# A LOCAL LIMITING ABSORPTION THEOREM IN A SINGULAR DISPERSIVE MEDIUM

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SUMMARY

This paper studies a generalization of the steady-state wave propagation problem derived from the classical wave equation

$$-iE(x)\,\partial_t u = \sum_{j=1}^{n} -iA_j \frac{\partial u}{\partial x_j} + B(x)u + f(x,t),$$

where $E(x)$ is a singular matrix. The solution is only approximate in the low-frequency case but is asymptotic to a projection of the free-space solution. For sufficiently high frequencies a 'classical solution' exists. The analysis parallels our previous work on the constant-coefficient case except that we assume that singular fields may not escape to become non-singular fields.

## 0. Introduction

ASPECTS of the spectral theory and solutions of first-order systems of partial differential equations have been studied extensively.

It is well known that the wave-propagation problems of classical physics may be posed as questions about first-order systems of Friedrichs–Wilcox type:

$$-iE(x)\,\partial_t u = A(D)u + Bu - f(x,t). \tag{0.1}$$

Here $E(x)$ is an $m \times m$ matrix function, $A(D) = -i\sum_{j=1}^{n} A_j\,\partial/\partial x_j$, the $A_i$ are constant $m \times m$ symmetric real matrices and $B(x)$ is a dispersion term of local type.

The case when $E$ is positive definite and $B \equiv 0$ has been particularly well studied since it falls under the classical theory of selfadjoint operators in Hilbert space. A few special cases of non-zero dispersion have been studied (see (**1**)).

If $E$ is not definite then (0.1) is not hyperbolic in the usual sense. The energy density

$$\tfrac{1}{2}(Eu, v) \tag{0.2}$$

is indeterminate.

The energy density identifies a certain set of disturbances which do not propagate. Such stationary fields will be called *singular*. Since the decreasing functions then for $f(x) \in \mathcal{S}$, $\hat{f}(p) = (\Phi f)(p)$ is defined by

$$(\Phi f)(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ip\circ x} f(x)\, dx.$$

The adjoint $(\Phi^* f)(p) = (\Phi f)(-p)$, and $\Phi$ has a unitary extension to $\mathcal{H}$. The projection $P_0$ in $\mathcal{H}$ has the form $\Phi^* \hat{P}_0(p)\Phi$. We define $P = I - P_0$.

The weighted spaces $L_{2,\pm\alpha}$ are defined by

$$\left\{ f \,\middle|\, \int_{\mathbb{R}^n} (1+|x|^2)^{\pm\alpha} |f(x)|^2\, dx < \infty \right\}.$$

Note that $L_{2,\alpha}$ and $L_{2,-\alpha}$ are dual to each other by the usual 'inner product'. Since $P$ is a $C^\infty$-projection on $\mathbb{C}^m$, an elementary argument shows that for $\alpha$ an integer,

$$\|Pf\|_{L_{2,\alpha}} = \|Pf\|_{H_{2,\alpha}} \leq C\, \|f\|_{H_{2,\alpha}} = C\, \|f\|_{L_{2,\alpha}}.$$

Interpolation gives the result for $\alpha$ real. Now $P$ is selfadjoint on $L_2$, and by duality of $L_{2,\alpha}$ and $L_{2,-\alpha}$, the restriction of $P$ to $L_{2,\alpha}$ has an $L_2$ inner product bounded adjoint $P^*$ on $L_{2,-\alpha}$,

$$(Pf, g)_{L_2} = (f, P^* g)_{L_2}; \qquad f \in L_{2,\alpha},\ g \in L_{2,-\alpha}.$$

This shows that $P^* = P$ on $L_2$ and therefore $P$ has a bounded unique extension to $L_{2,-\alpha}$.

It is evident from our introductory remarks that

$$\text{range}\,(P_{1_{\mathcal{H}_E}}) \subseteq \mathcal{H}_E.$$

We may and shall speak of similar weighted spaces relative to $\mathcal{H}_E$. Here $\mathcal{H}_E$ refers to the space defined by (1.7).

As the remarks in the introduction suggest, we define $E = E(x)$ as a matrix-valued function which is non-negative and Hermitian in $\mathbb{C}^m$,

$$(Ev, v)_{\mathbb{C}^m} \geq 0, \qquad v \in \mathbb{C}^m. \tag{1.5}$$

We further assume a certain uniformity: there is a constant non-negative matrix $E_1$, with

$$c(E_1 v, v) \leq (Ev, v) \leq d(E_1 v, v), \tag{1.6}$$

where $c, d > 0$.

As we noted in the introduction, we assume that the 'range' and 'null space' of $E$ are 'preserved' by $A$. The same therefore is true for $E_1$. The range and null space of $E$ (and $E_1$) are orthogonal in $\mathcal{H}$.

The notation $\mathcal{H} = L_2(\mathbb{R}^n, \mathbb{C}^m)$ refers to the collection of Lebesgue-measurable, square-integrable functions defined on $\mathbb{R}^n$ with range in $\mathbb{C}^m$. Now $E$ generates a seminorm on $\mathcal{H}$ via

$$\|f\|_E^2 = \int_{\mathbb{R}^n} (E(x)f(x), f(x))_{\mathbb{C}^m}\, dx. \tag{1.7}$$

Since $\ker(E)^{\perp} = \text{range}(E)$, it follows that $\|\cdot\|_E$ is a norm on range $(E)$ acting in $\mathcal{H}$.

It is a well-known fact that $A(D)$ is selfadjoint on $\mathcal{H}$ and by our assumption, on range $(E)$. Now $E(x)$ is an $m \times m$ matrix function and our assumption (1.6) implies that it has constant rank $r$, $0 < r \leq m$.

LEMMA 1.1. *There is an $r \times m$ matrix $C(x)$ with full rank and $E = C^*C$.*

LEMMA 1.2. *$F(x) = C^*(CC^*)^{-1}$ is a right inverse for $C$.*

LEMMA 1.3. *The operator $F: L_2(\mathbb{R}^n, \mathbb{C}^r) \to \text{range}(E)$ is unitary when range $(E)$ has the $\|\cdot\|_E$-norm.*

Define the domain of $FF^*A(D)$ by

$$\mathcal{D}(FF^*A(D)) = \{v \in \mathcal{H}_E \mid v \in \mathcal{D}(A(D))\}. \tag{1.8}$$

LEMMA 1.4. *$FF^*A(D)$ is a selfadjoint operator on $\mathcal{H}_E$ with the $\|\cdot\|_E$-norm.*

*Proof.* Suppose that $u$, $v$ belong to the domain of $FF^*A(D)$;

$$(FF^*A(D)n, Ev) = (A(D)u, FF^*Ev) = (u, A(D)FF^*Ev) = (u, A(D)v).$$

The last equality holds since if $v \in \text{range}(E)$,

$$FF^*Ev = FF^*EEk = C^*(CC^*)^{-1}(CC^*)^{-1}CC^*C^*k = C^*Ck = v. \tag{1.9}$$

However,

$$EFF^* = C^*CC^*(CC^*)^{-1}(CC^*)^{-1}C = C^*(CC^*)^{-1}C. \tag{1.10}$$

It is known that $A(D)$ is selfadjoint in $\mathcal{H}$. And since $A(D)$ fixes $\mathcal{H}_E$, the same is true on $\mathcal{H}_E$ with the $L_2$ norm. If $v \in \text{range}(E)$, then $A(D)v = Ek$ for some $k$;

$$EFF^*A(D)v = C^*(CC^*)^{-1}CC^*Ck = Ek = A(D)v. \tag{1.11}$$

By (1.11) and (1.9) $FF^*A(D)$ is symmetric on $\mathcal{H}_E$. Therefore $(FF^*A(D))^*$ is an extension of $FF^*A(D)$ on $\mathcal{H}_E$ with the $\|\cdot\|_E$ norm.

Suppose that $g \in \mathcal{D}((FF^*A(D))^*) \subseteq \mathcal{H}_E$. Then, for $u \in \mathcal{D}(FF^*A(D))$,

$$(FF^*A(D)u, g)_E = (u, h)_E, \qquad h = (FF^*A(D))^*g.$$

But $(FF^*A(D)u, g)_E = (FF^*A(D)u, Eg) = (A(D)u, g) = (u, h)_E = (u, Eh)$ but this implies that $g \in \mathcal{D}(A(D)^*) = \mathcal{D}(A(D))$ or $(u, A(D)g) = (u, h)_E$. Therefore $h = FF^*A(D)g$. This completes the proof.

It is easily seen that $EFF^*$ $(=FF^*E)$ is constant and equal to $E_1F_1F_1^*$ $(=F_1F_1^*E_1)$. We shall assume that, for some $\varepsilon > 0$,

$$|E - E_1| = O(|x|^{-1-\varepsilon}) \tag{1.12}$$

and

$$|B| = O(|x|^{-1-\varepsilon}) \tag{1.13}$$

as $|x| \to \infty$. Here $|\cdot|$ refers to the Euclidean matrix norm (or any equivalent norm). We also assume that $B$ and $E$ are $L^\infty$ functions. This is altogether reasonable from a physical point of view.

Upon replacing $E$ by $F_1^* E F_1$ and $A(D)$ by $F_1^* A(D) F_1$, we may and shall suppose that

$$E_1 = EFF^* = I_{1_{\mathcal{H}_E}},$$

the identity restricted to $\mathcal{H}_E$.

Note also that

$$|FFF^* - F_1 F_1^*| = |F_1 F_1^*(E_1 - E) F_1 F_1^*| = O(|x|^{-1-\varepsilon}), \qquad |x| \to \infty.$$

It is important to note that because $A(D)$ fixes $\mathcal{H}_E$, the roots of

$$\det\,(I_{1_{\mathcal{H}_E}} A(p) - \lambda I_{1_{\mathcal{H}_E}})$$

have the same properties as (1.2). This in turn means that our results are applicable when $F_1^* A(D) F_1$ replaces $A(D)$.

## 2. Properties of $A(D)$

It is important to develop certain properties of the resolvent of $A(D)$. As already noted, $A(D)$ is a selfadjoint operator when considered as acting in $\mathcal{H}$. Define $\mathbb{C}^+ = \{z \mid \operatorname{im} z \geq 0\}$, $\mathbb{C}^- = \{z \mid \operatorname{im} z \leq 0\}$. Yajima and Ikebe have shown the following (**7**, **6**).

THEOREM 2.1. *Suppose that* $\alpha, \beta > \frac{1}{2}$. *Then* $\lambda \to P(A(D) - \lambda I)^{-1}$ *is a continuous operator-valued function in* $\mathbb{C}^+ \setminus \{0\}$ *or* $\mathbb{C}^- \setminus \{0\}$. *The operators* $P(A(D) - \lambda I)^{-1}$ *are compact with domain* $L_{2,\alpha}$ *and range in* $L_{2,-\beta}$.

COROLLARY 2.2. $\lambda \to P(A(D) - \lambda I)^{-1}$ *is holomorphic in* $\mathbb{C}^+ \setminus \mathbb{R}$ *or* $\mathbb{C}^- \setminus \mathbb{R}$.

*Proof.* Note that $P(A(D) - \lambda I)^{-1}(L_{2,\alpha}) \subseteq \mathcal{H}$. Hence the first resolvent equation is quite valid.

The function $\lambda \to P(A(D) - \lambda I)^{-1}$ is discontinuous across the real axis. (It therefore is not an entire function.) Notice further that in $\mathcal{H}$

$$(A(D) - \lambda I)^{-1} = (P + P_0)(A(D) - \lambda)^{-1} \tag{2.1}$$

$$= P(A(D) - \lambda)^{-1} - (1/\lambda) P_0. \tag{2.2}$$

The same decomposition is then valid on $L_{2,\alpha}$ to

$$L_{2,-\alpha}, \qquad L_{2,\alpha} \supset \mathcal{H} \subset L_{2,-\beta}.$$

We are indebted to Professor J. C. Hoover for pointing out the existence of (**6**) making a rather awkward proof of Theorem 2.1 obsolete.

## 3. Limiting absorption for (0.1)

We recall equation (0.1):

$$-iE\,\partial_t u = A(D)u + Bu - f(x, t),$$

and its analogue in $\mathscr{H}_E$:

$$-I_{1\mathscr{H}_E} i\,\partial_t u = FF^*A(D)u + FF^*Bu - f'(x,t), \tag{3.1}$$

where $f' = FF^*f$.

If $f(x, t)$, the source field of the medium, is a separable sinusoidal disturbance, that is

$$f(x,t) = e^{-i\lambda t} f(x), \tag{3.2}$$

then (3.1) becomes

$$\lambda e^{-i\lambda t} u(x,\lambda) = FF^*A(D)u + FF^*Bu - e^{-i\lambda t} f'(x), \tag{3.3}$$

assuming the solution $u(x, t)$ takes the form $e^{-i\lambda t}u(x, \lambda)$ after any transient response has died away. Equation (3.3) is equivalent to

$$FF^*A(D)u(x,\lambda) + FF^*Bu(x,\lambda) - \lambda u(x,\lambda) = f'(x). \tag{3.4}$$

Actually $\lambda u$ means $\lambda I_{\mathscr{H}_E} u$ but since we are seeking a solution in $\mathscr{H}_E$ we write $u$ for $I_E u$.

It is wise to note the distinction between the usual theory of operators in Hilbert space and our present method. In formulating a limiting absorption principle we consider $A(D)$, $FF^*A(D) + FF^*B$, etc. acting from a space of type $L_{2,-\beta}$ to a space of type $L_{2,\alpha}$. Their 'resolvents' then reverse the action. Thus eigenvalue and eigenvector refer to a space of the type $L_{2,-\beta}$. It is a famous result of Agmon (see (**8**, **7**)) that $L_{2,-\beta}$ eigenvectors are indeed $L_2$ eigenvectors under certain conditions. (We do not need Agmon's result here but we believe it does extend to our problem.)

To devise a limiting absorption principle, we must solve the following equation for $u$:

$$FF^*A(D)u + F^*FBu - \lambda u = f. \tag{3.5}$$

Write $I_E$ for $I_{1\mathscr{H}_E}$, then (3.5) gives

$$I_E A(D)u + I_E Bu - \lambda Eu = Ef. \tag{3.6}$$

Thus

$$(A(D) - \lambda I)^{-1}[I_E A(D)u + I_E Bu - \lambda Eu] = (A(D) - \lambda I)^{-1} Ef. \tag{3.7}$$

Therefore,

$$I_E u + \lambda I_E (A(D) - \lambda I_E)^{-1} u + I_E (A(D) - \lambda I_E)^{-1} Bu - \lambda (A(D) - \lambda I_E)^{-1} Eu = (A(D) - \lambda I)^{-1} Ef \tag{3.8}$$

and

$$[I_E + \lambda (A(D) - \lambda I)^{-1}(I_E - E + I_E B\lambda^{-1})]u = (A(D) - \lambda I)^{-1} Ef. \tag{3.9}$$

By (2.2),

$$\left[I_E - I_E P_0(I_E - E + I_E B\lambda^{-1}) + I_E P\lambda (A(D) - \lambda I_E)^{-1}(I_E - E + I_E B\lambda^{-1})\right]u = (A(D) - \lambda I)^{-1} Ef. \tag{3.10}$$

Note that $I_E P_0 I_E = P_0 I_E$. At least formally,

$$u(x, \lambda)_{\pm} = [\ ]^{-1}(A(D) - \lambda I)_{\pm}^{-1} Ef(x). \tag{3.11}$$

In (3.11), [ ] means the left-hand side of (3.10). Compare this with (**4**, 2.14; **1**; **6**). The objective of our work is to give an interpretation of the right-hand side of (3.11). Note that the functions $E$ and $B$ act as bounded operators on the weighted spaces $L_{2,\pm\alpha}$. Using the Fourier transform we define the Sobolev spaces from $L_{2,\pm\alpha}$ spaces.

Definition (**9**). For each real number $\gamma$ define the Borel measure $\nu_\gamma$ on $\mathbb{R}^n$ by

$$d\nu_\gamma = (1 + |x|^2)^\gamma\, dx.$$

The Sobolev space of order $\gamma$ in $\mathcal{H}_E$ is

$$H^\gamma = \{f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^m) \mid \hat{f} \in L_2(d\nu_\gamma, \mathbb{R}^n, \mathbb{C}^m)\}, \tag{3.12}$$
$$\mathcal{H}_E^\gamma = I_E H^\gamma.$$

Note that $I_E$ easily extends to $H^\gamma$. Here $\mathcal{S}'$ is the Schwartz space of tempered distributions on $\mathbb{R}^n$ with 'values' in $\mathbb{C}^m$. A continuous linear operator $G: H^\gamma \to H^{\gamma+\alpha}$ is said to be of order $\alpha$ (see (**9**)).

We define

$$H_{i_G}^\gamma = \{f \in H^\gamma \mid f(x) = 0 \text{ almost everywhere in } R^n \backslash G\} \tag{3.13}$$

The following is a special case of a well-known result for Sobolev spaces.

Lemma 3.1. *Let $\chi_R$ be the characteristic function of the ball of radius $R$ in $\mathbb{R}^n$. If $\mathcal{K}$ is a mapping from $H^\gamma$ to $H^{\gamma+\alpha}$ (where $\gamma$ is real and $\gamma + \alpha > 1$) and $\tilde{I}$ is the injection of $H^{\alpha+2}$ into $L_2$, then $\chi_R \tilde{I}\mathcal{K} H^\gamma \to L_2$ is compact, and* $\chi_R \tilde{I}\mathcal{K}\Phi^* L_{2,\alpha} \to L_2$ is compact.

Lemma 3.2. *If $K_1(\lambda)$ and $K_2(\lambda)$ are bounded linear operators, then*

$$(I - K_1 + K_2)^{-1}$$
$$= (I - K_1)^{-1} - (I - K_1)^{-1}\{I + K_2(I - K_1)^{-1}\}^{-1} K_2 (I - K_1)^{-1} \tag{3.15}$$

*when the operator inverses on the right-hand side exist.*

Lemma 3.3. *Let $P_R = \Phi^* \chi_R \Phi$. Then the operator*

$$P_R I_E + P_R(A(D) - \lambda I)_{\pm}^{-1}(I_E B + \lambda(I_E - E)) \tag{3.16}$$

*has a bounded inverse on $P_R \mathcal{H}_{E,-\beta}$ for $\beta > \frac{1}{2}$ but sufficiently close to $\frac{1}{2}$, except possibly for a set of values of $\lambda$,*

$$\sum_R^{\pm} \subseteq \mathbb{C}^{\pm}.$$

*Proof.* Fix $\delta > 0$ and suppose that $|\lambda| \geq \delta$, $P_R I_E$ acts as the identity on

$P_R\mathcal{H}_{E,-\beta}$. By (2.2),

$$P_R(A(D)-\lambda I)^{-1} = -P_R\dot{P}_0\lambda^{-1} + P_R P(A(D)-\lambda I)^{-1} \tag{3.17}$$

on $\mathcal{H}_E$. The second term in (3.16) is then equal to

$$-P_R P_0\lambda^{-1}(I_E B + \lambda(I_E - E)) + P_R P(A(D)-\lambda I)^{-1}(I_E B + \lambda(I_E - E)). \tag{3.18}$$

Write $\chi_{R'_1}$ for $1-\chi_{R_1}$, then the first term in (3.18) is equal to

$$-P_R P_0(I_E B\chi_{R_1}\lambda^{-1} + (I_E - E)\chi_{R_1}) + P_R P_0(I_E B\chi_{R'_1}\lambda^{-1} + (I_E - E)\chi_{R'_1}). \tag{3.19}$$

By Lemma 3.1, the first term in (3.19) is a compact operator. The second term in (3.19) has small norm on $\mathcal{H}_{E,-\beta}$ if $R_1$ is large by (1.13) and (1.12). The second term in (3.18) is also compact by Theorem 2.1 since for $\beta > \frac{1}{2}$ but close to $\frac{1}{2}$, $I_E B$ and $I_E - E$ are bounded mappings from $\mathcal{H}_{E,-\beta}$ to $\mathcal{H}_{E,\beta}$. By Lemma 3.2, the invertability of (3.16) reduces to checking the right-hand side of (3.15) for

$K_1(\lambda) =$ the second term in (3.19),

$K_2(\lambda) =$ the first term in (3.19) plus the second term of (3.18).

For $R_1$ large, $(I-K_1)^{-1}$ exists via a Neumann expansion (on $P_R\mathcal{H}_{E,-\beta}$). The inverse $\{I + K_2(I-K_1)^{-1}\}^{-1}$ is of the form $\{I + M(\lambda)\}^{-1}$, where $M$ is a compact operator-valued function on $\mathcal{H}_{E,-\beta} = I_E L_{2,-\beta}$. Since $L_{2,-\beta} = L_2(dv_{-\beta}, \mathbb{R}^n, \mathbb{C}^m)$ we may regard $M(\lambda)$ as an operator in Hilbert space. An examination of the proof of (**4**, Lemma 3.8) or a similar argument in (**1**) leads to the conclusion that $\{I + M(\lambda)\}^{-1}$ exists for all $\lambda$ in $\mathbb{C}^{\pm}$ except for a discrete set of values where $\operatorname{im}\lambda \neq 0$ and a nowhere-dense set of linear measure zero, where $\operatorname{im}\lambda = 0$, assuming that $|\lambda| > \delta$. Let us call this set $\sum_{R,\delta}^{\pm}$. It is clear that $\sum_{R,\delta}^{\pm}$ is monotone increasing as $\delta \to 0$. Moreover,

$$\sum_R^{\pm} = \bigcup_{\delta\downarrow 0} \sum_{R,\delta}^{\pm}$$

intersects the real axis in a set of measure zero. Also since

$$\sum_{R,\delta_1}^{\pm} \cap \{z \mid |z| > \max\{\delta_1, \delta_2\}\} = \sum_{R,\delta_2}^{\pm} \cap \{z \mid |z| > \max\{\delta_1, \delta_2\}\},$$

$\sum_R^{\pm}$ is discrete in $\mathbb{C}^{\pm}\backslash\mathbb{R}$, and nowhere dense in $\mathbb{R}$.

This completes the proof of Lemma 3.3 and also proves the following result.

LEMMA 3.4. *$\overset{\pm}{\underset{R}{\Sigma}}$ has no limit point off the real axis and $\overset{\pm}{\underset{R}{\Sigma}} \cap \mathbb{R}$ has linear measure zero and is nowhere dense in $\mathbb{R}$.*

THEOREM 3.5 (Existence of local solutions). *If $R<0$ and $\lambda \notin \overset{\pm}{\underset{R}{\Sigma}}$, then for each $f \in \mathcal{H}_{E,\beta}$, $\beta > \frac{1}{2}$, but sufficiently close to $\frac{1}{2}$ there exists a function $r_{R_\pm} \in \mathcal{H}_{E,-\beta}$ such that*

$$u_R = P_R u_R, \tag{3.20}$$
$$P_R(I_E A(D) + I_E B - \lambda E)u_R = P_R Ef.$$

Notice that (3.20) is a localized form of (3.6).

*Proof.* Starting with (3.20), observe that $P_R$ commutes with $(A(D) - \lambda I)$. Upon multiplying both sides of (3.20) by $(A(D) - \lambda I)^{-1}$ we see that

$$P_R\big(I_E + (A(D) - \lambda I)^{-1}(I_E B + \lambda(I_E - E))\big)u_R = P_R(A(D) - \lambda I)^{-1}Ef. \tag{3.21}$$

By Lemma 3.3 there exists a function $u_{R_\pm}$ satisfying (3.20).

REMARK. Solutions of (3.4) are not initially restricted to $P_R \mathcal{H}_{E,-\beta}$ and $P_R$ does not commute with $E$ or $B$. Therefore $u_R$ is not just a projection of a solution to (3.6). the 'function'

$$R \longrightarrow \overset{\pm}{\underset{R}{\Sigma}}$$

may be quite complex. We shall give a few properties of its range below.

THEOREM 3.6. *If $|\lambda|$ is large, then a global solution to* (3.4) *exists.*

*Proof.* Repeating (3.10), we have

$$[I_E - I_E P_0(I_E - E + I_E B\lambda^{-1}) + I_E \lambda P(A(D) - \lambda I)^{-1}(I_E - E + I_E B\lambda^{-1})].$$

The first two terms may be rewritten as

$$I_E - P_0(I_E - E) - P_0 I_E B\lambda^{-1}. \tag{3.22}$$

We follow Ikebe (**6**) and generalize his argument to show $[I_E - P_0(I_E - E)]^{-1}$ exists on $\mathcal{H}_E$ and $\mathcal{H}_{E,-\beta}$. Since for $|\lambda|$ large, $\|P_0 I_E \beta\lambda^{-1}\|$ is small as an operator on $\mathcal{H}_E$ or $\mathcal{H}_{E,\mp\beta}$ and since the third term of (3.10) is compact by Theorem 2.1, the inverse of (3.10) is well defined as a bounded operator on $\mathcal{H}_{E,-\beta}$ except possibly for a set $\overset{\pm}{\Sigma}$ of large $\lambda$ values. Now $\overset{\pm}{\Sigma}$ has virtually the same properties as the sets $\overset{\pm}{\underset{R}{\Sigma}}$. This completes the proof.

COROLLARY 3.7. *If* $\lambda \notin \overset{\pm}{\Sigma}$ *and*

$$|\lambda| > \frac{\|P_0 I_E B\|}{\|I_E - P_0(I_E - E)\|}, \tag{3.23}$$

*then for each* $f \in \mathscr{H}_{E,\beta}$, $\beta > \frac{1}{2}$, *but close to* $\frac{1}{2}$ *there exists* $u \in \mathscr{H}_{E,-\beta}$ *such that* $u$, $f$ *satisfy* (3.4).

The solution of Corollary 3.7 is valid for sufficiently (3.23) high frequencies.

Low-frequency solutions may not be available in the global sense, when $A(D)$ is not elliptic. Suppose that $u \in P_0 \mathscr{H}_E$; then

$$FF^*A(D)u + FF^*Bu - \lambda u = f \tag{3.24}$$

and so

$$FF^*Bu - \lambda u = f,$$

$$u = (FF^*B - \lambda)^{-1} f \tag{3.25}$$

when such an inverse exists. This will be the case for $|\lambda|$ large *only*, in general, depending on the range of det $(FF^*B - \lambda I)$.

If $A(D)$ is elliptic then no stationary solution exists at '∞'. This makes low-frequency solutions possible.

COROLLARY 3.8. *If* $A(D)$ *is elliptic then* (3.4) *has solutions for low and high frequencies (outside a* $\overset{\pm}{\Sigma}$ *set).*

*Proof.* This follows from the fact that $P_0 = 0$ in all the previous arguments.

Returning to the non-elliptic case we have the following.

THEOREM 3.9. *Suppose that the field* $f$ *has the property* ($f \in \mathscr{H}_{E,\beta}$)

$$P_{R'}Ef = P_R Ef \qquad (R' > R). \tag{3.26}$$

*Then*

$$u_{R_\pm} = u_{R'_\pm} + \varepsilon(R, R'),$$

*where* $\varepsilon(R, R') \to 0$ *as* $R' - R \to 0$.

*Proof.* We have

$$\begin{aligned} P_R(I_E A(D) + I_E B - \lambda E)P_R u_R &= P_{R'}(I_E A(D) + I_E B - \lambda E)P_{R'} u_{R'} \\ &= P_R(I_E A(D) + I_E B - \lambda E)P_R u_{R'} + \\ &\quad + P_R(I_E A(D) + I_E B - \lambda E)(P_{R'} - P_R)u_{R'}. \end{aligned}$$

Thus,

$$\begin{aligned} P_R u_R = P_{R'} u_{R'} &+ (P_R u_{R'} - P_{R'} u_{R'}) + \\ &+ (P_R I_E + P_R (A(D) - \lambda I)^{-1} (I_E + I_E B \lambda^{-1} - E) P_R)^{-1_R} \times \\ &\times P_R (I_E A(D) + I_E B - \lambda E)(P_{R'} - P_R) u_{R'}. \end{aligned}$$

Noting that $P_R(A(D))$ is a bounded operator and $P_R$ is strongly continuous in $R$, the conclusion results.

If $\lambda \notin \overset{\pm}{\Sigma}_R$, $R > R_0$ and $\lambda \notin \overset{\pm}{\Sigma}$, then $u_{R_\pm}$ approximates the solution of (3.4) at high frequencies.

THEOREM 3.10. *Under the preceding conditions on* $\lambda$ *and when* $Ef \in P_{R_0}\mathscr{H}_{E,\beta}$, $\lim_{R\to\infty} u_{R_\pm}$ *exists and is a solution of* (3.5).

*Proof.* Let $C(\lambda)$ be the operator terms on the left-hand side of (3.6):

$$C(\lambda): \mathscr{H}_{E,-\beta} \longrightarrow \mathscr{H}_{E,\beta}, \qquad \beta > \tfrac{1}{2}, \text{ etc.}$$

For $|\lambda|$ large, there exists $u \in \mathscr{H}_{E,-\beta}$ such that (3.6) holds:

$$C(\lambda)u = Ef$$

for $\lambda$ fixed as above,

$$(P_R C(\lambda) P_R)^{-1} P_R C(\lambda) \longrightarrow I_E : \mathscr{H}_{E,-\beta} \longrightarrow \mathscr{H}_{E,-\beta}$$

strongly, as $R \to \infty$. Therefore

$$(P_R C(\lambda) P_R)^{-1} P_R \overset{s}{\longrightarrow} (C(\lambda))^{-1}.$$

This completes the proof.

It is important to establish that the set

$$\overset{\pm}{\Sigma} \cup \Big(\bigcup_{R>0} \overset{\pm}{\Sigma}_R\Big)$$

is limited in some sense so that a convergence theorem of the preceding sort is of some value. To this end we note that since $FF^*A(D)$ is a selfadjoint operator on $\mathscr{H}_E$,

$$\|(FF^*A(D) - \lambda I)^{-1}\| \leq 1/|\text{im } \lambda|.$$

If $|||\cdot|||$ respresents the operator norm for operators taking $\mathscr{H}_{E,\beta}$ to $\mathscr{H}_{E,-\beta}$ then it is easily seen that

$$|||(FF^*A(D) - \lambda I)^{-1}||| \leq \|(F^*FA(D) - \lambda I)^{-1}\|.$$

If we apply $(FF^*A(D)-\lambda I)^{-1}$ to both sides of (3.5) we see that for

$$\|B\|_* \leqslant (\|FF^*\|_{-\beta}\ |||(FF^*A(D)-\lambda I)^{-1}|||)^{-1} \tag{3.27}$$

or

$$\|B\|_* \leqslant \frac{|\mathrm{im}\ \lambda|}{\|FF^*\|_{-\beta}} \tag{3.28}$$

the solution to (3.5) exists in $\mathscr{H}_E$ (where $\|\cdot\|_*$ denotes the reverse norm of $|||\cdot|||$). Thus for $k$ large, $\overset{\pm}{\Sigma} \cap \{\lambda \mid |\mathrm{im}\ \lambda| > k\}$ is empty. (If $\|B\|_*$ is sufficiently small then (3.27) holds in addition away from a neighbourhood of $\overset{\pm}{\Sigma} \cap \mathbb{R}$.)

It is of some interest to regulate all the sets $\overset{\pm}{\Sigma}_R$ at the same time. This can be done easily if $E$ does not oscillate too much:

$$|E - I_E|_\infty = \alpha < 1. \tag{3.29}$$

Now write $G(\lambda) = (I_E - E + I_E B\lambda^{-1})$. If re $\lambda$ is fixed then

$$|\lambda|\ \|P_R\|\ \|(A(D)-\lambda I)^{-1}\|\ \|G(\lambda)\| \leqslant \frac{|\lambda|}{|\mathrm{im}\ \lambda|}|G(\lambda)|_\infty \to \alpha < 1 \quad \text{as } |\mathrm{im}\ \lambda| \to \infty.$$

If $M = |B|_\infty$ then

$$\frac{|\lambda|}{|\mathrm{im}\ \lambda|}|G(\lambda)|_\infty < \frac{|\lambda|\ \alpha}{|\mathrm{im}\ \lambda|} + \frac{M}{|\mathrm{im}\ \lambda|}.$$

A local or global solution then exists if $(\lambda = re^{i\theta})$

$$r < \frac{M}{|\sin\theta| - \alpha}. \tag{3.30}$$

THEOREM 3.11. *$\overset{\pm}{\Sigma}$ and $\overset{\pm}{\Sigma}_R$ are confined to the compliment of the region*

$$\left\{re^{i\theta} \mid r > \frac{M}{|\sin\theta| - \alpha}\right\}$$

*if* (3.29) *holds. In any case, $\overset{\pm}{\Sigma}$ is confined to*

$$\{z \mid |\mathrm{im}\ z| \leqslant \|B\|_*\ \|FF^*\|_{-\beta}\}.$$

REMARK. Since $(P_R C(\lambda) P_R)^{-1_R} P_R$ is bounded in $R$ $\left(\text{for } \lambda \text{ fixed and } \lambda \notin \overset{\pm}{\Sigma}_R\right)$ $u_R(\lambda, \cdot)$ is norm bounded in $R$ in $\mathscr{H}_{E,-\beta}$. Therefore

$$P_{R'}(I_E A(D) P_R u_R - \lambda I_E P_R u_R) = P_{R'} I_E f P_{R'} K(u_R, f).$$

For $|x|$ large, the coefficients of $K$ ($B$ and $I_E - E$) are uniformly small so that $u_R = u + O(|x|^{-1})$, $|x| \to \infty$, where

$$P_R(I_E A(D)u - \lambda I_E u) = P_R I_E f;$$

since $u - P_R u$ is small as $R \to \infty$, $u \approx u_R$ as $R \to \infty$, $|x| \to \infty$. Here we do not assume that $|\lambda|$ is large. Thus $u_R$ is asymptotically a projection of the free space solution.

It is of some interest to note that the trivial inequality (3.27) shows that the eigenvalues of the non-dispersive system ($B = 0$) are in a sense *breeders* for the point spectrum of the dispersive system. This applies both to the original setting and the problem cast in the various weighted spaces. We see in this the greater effect of the eigenvalue zero in comparison with those (outside the singular space) owing their existence to $E(x)$.

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