces as the range of Λ. However, if M is simply assumed to be quasicomplete (close bounded sets are complete), then the results above still go through because we may consider M as a subspace of the product of metrizable spaces. One may show that if $L^1(\Lambda)$ is defined as $\bigcap_i L^1(\Lambda_i)$ where Λ_i is the i^{th} projection of Λ into the product space above, then the results above go through. The justification of this is more or less standard and is left to the reader.

3 Scalar type operators in a topological vector space

We shall use the letters w.c.p. to stand for the "weak compactness property" of Proposition 2.1. That is, a mapping Λ has the w.c.p. if the image under Λ of weakly compact subsets is relatively compact.

Proposition 3.1 *Suppose M is a topological vector space. Let $B(M)$ be the set of continuous endomorphisms of M and assume $B(M)$ is quasicomplete in the "strong operator topology". (M is therefore quasicomplete.) Let $T \in B(M)$ be such that $\sigma(T)$ is real and compact. Suppose that the mapping which takes the space of polynomials $P(\sigma(T))$ in one real variable on $\sigma(T)$ with* supp norm *topology into $B(M)$ by $P \to P(T)$ is continuous and has the w.c.p. Then T is a scalar type operator.*

Proof. We assume as in the hypothesis that $B(M)$ is quasicomplete. $P(\sigma(T))$ is dense in $C(\sigma(T))$ by the Stone–Weierstrass theorem. Applying Proposition 2.1—Remark 2.2 to the mapping $f \to f(T)$, we obtain

$$T = \int_{\sigma(T)} \lambda d\mu(\lambda).$$

To see that μ is multiplicative, we note the continuity of $P \to P(T)$ shows that

$$\int fgd\mu = \int fd\mu \int gd\mu$$

for all $f,\ g \in C(\sigma(T))$. Now appealing to the same argument used to show uniqueness in Proposition 2.1, we may conclude that μ is multiplicative on the Borel sets of $\sigma(T)$, and we can extend μ "by zero" to the Borel sets of $\mathbb{C}$. This concludes the proof.

Proposition 3.2 *Let Λ be a continuous homomorphism of $C_0(X)$ into $B(M)$, and suppose Λ has the w.c.p. Then there exists a unique c.a. regular spectral measure on μ on the Baire subsets of X with*

$$\Lambda(f)x = \int_x fd\mu_x.$$

Proof. Since Λ has the w.c.p., the existence of μ follows from Proposition 1.5. Since $\Lambda(fg)x = \Lambda(f)\Lambda(g)x$, we have $\int fgd\mu = \int fd\mu \int gd\mu$. Processing as in the proof that previous proposition, we obtain the fact that

$$\mu(\delta_1 \cap \delta_2) = \mu(\delta_1)\mu(\delta_2)$$

for δ_1 and δ_2 compact G_δ sets. Therefore the proposition follows.

Proposition 3.3 *Let T be linear operator in M (perhaps unbounded and/or discontinuous), and suppose T is closed and densely defined with non-empty resolvent set $\rho(T)$. Suppose that for some $\lambda \in \rho(T)$, $(\lambda I - T)^{-1}$ satisfies the hypothesis of Proposition 3.1. Then T is a scalar operator.*

Proof. Proposition 3.1 shows that $(\lambda I - T)^{-1}$ is a scalar operator and an application of (5.91) shows that T is a scalar operator. This completes the proof.

Remark 3.1 In the case of Hilbert space the "Cayley transform" of an unbounded operator can be used to prove the spectral theorem for unbounded operators. (See Riesz–Sz.-Nagy [31, p.320], and von Neumann [28].) It can be used in our context as well. If T is a scalar operator with $\sigma(T)$ in the unit circle, we shall refer to T as a "unitary" operator. If we have

$$T = \int \lambda d\mu,$$

then let us denote the mapping $\int \overline{\lambda} d\mu$ by T^* in analogy to the locally convex case (Hilbert space).

Assuming $\mu(\mathbb{C}) = I$, suppose that T is a scalar operator. It is unitary iff $TT^* = I$. To show this, suppose T is unitary. Then

$$TT^* = \int_{\sigma(T)} \lambda\overline{\lambda} d\mu(\lambda) = \int_{\sigma(T)} |\lambda|^2 d\mu(\lambda) = \int_{\sigma(T)} 1 d\mu(\lambda)$$

$$= \int_{\sigma(T)} d\mu(\lambda) = \mu(\sigma(T)) = \mu(\mathbb{C}) = I.$$

If $TT^* = I$, then we have

$$TT^* - f(T) = \int_{\sigma(T)} f(\lambda) d\mu,$$

where $f(\lambda) = |\lambda|^2$. The spectral mapping theorem (4.81) shows that $f(\rho(T)) = \{1\}$. Hence if $\lambda \in \sigma(T)$, $f(\lambda) = 1$. Now we consider

Proposition 3.4 *T is a scalar operator with real spectrum if and only if its Cayley transform in unitary.*

Proof. If T is scalar with spectral measure μ, then $\displaystyle\int \frac{z-i}{z+i} d\mu$ is scalar. We know $\dfrac{z-i}{z+i}$ takes the real line into the unit circle. If $(T - iI)(T + iI)^{-1}$ is unitary, then a similar argument shows that T is scalar. For let

$$f(z) = \frac{z+1}{i(z-1)}.$$

If

$$(T - iI)(T + iT)^{-1} = \int \lambda d\mu,$$

then
$$T = \int f(\lambda) d\mu(\lambda)$$
and therefore T has real spectrum. This completes the proof.

Remark 3.2 Before going on we observe that a number of results of the kind we have given above can be stated in a somewhat simpler form in the locally convex case. For example:

Proposition 3.5 *Let M be locally convex and for f Lebesgue integrable ($f \in L^1(\mathbb{R})$, $\mathbb{R}$ = the real line) let f denote its Fourier transform. T is a scalar operator with real spectrum on M if for each $x \in M$ the set*
$$\left\{ \int_{\infty}^{\infty} f(t) e^{-2\pi i t T} x dt | \; |f|_\infty \leq 1, \; f \in L^1(\mathbb{R}) \right\}$$
is weakly relatively compact in M.

The idea of course is that the set above is the range of a transformation whose domain is $L^1(\mathbb{R})$, a superset of $C_0(\mathbb{R})$ and the transformation is continuous and thereby the compactness criteria. The integral which appears above may be interpreted as a Pettis integral. The compactness condition is equivalent to the w.c.p. ([15]). A proof is found in [38].

4 Product measures and functions of several operators

Remark 4.1 Suppose $\mu_1, \ldots, \mu_n$ are commuting complex spectral measures in the sense of Remark 1I. Then as in Definition 4I we can define the "product" of $\mu_1, \ldots, \mu_n$ on the ring of "elementary sets" generated by $\overset{n}{\underset{i=1}{\times}} B(\mathbb{C})$ ($B(\mathbb{C})$ is the Borel field for $\mathbb{C}$). If $\overset{n}{\underset{i=1}{\times}} \mu_i$ has bounded semivariation on $\overset{n}{\underset{i=1}{\times}} B(\mathbf{C})$, then we can define
$$\int f(z_1, \ldots, z_n) \, d \left(\overset{n}{\underset{i=1}{\times}} \mu_i \right)$$
for continuous functions which are linear combinations of continuous functions of one variable with compact support. Since we assume μ_i to be defined

on all of $B(\mathbf{C})$, and $\mu_i(\mathbf{C}) = I$, $\overset{n}{\underset{i=1}{\times}} \mu_i$ will be defined on $\overset{n}{\underset{i=1}{\times}} \mathbf{C}$ and be the identity there. We may therefore, extend (by a standard density argument) the integral to $C_0\left(\overset{n}{\underset{i=1}{\times}} \mathbf{C}\right) = C_0(\mathbf{C}_n)$. (See the remarks preceeding Remark 5I). Let us refer to the mapping $f \to \int f d\left(\overset{n}{\underset{i=1}{\times}} \mu_i\right)$ as

$$\Gamma(f) = \int_{C^n} f(z_1, \ldots, z_n)\, d\left(\overset{n}{\underset{i=1}{\times}} \mu_i\right).$$

Proposition 4.1 *Suppose* Γ *has the w.c.p. Then there exists a unique regular c.a. measure on* $B(\mathbb{C}^n)$ *(the Borel field) such that*

$$\Gamma(f) = \int_{C^n} f d\mu \qquad (f \in C_0(\boldsymbol{C}^n)).$$

Furthermore, $\mu(E) = \left(\overset{N}{\underset{i=1}{\times}} \mu_i\right)(E)$ *for any elementary set* E*. (See the remark following Proposition 4I).*

Proof. The existence of μ follows from Proposition 2.1. The fact stated concerning μ and $\overset{\mathbf{n}}{\underset{i=1}{\times}} \mu_i$ follows from considering elements of $C_0(\mathbf{C}^n)$ of the form

$$f_1(z_1) f_2(z_2) f_3(z_3) \ldots f_n(z_n)$$

and using an argument similar to the uniqueness argument of Proposition 2.1. This completes the proof.

Remark 4.2 We have not addressed the problem of uniqueness of spectral measures, and we will now do this before proceeding further. For unbounded operators with nonempty resolvent set we can reduce the problem to the case of compact spectrum by considering the spectral measure(s) of the resolvent operator at some fixed point in the resolvent set. Suppose then that

$$T = \int_{\sigma(T)} \lambda d\mu_1 = \int_{\sigma(T)} \lambda d\mu_2$$

with $\sigma(T)$ compact. Let us also suppose that T can be decomposed *uniquely* as the sum of scalar operators with real spectrum, T_1 and T_2 so that

$$T = T_1 + iT_2$$

and the product of the spectral measures for T_1 and T_2 exists. Such a decomposition obviously *exists*. Let

$$\mu_1 = \mu_1' \times \mu_1'', \; \mu_2 = \mu_2' \times \mu_2''$$

and

$$S_1 = \int \lambda d\mu_2', \; S_2 = \int \lambda d\mu_2'', \; T_1 = \int \lambda d\mu_1', \; T_2 = \int \lambda d\mu_2'',$$

then uniqueness implies that

$$\int \lambda d\mu_1' = \int \lambda d\mu_2'$$

and the Weierstrass theorem, together with the considerations of Proposition 2.1, imply that $\mu_1' = \mu_2'$ and, so, $\mu_1 = \mu_2$. In less general settings the proof of uniqueness is based on Liouvilles theorem and the so-called "single-valued extension property".

Unfortunately, Liousville's theorem is false without local convexity (see [48, p.196]), and if we look this argument it appears that using some further results of [48, Chap.VIII], one might construct examples of operators with nonunique spectral measures. We shall meet this problem again below. In any case, if we agree that a function calculus may depend on the spectral measure involved, we shall not have troubles with ambiguity. We state the following for completeness (the function theoretic notions used here are defined for example in [17]).

Proposition 4.2 *Let T be a scalar operator with compact spectrum, and suppose that the function algebra $R(\sigma(T))$ (uniform closure of rational functions with poles off $\sigma(T)$) has one of the following properties:*

(i) $\sigma(T)$ is the Choquet boundary of $R(\sigma(T))$

(ii) The planar measure of $\sigma(T)$ is zero.

(iii) $\sigma(T) \setminus E$ has planar measure zero where E is the set of peak points of $R(\sigma(T))$.

Then the spectral measure for T is unique.

The proof consists in noting that any of (i), (ii) or (iii) implies that $R(\sigma(T)) = C(\sigma(T))$. (Unfortunately, (i), (ii) and (iii) are not "topological" properties of $\sigma(T)$; that is, not invariant under homeomorphism, and neither is "$R(\sigma(T)) = C(\sigma(T))$" a topological property of $\sigma(T)$, and there is no "geometric" property of $\sigma(T)$ known to be equivalent to "$R(\sigma(T)) = C(\sigma(T))$".

Remark 4.3 Let $T_1, \ldots, T_n$ be continuous scalar operators with compact spectrum and let f be a Borel measurable function of n complex variables. Then we define

$$f(T_1, \ldots, T_n)x = \int_{C^n} f(z_1, \ldots, z_n)\, d\left(\mathop{\times}_{i=1}^{N} \mu_i\right)$$

provided $\mathop{\times}_{i=1}^{n} \mu_i$ exists and

$$\int f(z_1, \ldots, z_n) d\left(\mathop{\times}_{i=1}^{N} \mu_i\right)$$

exists (Remark 4.2). It is also possible to define functions of scalar operators with unbounded spectrum. In this case the definition above results in a slight problem: the case where polynomial functions are considered. The definition above gives an extension of the natural operator one associates with the polynomial. We state this as:

Proposition 4.3 *Let $T_1, \ldots, T_n$ be scalar operators whose spectral measures commute and whose product exists as in Remark 4.3. If f is a Borel measurable function on $\mathbb{C}^n$, then $f(T_1, \ldots, T_n)$ is a scalar operator with complex spectral measure, and for any polynomial Q in n variables, $Q(T_1, \ldots, T_n)$ defines an operator with a scalar extension.*

Proof. If $\mu = \mathop{\times}_{i=1}^{n} \mu_i$, and

$$f(T_1, \ldots, T_n)x = \int_{\mathbb{C}^n} f d\mu_x,$$

then Remark 1.8 implies that $f(T_1, \ldots, T_n)$ is a scalar operator with spectrum in $f(\text{supp}(\mu))$.

We can extend Proposition 1.16 as

Proposition 4.4 *Let the hypothesis of Proposition 4.3 on* $\mu = \overset{n}{\underset{i=1}{\times}} \mu_i$ *hold, and suppose* f *and* g *are measurable on* $\mathbb{C}^n$*. Then*

(a) $D(f(T_1, \ldots, T_n)) = D(|f|(T_1, \ldots, T_n))$.

(b) $D(f(T_1, \ldots, T_n)) \subseteq D(g(T_1, \ldots, T_n))$ *if* $|f| \leq |g|$.

(c) $f(T_1, \ldots, T_n) \in L(M)$*, if* f *is bounded.*

(d) $(f + g)(T_1, \ldots, T_n) \supseteq f(T_1, \ldots, T_n) + g(T_1, \ldots, T_n)$.

(e) $D(f(T_1, \ldots, T_n) + g(T_1, \ldots, T_n)) = D((f+g)(T_1, \ldots, T_n)) \cap D(f(T_1, \ldots, T_n))$.

(f) $D((fg)(T_1, \ldots, T_n)) \cap D(g(T_1, \ldots, T_n)) = D(f(T_1, \ldots, T_n) g(T_1, \ldots, T_n))$.

(g) $f(T_1, \ldots, T_n)\mu_i(\delta) \supseteq \mu_i(\delta) f(T_1, \ldots, T_n)$ *for all Borel sets* δ *and all* $i = 1, \ldots, n$.

(h) $f(T_1, \ldots, T_n) \supseteq \mu(\delta) f(T_1, \ldots, T_n)$.

Remark 4.4 In Proposition 4.4 the order in which integration is carried out is immaterial. That is, $\mu(E) = (\overset{n}{\underset{i=1}{\times}} \mu_i)(E)$ where the factors μ_i may occur in any order.

A useful addition to Proposition 4.4 is to allow the functions to be $\mathbb{C}^n$-valued. The argument may be carried out component-wise.

5 A Radon–Nikodym property

The following interesting result is related to the Radon–Nikodym theorem.

Proposition 5.1 *Let* μ_1 *and* μ_2 *be complex spectral measures with the range of* μ_1 *containing the range of* μ_2 *in* $L(M)$*. Then there exists a Borel measurable function* f *s.t.*

$$\mu_2(E) = \int \chi_E \circ f d\mu_1$$

for all Borel sets E τ *i.e.,* $\mu_2(E) = \mu_1(f^{-1}(E))$.

Proof. Let $\{E_i^n\}_{i=1}^{\infty}$ be a Borel partition of the support of μ_2, with diameter $(E_i^n) \leq \frac{1}{n}$, and let $Z_i^n \in E_i^n$ for each i and n. We shall suppose that $\{E_i^m\}$ refines $\{E_i^n\}$ if $m > n$. Since μ_1 and μ_2 are spectral measures, it is easy to show that we may associate Borel sets $\{F_i^n\}$ for each n to $\{E_i^n\}$ such that if $m > n$, then

(i) for each F_i^m there is a unique set F_{ji}^n such that $F_{ji}^n \supset F_i^m$ and

(ii) $F_i^m \subset F_j^n$ if and only if $E_i^m \subset E_j^n$.

(iii) $\mu_1(F_i^n) = \mu_2(E_i^n)$ for all i, n.

(iv) $F_i^n \cap F_j^n = \emptyset$ if $i \neq j$ (if not, the standard procedure for constructing disjoint sequences preserves (i), (ii), and (iii); (i), (ii), (iii) follow easily from the range assumption on μ_1 and μ_2).

Define $f_n(z) = \sum_{i=1}^{\infty} z_i^n \chi_{F_i^n}(z)$, $z_i^n \in E_i^n$, then $f_n(z) \to 0$ for $z \in \mathbf{C} \setminus \bigcap_{n-1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n$ and $|f_m(z) - f_n(z)| \leq \frac{1}{n}(m > n)$, on $\bigcap_{n-1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n$, so $f_n \to f$ uniformly on $\bigcap_{n-1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n$, and f is bounded and Borel measurable. Let g be continuous with compact support. Let $g_n(z) = \sum_{i=1}^{\infty} g(z_i^n) \chi_{E_i^n}(z)$, we have

$$\int g_n d\mu_2 = \int g \circ f d\mu_1.$$

From 7.2.7 of [47], we may show that $g d\mu_2 = g \circ f d\mu_1$ and so by the argument for uniqueness in Proposition 2.1,

$$\mu_2(E) = \int_{\mathbb{C}} \chi_E \circ f d\mu_1$$

This concludes the proof.

6 Measures with values among the discontinuous operators

6.1 Review

We have shown above that if $\sigma(T)$ (spectrum T) is defined as the complement of

$$\begin{gathered}\{\lambda \mid \text{there is a disc } D_\lambda \text{ centered at } \lambda \text{ and for each piecewise} \\ \text{smooth simple closed curve } c \text{ in } D_\lambda,\ \int_c (zI - T)^{-1} dz = 0, \\ (zI - T)^{-1} \text{ is a continuous operator for } z \in D_\lambda\},\end{gathered} \tag{6.1}$$

then the support of the spectral measure for T is $\sigma(T)$. This was also shown to be the case for discontinuous operators as well. Part of the definition is that the Riemann integral above exists. The Fubini theorem has also been shown to be valid in the cases where we require it.

The Riesz theorem was proved:

Theorem 6.1 *Suppose X is a quasicomplete TVS, $B(X)$ is sequentially complete for the strong operator topology. Let Λ be a continuous mapping from the continuous functions vanishing at "∞" with* sup *norm topology on a locally compact Hausdorff space Ω to $B(X)$. Suppose Λ maps weakly compact sets to relatively compact sets in $B(X)$. Then there exists a regular countably additive Baire measure μ of bounded semivariation such that*

$$\Lambda(f) = \int_\Omega f d\mu. \tag{6.2}$$

Further, if $\Lambda(fg) = \Lambda(f)\Lambda(g)$ then μ is multiplicative.

We shall require a slightly different version to be stated in Section 8 below. We now begin the extension of the theory to a wider class of operators.

6.2 Extension of the Abstract Theory

Let X be a (not necessarily locally convex) topological vector space which is quasicomplete. $B(X)$ denotes the continuous endomorphisms of X. R shall be the class of the Borel sets of a metrizable locally compact Huasdorff space

H (usually the plane). Suppose μ is a set function on R with values in the set of linear mappings from A into X where A is a subspace of X which depends on the value of μ under consideration. If $D(\mu(\delta))$ represents the domain of the operator $\mu(\delta)$ we suppose that there is a subspace K of X such that

$$\bigcap_{\delta \in R} D(\mu(\delta)) \supseteq K \tag{6.3}$$

and that K has the property that if $x \in K$, $\mu(\cdot)(x)$ is a countably additive X-valued measure which has bounded semivariation. We say that μ is *densely defined* if K is dense in X and if $\mu(\delta)(x) \in K$ for all $\delta \in R$ and $x \in K$. If, in addition,

$$\mu(\delta_1)(\mu(\delta_2)(x)) = \mu(\delta_1 \cap \delta_2)(x) \tag{6.4}$$

for all δ_1, δ_2 in R and x in K, then we say μ is a densely defined spectral measure. It is not necessary to assume K is dense in X but then our results become localized to a greater extent. Such measures where K is not dense will be called "locally defined."

Multiplicative nature of $\int f d\mu$.

We write $\mu_x(\cdot)$ for $\mu(\cdot)(x)$ where μ is some densely defined or locally defined spectral measure, and if the expression for x is complicated then we write $\mu_{(\ )}(\cdot)(z)$ where z is the expression.

Suppose f and g are simple Borel functions. Then it is easily shown that ($x \in K$)

$$\int f d\mu_{(\ )}\left(\int g d\mu_x\right) = \int f g d\mu_x = \int g d\mu_{(\ )}\left(\int f d\mu_x\right). \tag{6.5}$$

Now let g approach a bounded Borel function (which we still call g) uniformly. Then (6.5) defines

$$\int f d\mu_{(\ )}\left(\int g d\mu_x\right). \tag{6.6}$$

In particular, for $f = \chi_\delta$ (characteristic function of $\delta \in R$) (6.5) defines

$$\mu_{(\ )}(\delta)\left(\int g d\mu_x\right). \tag{6.7}$$

From (6.5) it follows that the integral (6.6) is a map, continuous on the simple functions with the topology of uniform convergence and therefore

$$\mu_{(\ )}(\cdot)\left(\int g d\mu_x\right) \tag{6.8}$$

has bounded semivariation. We may and shall suppose therefore that

$$\mu_{(\ )}(\delta)\left(\int g d\mu_x\right) \in K. \tag{6.9}$$

Definition 6.1 If μ is a densely defined spectral measure, the *support* of $\mu = \operatorname{supp}(\mu)$ is defined as the closure of the union of the supports of the measures μ_x, $x \in K$.

Remark 6.1 It will be convenient in applications to consider an operator T as "having a densely (or locally) defined spectral measure μ" if $D(T) \subseteq X' \subseteq X$ and

$$Tx = \int_H \lambda d\mu_x \quad (x \in D(T) \cap K)$$

where X' is a subspace of X with its induced topology and X' is not necessarily invariant for values of μ. It is also possible that we may wish to place some "weaker" topology on X. In any case we have

Theorem 6.2 *If $T \in L(X)$ and*

$$Tx = \int_H \lambda d\mu_x \quad (x \in K) \tag{6.10}$$

(H is the plane) then

$$\operatorname{supp}(\mu) \subseteq \sigma(T).$$

(Inclusion may be strict.)

Note first that the following lemma can be proved as in Proposition 1.7 and Remark 1.8 above.

Lemma 6.1 *Let f be a Borel function and g be a bounded Borel function, and for $x \in K$, define (B, F are Borel sets)*

$$m_{f,x}(B) = \mu_x(f^{-1}(B) \cap F). \tag{6.11}$$

Then

$$\int_F g \circ f d\mu_x = \int_{f(F)} g dm_{f,x}. \tag{6.12}$$

Proof of Theorem 6.2. If $\lambda \in \rho(T)$ then there exists an open disc D_λ, such that

$$\int_c (z-T)^{-1} dz = 0 \quad (\forall c \subseteq D_\lambda). \tag{6.13}$$

If, on the other hand,

$$\lambda \in \left(\bigcap_{x \in X} \{z \mid \text{there exists } D_z \text{ such that for all } c \subseteq D_z, \right. \tag{6.14}$$

$$\left. \int_c (z - T^{-1}) x dz = 0\} \right)^0,$$

then $\lambda \in \rho(T)$. Here, S^0 means the "interior" of S. However the set in (6.14) is contained in

$$\left(\bigcap_{x \in K} \{z \mid \text{same as in(6.14)}\} \right)^0. \tag{6.15}$$

If $z \in (\text{supp } (\mu))^c$, then $(z-T)^{-1}x$ is

$$\int \frac{d\mu_x(\lambda)}{z-\lambda} \quad (x \in K) \tag{6.16}$$

by (6.5) if z is in (6.15) we may choose a disc D_z with center z such that

$$\int_c (\lambda - T)^{-1} x d\lambda = 0 \quad (c \subseteq D_z,\ x \in K). \tag{6.17}$$

Let $D'_z \subset D_z$ and define

$$\mu_x = \mu_x|_{D'_z} + \mu_x|_{D'^c_z} \quad (x \in K) \tag{6.18}$$

we have, by an argument similar to that of Proposition 1.9 above, that

$$\mu_{x_{D'_z}}(D'_z) = 0 \tag{6.19}$$

so $z \in (\text{supp } (\mu))^c$ and this gives the result.

Remark 6.2 It can be shown that for each $x \in K$ there is a unique open maximal continuous extension of

$$\int \frac{d\mu_x(z)}{\lambda - z} \quad (\lambda \notin \text{supp}(\mu)). \tag{6.20}$$

Let $f_1(\lambda),\ f_2(\lambda)$ be any two extensions with range in K and such that $\mu(\cdot)$ is continuous relative to $f_1(\lambda),\ f_2(\lambda)$. Define $f(z) = f_1(z) - f_2(z)$ in the intersection of the domains of f_1 and f_2. Suppose D_{z_0} is a disc such that $f(z) \neq 0$ in D_{z_0}. Choose $z_n \to z_0$ in D_{z_0}. $(z_0 I - T)f(z_0) = 0$ and

$$\int_M \frac{d\mu_{()}(z)}{z_0 - z}(z_0 I - T)\mu_x(M) = \mu_x(M) \tag{6.21}$$

where M is closed, $z_0 \notin M$. For $x = f(z)$,

$$(z_0 I - T)\mu_x(M) = \mu_{()}(M)(z_0 I - T)(x) = 0 \tag{6.22}$$

hence $\mu_x(M) = 0$. Whence taking $M = \{z_n\}$ and $x = f(z_n)$,

$$0 = \mu\{z_0\}(\mu(\{z_n\})f(z_n)) = \mu(\{z_0\})f(z_n) \to \mu(\{z_0\})f(z_0). \tag{6.23}$$

Let M_n be an increasing sequence of closed sets, $\bigcup_n M_n = \mathbf{C} \setminus \{z_0\}$. Then

$$0 = \mu(M_n)f(z_0) \to \mu\left(\bigcup_n M_n\right) f(z_0) = 0 \tag{6.24}$$

so

$$\mu(\{z_0\})f(z_0) = f(z_0) = 0,$$

a contradiction.

Theorem 6.3 *Suppose T is an operator with dense domain $D(T)$ where $D(T) \subseteq X' \subseteq X$. Suppose also that T has a densely defined spectral measure μ so that*

$$T(x) = \int \lambda d\mu_x \ (x \in K \subseteq D(T)), \tag{6.25}$$

then

(i) $\text{supp}(\mu) \subseteq \sigma(T)$.

(ii) *For any measurable complex-valued function f, there exists a dense subspace K_f of K such that if $x \in K_f$, then*

$$\int f d\mu_x \tag{6.26}$$

exists,

(iii) *If f is as in (ii), then there exists a densely defined measure v_f such that*

$$v_{f,x}(B) = \mu(f^{-1}(B) \cap F)(x); \int_F f d\mu_x = \int_{f(F)} \lambda dv_{f,x} \ (x \in K_f).$$

(iv) *For polynomials P,*

$$P(\text{supp } (\mu)) \subseteq \sigma(P(T)). \tag{6.27}$$

(v) *If f and g are measurable functions, there exist dense subspaces K_f, $K_{f,g}$, etc., such that in and for these subspaces*

(a) $K_f = K_{|f|}$,

(b) $K_{f,g} \subset K_{f+g}, \ K_{f,g} \subset K_{fg}$,

(c) $\int f d\mu_x$ *exists if* $x = \int g d\mu_y, y \in K_{f,g}$,

(d) $\int f \circ g d\mu_x = \int f dv_{g,x}, \quad (x \in K_{f\circ g})$,

(e) $\int (f+g) d\mu_x = \int f d\mu_x + \int g d\mu_x, \quad (x \in K_{f,g})$,

(f) $\int fg d\mu_x = \int f d\mu_{()}(\int g d\mu_x), \quad (x \in K_{f,g})$,

(g) *if* $f_n(t) \to f(t), \ f_n, \ f$ *Borel functions then*

$$\int f_n d\mu_x \to \int f d\mu_x$$

for all x in a dense subset of K,

(h) *if* $|f| \leq |g|$ *then* $K_f \supset K_g$, *and if* $f_n(t) \to f(t), |f_n(t)| \leq |g(t)$ *then* $\int f d\mu_x$

exists for all x for which $\int g d\mu_x$ exists and $\lim\limits_{n\to\infty} \int f_n d\mu_x = \int f d\mu_x$ for such x.

Proof.

(ii) Let

$$K_f = \left\{ x \,|\, x \in \bigcup_{n=1}^{\infty} K_n \right\},\ K_n = \mu(\delta_n)(K),\ \delta_n = \left\{ t \big| \, |f(t)| \le n \right\}.$$

Since $\delta_n \to \mathbb{C}$ if $x \in K$, then there is a sequence $\{x_n\} \subseteq K_f$ such that $x_n \to x$; therefore, $K_f \supseteq K$.

(iii) For $x \in K_f$, $\int_F f d\mu_x$ exists and in the right-hand integral we are integrating over $f(F) \cap f(\delta_n)$ for some n. The equation follows from Lemma 6.1.

(iv) If $p(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ then

$$\int p(\lambda) d\mu_x = \int_{p(\mathrm{supp}(\mu))} \lambda dv_{p,x} \tag{6.28}$$

and Theorem 6.2 shows (i) with only a slight modification.

(v) If f and g are measurable, $K_f = K_{|f|}$ is obvious. Define $K_{f,g} = \bigcup_1^{\infty} K_n$ where

$$K_n = \mu(\delta_n)(K),\, \delta_n = \{t \,\big|\, |f(t)| \le n\} \cap \{t \,\big|\, |g(t)| \le n\} \tag{6.29}$$

then (b) holds. For (c) we note that

$$\int fg d\mu_x = \int f d\mu_{(\ )} \left(\int g d\mu_x \right) \tag{6.30}$$

for simple f(g is bound for all practical purposes) and the result follows since the left-hand side is continuous in uniformly convergent sequences $\{f_n\}$. Thus (f) follows too. (e) certainly holds by definition of $K_{f,g}$. For (d) choose

$$\delta_n = \{s \,\big|\, |f(s)| \le n\} \cap f^{-1}\{s \,\big|\, |f(s)| \le n\}. \tag{6.31}$$

For (g) choose

$$\delta_n = \{s \,\big|\, |f(s)| \le n\} \cap \{s \,\big|\, |f_k(s) - f(s)| < 1\}. \tag{6.32}$$

(h) follows from the proof of Proposition. This completes the proof.

NOTE. Stronger results can be had provided one is willing to assume something of the sort:

$$\lim_{n\to\infty}\int f_n d\mu_x \text{ exists} \tag{6.33}$$

implies

$$\lim_{n\to\infty}\int g d\mu_{(\ )}\left(\int f_n d\mu_x\right) \text{ exists} \tag{6.34}$$

for bounded g. See Proposition 1.13 above.

6.3 Examples

Theorem 6.3 develops the rudiments of an operational calculus. We might define

$$f(T)(x) = \int f d\mu_x \quad (x \in K_f)$$

for a Borel function f. It can be shown that $f(T)$ may be discontinuous even when f is bounded on $\operatorname{supp}(\mu)$ and T is continuous. This is because the representation

$$f(T)(x) = \int f d\mu_x$$

may not be defined for all $x \in D(T)$.

We now illustrate the foregoing theory with a family of examples in a non-locally convex space.

Example 6.1 Consider the operators T_f defined on $L^p[0,1](0 < p < 1)$ by

$$(T_f x)(t) = x(F(t))f(t) \quad (x \in L^p) \tag{6.35}$$

$$F(t) = \int_0^t |f(s)|^p ds, \quad F(1) = 1, \quad f > 0 \text{ in } \mathrm{L^p}. \tag{6.36}$$

The lack of duality theory for $L^p(0 < p < 1)$ has made these spaces fill a pathological role mathematically (c.f. (1.2) of [16]). The continuous mappings on $L^p(0 < p < 1)$ are essentially of the character of this example.

Theorem 6.4 *Suppose $X = L^p[0,1]$ and $T \in B(X)$. Then $\sigma(T)$ is compact. Suppose $\sigma(T)$ is contained in a Jordan arc, and consider the map (x is fixed)*

$$P_{|\sigma(T)} \to P(T)x \in X, \tag{6.37}$$

where P is a polynomial on $\sigma(T)$. Suppose the map is continuous when its domain is given the sup *norm topology. Then there exists a unique X-valued Borel measure μ_x such that*

$$P(T)x = \int_{\sigma(T)} p(\lambda) d\mu_x(\lambda). \tag{6.38}$$

Proof. Follows from a result in [37] and Theorem 2.2 above. The compactness of $\sigma(T)$ is shown in [11].

Lemma 6.2 *If P and Q are polynomials and the hypotheses above hold on T and further if K_0 is some nontrivial subspace of X and*

$$P_{|_{\sigma(T)}} \to P(T)x \tag{6.39}$$

is continuous for all x in K_0, then

$$x \to \mu_x(\delta) \tag{6.40}$$

defines a linear map with range in X for all δ.

Proof. Consider $x \to \mu_x(\delta)$, which is well defined by uniqueness (see Proposition 1.5) for fixed δ. $x_1 + x_2$ certainly makes $P \to P(T)(x_1 + x_2)$ continuous if x_1 and x_2 do, so we may indeed say K_0 is linear.

$$\begin{aligned} P(T)(x_1 + x_2) = \int_{\sigma(T)} P(\lambda) d\mu_{x_1+x_2} &= \int_{\sigma(T)} P(\lambda) d\mu_{x_1} \\ + \int_{\sigma(T)} P(\lambda) d\mu_{x_2} &= P(T)x_1 + P(T)x_2 \end{aligned} \tag{6.41}$$

and therefore

$$\mu_{x_1+x_2}(B) = \mu_{x_1}(B) + \mu_{x_2}(B)$$

for all compact G_δ sets B and thus for all Borel sets.

Definition 6.2 Let K be the smallest linear manifold containing K_0 which is invariant under all the mappings $\int f d\mu_{(\ )}$ where f is a bounded Borel function.

We must now establish the existence of a subspace such as the one in the Definition 6.2. Theorem 6.4 and the reasoning leading to expression (6.8) show that when for some x, $P \to P(T)x$ is continuous then $\mu_{(\ )}(\delta)(\int f d\mu_x)$ is defined for all bounded Borel functions f. Now if we take $\tilde{K}$ to be the *largest* subspace with the property that $x \in \tilde{K}$ implies

$$P \to P(T)x \tag{6.42}$$

is continuous, then $\tilde{K}$ is invariant under the maps $\int f d\mu_{(\ )}$. This argument extends beyond L^p if the weak compactness property is added to the hypothesis. We have no need of this however. Thus the collection of linear manifolds containing K_0 with the invariance property of K is nonempty and we have

Lemma 6.3 *The subspace* K *exists.*

Lemma 6.4

$$\mu(\delta_1 \cap \delta_2)x = \mu(\delta_1)\mu(\delta_2)x \quad (x \in K). \tag{6.43}$$

Proof. For δ_1, δ_2 compact G_δ sets there exists polynomials P_n, Q_n such that $P_n \to \chi_{\delta_1}$, $Q_n \to \chi_{\delta_2}$ in bounded fashion on $\sigma(T)$. Then

$$\int P_k(\lambda)Q_n(\lambda)d\mu_x \to \int \chi_{\delta_1 \cap \delta_2} d\mu_x,$$

$$\int P_k Q_n d\mu_x = \int P_k d\mu_{Q_n(T)x} \to \int \chi_{\delta_1} d\mu_{Q_n(T)x} = \int Q_n d\mu_{\mu(\delta_1)x} \to \int \chi_{\delta_2} d\mu_{\mu(\delta_1)x}$$

by the proof of Proposition 1.4. Thus

$$\mu(\delta_1 \cap \delta_2)x = \mu(\delta_1)\mu(\delta_2)x$$

and the lemma follows.

Lemma 6.5

$$\int f d\mu_{(\ )}(\int g d\mu_x) = \int fg d\mu_x$$

for all bounded Borel functions and $x \in K$.

Proof. This follows by Lemma 6.4 and the results of section 6.2.

Remark 6.3 Consider again T_f defined in Example 6.1 above. It is easily seen that $\sigma(T_f) = \mathbb{C}_1$, the unit circle. For computational purposes we select a specific $f(t)$:

$$f(t) = (1+kp)^{1/p} t^k, \; 0 < kp+1 < 1.$$

Note that

$$\big|\big|T_f(x)\big|\big|_{L^p}^p = \big|\big|x\big|\big|^p.$$

T_f^{-1} exists and is defined by

$$(T_f^{-1}x)(t) = \frac{X(F^{-1}(t))}{f(t)}, \tag{6.44}$$

F^{-1} is the inverse of F. T_f has a decomposition not unlike that of a spectral operator and is the limit of operators with densely defined measures. To show this, let $1/2 > \delta > 0$. Define

$$(P_\delta x)(t) = \chi_\delta(t)x(t) \tag{6.45}$$

where χ_δ is the characteristic function of $(\delta, 1-\delta)$. The spectrum of $P_\delta T_f$ is contained in the unit disc and includes $\mathbb{C}_1$. Consider $P_\delta L^p[0,1]$ and let T_f act on this subspace of $L^p[0,1]$. $P_\delta L^p$ is not invariant under T_f so we consider the space $P_\delta B(X)$ ($X = L^p$) as acting on $P_\delta L^p$. We shall show that $P_\delta T_f$ has a densely defined measure μ_δ supported on $\mathbb{C}_1$ (thus verifying the parenthetical statement in Theorem 6.2). First suppose $Q(x) = \sum_{-n}^{n} a_k z^k$ is a trigonometric polynomial ($|z| = 1$). Consider the mapping

$$Q \to P_\delta Q(T_f)x, \quad x \in P_\delta(L^p) \tag{6.46}$$

We write T for T_f in what follows.

$$\big|\big|P_\delta Q(T)x\big|\big|^p = \int |\chi_\delta(t)(a_0x(t)+a_{\pm1}T^{\pm1}x(t)+\cdots+a_{\pm n}T^{\pm n}x(t))|^p dt, \tag{6.47}$$

$$T^n x(t) = x(F^n(t)) \prod_{m=0}^{n-1} f(F^m(t)), \; T^{-n}x(t) = \frac{x(F^{-n}(t))}{\prod_{m=0}^{n-1} f(F^{m-n}(t))}$$

where

$$F^n(t) = F(F^{n-1}(t)),\ F^0(t) = f(t)F^{-n}(t) = F^{-1}(F^{-n+1}(t)).$$

It can be shown that $P_\delta T^{\pm k}x(t) = 0$ for all that t and all large $\pm k$ (how large depends on δ, see 6.56 below). Parseval's equality therefore shows that $Q \to P_\delta P(T)x$ is continuous and therefore by (a slight modification of) Theorem 6.4, we might have our measure μ_δ. It can be shown that the same argument holds for $x \in P_\beta L^p[0,1]$ whether $\beta = \delta$ or not. Thus μ_δ is densely defined ($K = \bigcup_{1/2>\delta>0} P_\delta L^p[0,1]$). If χ_δ, is the characteristic function of $[0,1]\backslash(\delta,\, 1-\delta)$, then

$$T_f = P_\delta T_f + p_{\delta'} T_f. \tag{6.48}$$

Define $N_\delta = P_{\delta'} T_f$. Then

$$T_f = \int_{\mathbb{C}_1} \lambda d\mu_\delta(\lambda) + N_\delta \tag{6.49}$$

(on K). Therefore T_f has a decomposition on a dense subspace similar to that of a spectral operator. It is clear that $N_\delta \to 0$ as $\delta \to 0$. However N_δ is not "quasi-nilpotent", $P_\delta T_f$ (pointwise) is! We do have

$$\lim_{\delta\to 0} \int_{\mathbb{C}_1} \lambda d\mu_\delta\, x = T_f x. \tag{6.50}$$

The question of computing μ is a vital one in any application and since the solution of this problem for T_f leads to an interesting functional equation we include it here.

Proposition 6.1 *Let $T = T_f$. If (e^{is_1}, e^{is_2}) is an "interval" (counterclockwise) on C_1, then $x \in K$ implies*

$$\mu_\delta(s_1, s_2)\, x = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{s_2}^{s_1} [R_\delta(T,\ (1+\varepsilon)e^{i\theta}) - R_\delta(T, (1-\varepsilon)e^{i\theta})]\, x d(e^{i\theta}). \tag{6.51}$$

Proof. Follows immediately from (6.39)-(6.43) ($R_\delta = P_\delta R$) and the Cauchy integral theorem (see Proposition 1.5).

To use (6.51) we must compute $(\lambda I - T)^{-1}$ directly since in $L^p(0 < p < 1)$ there is no hope for a kernel representation. This means that we must solve the equation

$$\lambda x(t) - x(F(t))f(t) = U(t) \tag{6.52}$$

for $x \in L^p$, $\lambda \in \mathbb{C}$. It is natural to expect two solutions, one for $|\lambda| > 1$ one for $|\lambda| < 1$. They are

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} \lambda^{-(n+1)} U(F^n(t)) \prod_{m=0}^{n-1} f(F^{n-1-m}(t)) \quad (|\lambda| > 1), \\ -x(t) &= \sum_{n=0}^{\infty} \lambda^n \frac{U(F^{-(n+1)}(t))}{\prod_{m=0}^{n} f(F^{m-(n+1)}(t))} \quad (|\lambda| < 1). \end{aligned} \tag{6.53}$$

Formally then

$$\begin{aligned} &(\mu_\sigma(s_1, s_2)x(t)) = \\ &= \lim_{\epsilon \to 0} P_\delta \sum_{n=0}^{\infty} \left\{ \left(\frac{b_{n+1}(t)(1+\varepsilon)^{n+1}}{n+1} + \frac{a_n(t)}{(1+\epsilon)^{n+1}(n+1)} \right) [\cos(n+1)s_2 - \right. \\ &\quad - \cos(n+1)s_1] + \left(\frac{a_n(t)}{(1+\epsilon)^{n+1}(n+1)} - \frac{b_{n+1}(t)(1-\epsilon)^{n+1}}{n+1} \right) \\ &\quad \left. i[\sin(n-1)s_2 - \sin(n+1)s_1] \right\} + s_2 - s_1 \end{aligned} \tag{6.54}$$

where

$$\begin{aligned} a_n(t) &= U(F^n(t)) \prod_{m=0}^{n-1} f(F^{n-1-m}(t)) \\ & \qquad (n = 1, 2, \ldots, t \in (0,1)) \\ b_n(t) &= \frac{U(F^{-n-1}(t))}{\prod_{m=0}^{n} f(F^{m-(n+1)}(t))} \end{aligned} \tag{6.55}$$

Given our choice of $f(t)$, we have

$$\begin{aligned} a_n(t) &= U(t^{(kp+1)^n})(1+kp)^{n/p}t^k \left(\frac{1-(kp+1)^n}{-kp}\right), \\ b_n(t) &= U(t^{(kp+1)^{-(n+1)}}) \cdot \frac{1}{(1+kp)^{(n+1)/p}t^{1/p}\dfrac{1-(kp+1)^{-(n+1)}}{1-(kp+1)^{-1}}} \end{aligned} \tag{6.56}$$

For

$$U \in K = \bigcup_{\frac{1}{2}>\delta>0} P_\delta(L^p[0,1])$$

we have $U(t) \equiv 0$ for

$$t \in (0, \delta(U)) \cup (1-\delta(U), 1)$$

where $\delta(U)$ depends on U. Since $t^{(kp+1)^n} \to 1$ and $t^{(kp+1)^{-n}} \to \infty$, we see that if $t \in (\delta, 1-\delta)$ then the terms in (6.56), and hence in (6.54) are eventually zero. That is,

$$(P_\delta a_n(t))^2 + (P_\delta b_n(t))^2 = 0.$$

It is *not* true that $(a_n(t))^2 + (b_n(t))^2$ vanishes. Thus μ_δ may be computed explicitly and so many projections of solutions to (6.53). Of course, the series (6.53) converges in L^p. It may diverge pointwise as (6.56) shows.

Theorem 6.5 *Let T be an isometry of $L^p[0,1]$ and suppose $\sigma : [0,1] \to [0,1]$ is measure preserving and $g : [0,1] \to \mathbb{R}^1$ is Lebesgue measurable with $g \geq 0$. Suppose*

$$(Tx)(t) = x(\sigma(t))g(t)$$

where $\sigma(t) \neq t$ except perhaps on a nowhere dense (countable) set. Suppose $g(t)$ is bounded away from zero except on a set of a measure zero. Then T has a "spectral" decomposition of the form

$$T = \int_{\sigma(T)} \lambda d\mu_\delta + N_\delta \tag{6.57}$$

with $N_\delta \to 0$ as $\delta \to 0$.

Proof. A Relatively easy modification of the previous discussion. The points $\sigma(t) = t$ become "nodal points" which must be "projected out". The result may be extended in various ways at the cost of simplicity (to Orlicz spaces for example). We may extend

$$f(T) = \lim_{\delta \to 0} \int f(\lambda) d\mu_\delta \tag{6.58}$$

(f a Borel function), when it exists, to all of L^p in case f is analytic on $\sigma(T)$ (via the Dunford integral, see [11] to see that the integral of the resolvent exists). This is also the case with many differential operators, i.e., we must sacrifice generality of the operational calculus to obtain everywhere defined spectral integrals.

7 Differential operators

For differential operators, the situation is somewhat different than in the preceding examples. The reason for this is that most differential operators are treated in the context of Hilbert spaces. "Most" of the projectors in the range of the associated measure are bounded (but not Hermitian in general). This makes it possible to obtain *complete* eigenfunction expansions in certain cases (see [12]).

For a detailed treatment of some of these problems, we refer the reader to [41] and [43].

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