and define

$$\varphi(u, y, r) = \sum_{i=1}^{m} r\delta(u_i)^2. \tag{3.26}$$

It is easy to verify as in Example 2, that assumptions A.1, A.2, A.3, A.4, and A.5 are satisfied.

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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **121**, 57–78 (1987)

Perturbation of Invariant Subspaces of the Equations of Elasticity: Spectral Theory

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Submitted by C. L. Dolph

Received May 30, 1985

This paper is a discussion of the perturbation of operators in certain of their invariant subspaces. The program is carried out by the study of the simple nontrivial example, the equations of linear elasticity.

0. INTRODUCTION

The perturbation of operators has been studied in many ways. We introduce one more here. The method is to study the spectral properties of operators when certain of their *invariant subspaces* are perturbed and *coupled by compact mappings.*

One of the simplest nontrivial examples which displays just enough complexity to be useful is given by the equations of linear elasticity. Rather than propose a general abstract theory we choose to study these equations. The invariant subspaces of interest are determined by the pressure, shear, and stationary modes of propagation.

The propagation of waves in an elastic medium may be studied formally through the equation

$$\partial_t u = Au + f(x, t), \tag{0.1}$$

where A is determined by

$$A = E(x)^{-1} \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + Bu. \tag{0.2}$$

The A_j being symmetric constant matrices and $E(x)$ a positive real symmetric matrix (see [3, 8]). The difference $A - A_1$, of the operator (0.2) and A_1, the isotropic version, is assumed to be an integral operator of Fourier type. If this difference has special properties relative to the isotropic pressure and shear waves then A may be studied using the spectral theory for nonself-adjoint operators. Using some techniques from [8] we show how to apply the abstract theory of [11].

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0022-247X/87 $3.00

The technique probably applies to any appropriate perturbation of an equation decomposable as a direct sum of wave equations.

The study of $A = A_1 + (A - A_1)$ may be thought of in terms of the steady-state solution of (0.1). In Section 1 we outline material mostly taken from [8]. In Section 2 we examine a portion of the steady-state solution and this is completed in Section 3. In Section 3 the existence of a local spectral measure is shown, and in Section 4 some related problems are mentioned.

1. Elastic Waves in Unperturbed Media

We discuss now the case of A_1. The equations for elastic waves in an isotropic medium may be written in the form (see [8]),

$$\partial_t^2 w = \mu \Delta w + (\lambda + \mu) \nabla(\nabla \circ w), \tag{1.1}$$

where w is the displacement vector, λ and μ are the so called Lamé parameters of the medium. Equations (1.1) may be written as a first order system and it is this form we will study. Setting $\{e_{ij}\}$ equal to the components of the strain tensor and $v = (v_1, v_2, v_3)$ the velocity vector, (1.1) may be written as

$$\partial_t u = \begin{pmatrix} & \tilde{A} \\ \tilde{A}^* & \end{pmatrix} u = A(D)\, u = A_1 u, \tag{1.2}$$

where

$$u = E(e_{11} + e_{22} + e_{33},\, e_{11} - e_{22} - 2e_{33},\, e_{11} - e_{22},\, 2e_{12},\, 2e_{13},\, 2e_{13},\, v_1 v_2,\, v_3). \tag{1.3}$$

Here $e_{ij} = \frac{1}{2}{}^t(\partial_i w_j + \partial_j w_i)$, $v_j = \partial_t w_j$ and $E = C^{1/2}$ with

$$C = \operatorname{diag}(\lambda + \tfrac{2}{3}\mu,\, \mu/3,\, \mu,\, \mu,\, \mu,\, \mu,\, 1,\, 1,\, 1). \tag{1.4}$$

We set $C_0 = C$ with the last three rows and columns deleted. $\tilde{A}$ is the operator

$$C_0^{1/2} \begin{bmatrix} D_1 & D_2 & D_3 \\ D_1 & D_2 & -2D_3 \\ D_1 & -D_2 & 0 \\ D_2 & D_1 & 0 \\ D_3 & 0 & D_1 \\ 0 & D_3 & D_2 \end{bmatrix}. \tag{1.5}$$

D_j has the meaning $i(\partial/\partial x_j)$, $i = \sqrt{-1}$. The symbols $\mathscr{D}(\mathbb{R}^n, \mathbb{C}^m)$ and $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^m)$ stand for the Schwartz spaces and for $f \in \mathscr{S}$ the Fourier transform

$$(\Phi f)(p) = \int_{\mathbb{R}^n} e^{-ip \circ x} f(x)\, dx = \hat{f}(p) \tag{1.6}$$

of f defines an isomorphism of $\mathscr{S}$ which extends by continuity to $L^2(\mathbb{R}^n, \mathbb{C}^m)$ and by duality to the space of tempered distributions. The relation of Φ to its adjoint Φ^* is

$$(\Phi f)(p) = \hat{f}(p) = (\Phi^* f)(-p) = \hat{f}(-p). \tag{1.7}$$

The symbol of $A(D)$, $A(p) = \Phi A(D)$ is 9×9 matrix-valued function of (μ, λ, p), $p = (p_1, p_2, p_3) \in \mathbb{R}^3 \setminus \{0\}$. The eigenvalues of $A(p)$ are $\xi_\pm^1(p) = \pm(\lambda + 2\mu)^{1/2}\, |p|$, $\xi_\pm^2 = \pm\mu^{1/2}\, |p|$, $\xi_0(p) = 0$, each of $\xi_\pm^2$ having multiplicity two while $\xi_0(p)$ has multiplicity three and $\xi_\pm^1$ has multiplicity one. Corresponding to each of these eigenvalues is an orthoprojector $\hat{P}_{\pm 1}$, $\hat{P}_{\pm 2}$, $\hat{P}_0$ satisfying

$$A(p)\, \hat{P}_i = \xi_i(p)\, \hat{P}_i. \tag{1.8}$$

These projectors give the resolution of the identity for $A(p)$,

$$I = \hat{P}_{-1} + \hat{P}_1 + \hat{P}_{-2} + \hat{P}_2 + \hat{P}_0. \tag{1.9}$$

The 9×9 matrices representing P_i may be computed by the formula

$$P_i(x) = (-2\pi i)^{-1} \int_{|\xi_i(x) - \xi| = \delta} [A(x) - \xi I]^{-1}\, d\xi. \tag{1.10}$$

The computation is tedious and we simply list the results

$$\hat{P}_{\pm 2}(w) = \frac{1}{2} \begin{bmatrix} C_{6 \times 6}^2(w) & \pm D_{6 \times 3}^2(w) \\ \pm E_{3 \times 6}^2(w) & F_{3 \times 3}^2(w) \end{bmatrix}, \tag{1.11}$$

where $p/|p| = w$ and

$$F_{3 \times 3}^2(w) = \begin{bmatrix} 1 - w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & 1 - w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & 1 - w_3^2 \end{bmatrix} \tag{1.12}$$

with

$$C^2_{6\times 6}(w)=\tilde{A}(w)\left(\frac{F^2(w)}{\mu}\right)\tilde{A}^*(w), \tag{1.13}$$

$$E^2_{3\times 6}(w)=\left(\frac{F^2(w)}{\mu}\right)(\tilde{A}^*(w)), \tag{1.14}$$

$$D^2_{6\times 3}(w)=\tilde{A}(w)\left(\frac{F^2(w)}{\mu}\right). \tag{1.15}$$

$\hat{P}_{\pm 1}$ may be computed by substitution of $F^1=I-F^2$ for F^2 in (1.11) and $\lambda+2\mu$ for μ in (1.13)–(1.15). $P_0(w)$ may then be computed by (1.9).

We shall set $H=L^2(\mathbb{R}^3, \mathbb{C}^9)$ and consider solutions of (1.3) in terms of H. The orthoprojectors $\hat{P}_i$ define selfadjoint projectors on H via the operators

$$P_{\pm j}=\Phi^*\hat{P}_{\pm j}(w)\,\Phi, \qquad j=0, 1, 2. \tag{1.16}$$

From this point on we write

$$P_j=P_{-j}+P_{+j}. \tag{1.17}$$

These projections commute with $A(D)$ where the domain of $A(D)$ is taken as

$$D(A(D))=\{u\in L^2(\mathbb{R}^3, \mathbb{C}^9)\mid A(D)\,u\in L^2(\mathbb{R}^3, \mathbb{C}^9)\}. \tag{1.18}$$

The derivatives in (1.18) are taken in the distributional sense.

The projections P_j determine the modes of propagation of solutions to (1.2) with P_0H being the stationary data. To obtain a solution to the wave propagation problems considered below for so-called globally acting perturbations, it appears that the difference operator B must be restricted somewhat in terms of its distortion of the pressure, shear, and stationary waves. Basically, we allow for such waves to be "globally coupled" by B only in a restricted fashion (though not necessarily with bounded support). Set $H_1=P_1H$, $H_2=P_2H$. There exist mappings σ_1, σ_2 on H,

$$\sigma_1\colon\quad H\to \mathrm{BL}(\mathbb{R}^3, (\lambda+2\mu)^{1/2}, \mathbb{C})\oplus L^2(\mathbb{R}^3, \mathbb{C}), \tag{1.19}$$

$$\sigma_1^*\colon\quad \mathrm{BL}(\mathbb{R}^3, (\lambda+2\mu)^{1/2}, \mathbb{C})\oplus L^2(\mathbb{R}^3, \mathbb{C})\to H, \tag{1.20}$$

$$\sigma_2\colon\quad H\to \mathrm{BL}(\mathbb{R}^3, \mu^{1/2}, \mathbb{C}^2)\oplus L^2(\mathbb{R}^3, \mathbb{C}^2), \tag{1.21}$$

$$\sigma_2^*\colon\quad \mathrm{BL}(\mathbb{R}^3, \mu^{1/2}, \mathbb{C}^2)\oplus L^2(\mathbb{R}^3, \mathbb{C}^2)\to H, \tag{1.22}$$

with $P_1\sigma_1^*=\sigma_1^*$, $\sigma_1P_1=\sigma_1$ and $P_2\sigma_2^*=\sigma_2^*$, $\sigma_2P_2=\sigma_2$. Here BL refers to Beppo–Levi space (the distributions on $\mathbb{R}^3$ with square integrable

gradient). The mappings σ_1 and σ_2 are given by the pseudodifferential operator kernels $\hat{\sigma}_1$, $\hat{\sigma}_2$, where

$$(2(\lambda+2\mu))^{1/2}\,\hat{\sigma}_1(w)=\begin{bmatrix} a_p(w)_{1\times 6} & 0_{1\times 3} \\ 0_{1\times 6} & ib_p(w)_{1\times 3}\end{bmatrix}_{2\times 9}, \tag{1.23}$$

$$(2(w_1^2+w_2^2))^{1/2}\,\hat{\sigma}_2(w)=\begin{bmatrix} a_s(w)_{2\times 6} & 0_{2\times 3} \\ 0_{2\times 6} & ib_s(w)_{2\times 3}\end{bmatrix}_{4\times 9}, \tag{1.24}$$

$$a_p(w)=\left(\left(\lambda+\frac{2\mu}{3}\right)^{1/2}, \left(\frac{\mu}{3}\right)^{1/2}(1-3w_3^2), \mu^{1/2}(w_1^2-w_2^2), 2\mu^{1/2}w_1w_2,\right. \tag{1.25}$$

$$\left. 2\mu^{1/2}w_1w_3, 2\mu^{1/2}w_2w_3\right),$$

$$b_p(w)=((\lambda+2\mu)^{1/2}\,w_1, (\lambda+2\mu)^{1/2}\,w_2, (\lambda+2\mu)^{1/2}\,w_3), \tag{1.26}$$

$$a_s(w)=\begin{bmatrix} 0 & 0 & 2w_1w_2 & w_2^2-w_1^2 & w_2w_3 & -w_1w_3 \\ 0, & 3w_3(w_1^2+w_2^2) & w_3(w_1^2-w_2^2) & 2w_1w_2w_3 & w_1(2w_3^2-1) & w_2(2w_3^2-1)\end{bmatrix}, \tag{1.27}$$

$$b_s(w)=\begin{bmatrix} w_2 & -w_1 & 0 \\ w_1w_3 & w_2w_3 & -(w_1^2+w_2^2)\end{bmatrix}. \tag{1.28}$$

Here a_s, a_p, b_s, b_p are obtained as eigenvectors spanning the subspaces of $\mathbb{C}^9$ generated by P_1 and P_2 as in [8]. (Note that $A(D)$ has rank 6.)

The mappings σ^*, and σ_2^* are obtained by substitution of the adjoints ${}^ta_s$, ${}^tb_s$, ${}^ta_p$, ${}^tb_p$ for a_s, etc., in (1.23)–(1.24) (change i to $-i$).

Now we make the following assumptions on $B=A-A_1$:

(1) B is a closed operator and $D(B)\supseteq D(A)$.

(2) P_iB is closed.

(3) P_iBP_j is compact if $i\neq j$.

(4) P_jBP_j is the "preimage" of an L^2 matrix function ($j\neq 0$) (K^j_{il}) on σ_jH_j. (This is the form a perturbation would take in the "wave equation" format (1.1).)

2. The Projected Problem

Definition 2.1. Set

$$\sigma=\sigma_1\otimes\sigma_2, \tag{2.1}$$

$$\sigma^*=\sigma_1^*\otimes\sigma_2^*; \tag{2.2}$$

where

$$\sigma\colon\ H \to (\mathrm{BL}\times L^2)_1 \otimes (\mathrm{BL}\times L^2)_2, \tag{2.3}$$

$$\sigma^*\colon\ (\mathrm{BL}\times L^2)_1 \otimes (\mathrm{BL}\times L^2)_2 \to H, \tag{2.4}$$

by

$$\sigma f = \sigma_1 f \otimes \sigma_2 f, \tag{2.5}$$

$$\sigma^*(h_1 \otimes h_2) = \sigma_1^* h_1 + \sigma_2^* h_2. \tag{2.6}$$

Note that

$$\sigma(I - P_0) = \sigma, \tag{2.7}$$

$$(I - P_0)\,\sigma^* = \sigma^*. \tag{2.8}$$

Consider the equation

$$i\partial_t Pu = P(A(D)\,u) + PBu, \tag{2.9}$$

where $P = I - P_0$. We then have

$$i\partial_t Pu = A(D)\,Pu + PBPu + PBP_0 u, \tag{2.10}$$

$$\begin{aligned}
&i\partial_t P_1 u = A(D)\,P_1 u + P_1 BP_1 u + P_1 BP_2 u + P_1 BP_0 u,\\
&i\partial_t P_2 u = A(D)\,P_2 u + P_2 BP_1 u + P_2 BP_2 u + P_2 BP_0 u,\\
&A(D)\,P_1 u + P_1 BP_1 u + P_1 BP_2 u + P_1 BP_0 u - \xi P_1 u = P_1 f,\\
&A(D)\,P_2 u + P_2 BP_1 u + P_2 BP_2 u + P_2 BP_0 u - \xi P_1 u = P_2 f.
\end{aligned} \tag{2.11}$$

The second two equations in (2.11) are the steady-state equations for frequency ξ and source function f.

We have

$$P_i BP_j u = \sum_{l=1}^{\infty} \mu_l^{i,j}(u, \Psi_l^{i,j})\,\Phi_l^{i,j} \qquad (i \neq j), \tag{2.12}$$

where $(\Psi_l^{i,j}, \Phi_m^{i,j}) = \delta_{lm}$ and (,) denotes the inner product of H.

We shall consider the problems of (2.11). To begin, we shall suppose that the operator

$$A(D) + \sum_{j\neq 0} P_j BP_j - \xi I = T - \xi I \tag{2.13}$$

has a bounded inverse $(T-\xi I)^{-1}$ on H for all ξ in $\mathbb{C}$ except for points on the real axis together with points in a discrete set of complex numbers which generally lies both above and below the real axis. (These assumptions will be justified below.) Now with the equations

$$PA(D)\,u + PBu - P\xi u = Pf, \tag{2.14}$$

$$A(D)\,Pu + PBu - \xi Pu = Pf,$$

$$(T - \xi I)\,u + Au = Pf, \tag{2.15}$$

$$u + (T-\xi I)^{-1} Au = (T-\xi I)^{-1} Pf, \tag{2.16}$$

$$u = (I + (T-\xi I)^{-1} A)^{-1} (T-\xi I)^{-1} Pf. \tag{2.17}$$

We see that a solution u to (2.14) exists if the inverse operators in (2.17) exist. It is therefore not to be expected that a solution to (2.14) will exist in H when ξ is real. Our procedure is to first establish conditions under which $(T-\xi I)^{-1}$ and $(I + (T-\xi I)\,A^{-1})^{-1}$ exist and then to apply this information to the solution of (2.1) via the limiting absorption principle. We shall require certain information concerning $A(D)$.

LEMMA 2.1. *Suppose* $\xi \notin \mathbb{R}^1$. *Then for* $R_0(\xi, \eta)\,\sigma = \Phi\sigma R_0(\xi) = \Phi\sigma(A(D) - \xi I)^{-1}$,

$$R_0(\xi,\eta) = \begin{bmatrix}
\dfrac{\xi}{\mu\,|\eta|^2 - \xi^2} & 0 & \dfrac{1}{\mu\,|\eta|^2 - \xi^2} & 0\\
0 & \dfrac{\xi}{\mu\,|\eta|^2 - \xi^2} & 0 & \dfrac{1}{\mu\,|\eta|^2 - \xi^2}\\
\dfrac{\mu\,|\eta|^2}{-\xi^2 + \mu\,|\eta|^2} & 0 & \dfrac{\xi}{\mu\,|\eta|^2 - \xi^2} & 0\\
0 & \dfrac{\mu\,|\eta|^2}{-\xi^2 + \mu\,|\eta|^2} & 0 & \dfrac{\xi}{\mu\,|\eta|^2 - \xi^2}
\end{bmatrix}$$

$$\otimes \begin{bmatrix}
\dfrac{\xi}{-\xi^2 + (\lambda+2\mu)\,|\eta|^2} & \dfrac{1}{-\xi^2 + (\lambda+2\mu)\,|\eta|^2}\\
\dfrac{(\lambda+2\mu)\,|\eta|^2}{-\xi^2 + (\lambda+2\mu)\,|\eta|^2} & \dfrac{\xi}{-\xi^2 + (\lambda+2\mu)\,|\eta|^2}
\end{bmatrix}. \tag{2.18}$$

We now take up the task of computing the "resolvent" of the operator T designated above. Since existence–uniqueness results for

$$Tu - \xi u = f \tag{2.19}$$

are not available in immediate form, our task requires us to examine $A(D)$ and its relation to T carefully.

3. The Helmholtz Equation for T

To begin, we use the transform σ on (2.19) to obtain

$$\sigma T\sigma^* g - \lambda g = \sigma f, \tag{3.1}$$

where we suppose $\sigma^* g \in D(T)$. For the moment let us suppose that $B \equiv 0$. Then

$$g = (\sigma T \sigma^* - \lambda I)^{-1} \sigma f. \tag{3.2}$$

The inverse of $\sigma T\sigma^* - \lambda I$ being defined by (2.18). In the general case $B \neq 0$ the method of solution is to seek g in terms of the solution of (3.2) when $B = 0$. If the solution of (3.2) in the general case is to be anything like the simple solution, some asymptotic conditions must be placed upon

$$\sigma \sum_{i=1}^{2} P_i B P_i \sigma^*. \tag{3.3}$$

We have already assured that (3.3) is determined by an L^2 function. More specifically, we have

$$\sigma \sum_{i=1}^{2} P_i B P_i = K_i \otimes K_2 \sigma, \tag{3.4}$$

where

$$\sum_{i=1}^{2} \int_{\mathbb{R}^3} (1+|x|)^{1+\varepsilon} \|K_i(x)\|^2 \, dx < \infty \tag{3.5a}$$

and

$$(1+|x|)^{(1+\varepsilon)/2} K_i\colon \quad H^2 \times H^1 \subset H^1 \times L^2 \tag{3.5b}$$

for some $\varepsilon > 0$. The condition (3.4) has a natural interpretation when B is similar to an integral operator in $L^2(\mathbb{R}^3, \mathbb{C}^9)$. We shall further require some conditions on certain of the rows and columns of (3.4). Such will be specified below. The condition (3.5) is a measure of the influence of B (in the appropriate subspaces of $L^2(\mathbb{R}^3, \mathbb{C}^9)$). Condition (3.5) does not necessarily indicate that the influence of B *decreases* with distance from the origin.

The following lemmas are required.

Lemma 3.1. *There exist bounded matrix-valued functions K_1', K_2' such that*

$$(K_i')^{-1} K_i \in L^2(\mathbb{R}^3, \mathbb{C}^{n_i}), \qquad i = 1, 2, \; n_i = 4, 2$$

and

$$\sup_{y \in \mathbb{R}^3} \sum_{i=1}^{2} \int \frac{\|K_i'(x)\|^2}{|y-x|^2} \, dx < \infty. \tag{3.6}$$

Lemma 3.2.

$$K_1 \otimes K_2\colon \quad (H^2 \times H^1)^4 \otimes (H^2 \times H^1)^2 \to (H^1 \times L^2)^4 \otimes (H^1 \times L^2)^2.$$

Lemma 3.3.

$$(\Phi^* \hat{R}_0(P, \xi))(x-y) = \tilde{R}_0(x, y) + \tilde{R}_0(x, y, \xi),$$

where

$$\tilde{R}_0(x, y) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta(x-y) & 0 & 0 & 0 \\ 0 & \delta(x-y) & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ \delta(x-y) & 0 \end{bmatrix} \tag{3.7}$$

and

$$\tilde{R}_0(x, y, \xi) = \begin{cases} \tilde{R}_0^+(x, y, \xi) & \operatorname{im} \xi > 0, \\ \tilde{R}_0^-(x, y, \xi) & \operatorname{im} \xi < 0, \end{cases} \tag{3.8}$$

where

$$R_0^{\pm} = \mu_0^{-1/2} C \, \frac{\exp(\pm i \mu_0^{1/2} \xi |x-y|}{|x-y|} \begin{bmatrix} \xi & 0 & 1 & 0 \\ 0 & \xi & 0 & 1 \\ \xi^2 & 0 & \xi & 0 \\ 0 & \xi^2 & 0 & \xi \end{bmatrix}$$
$$\otimes (\lambda + 2\mu)^{-1/2} \, C \, \frac{\exp(\pm i(\lambda+2\mu)^{1/2} \xi |x-y|}{|x-y|} \begin{bmatrix} \xi & 1 \\ \xi^2 & \xi \end{bmatrix}. \tag{3.9}$$

Let $\tilde{R}_{0,0}$ be the operator generated by $R_0(x, y)$, $R_{0,1}$ the operator generated by $\tilde{R}_0^{\pm}$.

The proofs of the preceding lemmata are elementary computations based on the assumptions about B and (2.18). Note that (2.18) and Lemma 3.3 give another proof that the spectrum of $A(D)$ is entirely continuous except for $\xi = 0$ which is a point in the discrete spectrum with infinite multiplicity. In the following lemmas unless otherwise stated, we consider the domain of an operator to be in $L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes L^2(\mathbb{R}^3, \mathbb{C}^2)$.

LEMMA 3.4. *Let*

$$K_1' \otimes K_2' \tilde{R}_0^{\pm}(K_1')^{-1} K_1 \otimes (K_2')^{-1} K_2 = (s_{ij}^{\pm}(x, y, \xi)) \otimes (a_{ij}^{\pm}(x, y, \xi)).$$

Then

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |s_{ij}^{+}(x, y, \xi)|^2 \, dx \, dy \leqslant \begin{cases} M(\operatorname{im} \xi)^{-1} \max\{|\xi|^4, 1\}, & \operatorname{im} \xi > 0 \\ N \sum_{l=1}^{4} \|K_{il}^1\|_{L^2}^2 \max\{|\xi|^4, 1\}, & \operatorname{im} \xi > 0, \end{cases} \tag{3.10}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |a_{ij}^{+}(x, y, \xi)|^2 \, dx \, dy \leqslant \{\textit{similar to } (3.10);$$

a similar set of inequalities holds for s^-, a^-.

LEMMA 3.5. *Let*

$$\int_{\mathbb{R}^3} (K_1' \otimes K_2') \tilde{R}_0(x, y)((K_1')^{-1} K_1) \otimes ((K_2')^{-1} K_2) \, u \, dy = \mathscr{K} u. \tag{3.11}$$

Then for $K_1 \otimes K_2 = (k_{ij}^1) \otimes (k_{ij}^2)$,

$$\mathscr{K} u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ k_{11}^2 & k_{12}^2 \end{bmatrix}. \tag{3.12}$$

Suppose that $|\mathscr{K}(x)| \sim O(|x|^{-1-\varepsilon})$, $|x| \to \infty$, $\varepsilon > 0$, *and zero is not in the essential range of*

$$(\gamma - k_{13}')(\gamma - k_{24}') - k_{14}' k_{23}' \qquad \textit{or} \qquad \gamma^2 - \gamma k_{12}^2,$$

then $(\gamma I - \mathscr{K})^{-1}$ *exists as a bounded operator on* $L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes L^2(\mathbb{R}^3, \mathbb{C}^2)$.

Proof of Lemma 3.4. We do only the + case for $\operatorname{im} \xi \geqslant 0$. To simplify notation we write $(K^{1'})^{-1}(K^1) = (k_{ij}'')$,

$$|s_{ij}^{+}(x, y, \xi)|^2 = \left(\frac{\pi}{2}\right) \frac{\exp(-2 \operatorname{im} \xi |x-y|)}{|x-y|^2} |k_{ij}^{1'}(x)|^2 \left| \sum_{l=1}^{4} \delta_{il}(\xi) k_{lj}''(y) \right|^2$$

$$\leqslant \left(\frac{\pi}{2}\right) \frac{\exp(-2 \operatorname{im} \xi |x-y|)}{|x-y|^2} |k_{ij}^{1'}(x)|^2$$

$$\times \left(\sum_{l=1}^{4} |\delta_{il}(\xi)|^2 \right) \left(\sum_{l=1}^{4} k_{lj}''(y) \right) \Bigg|^2, \tag{$*$}$$

where $\delta_{il}(\xi) = 1, 0, \xi$, or ξ^2. Equation ($*$) is equal to

$$\sum_{l=1}^{4} \sum_{l=1}^{4} (\pi/2) \frac{\exp(-2 \operatorname{im} \xi |x-y|)}{|x-y|^2} |k_{ij}^{1'}(x)|^2 |\delta_{il}(\xi)|^2 |k_{lj}''(y)|^2$$

$$\times |\delta_{il}(\xi)|^2 \iint \frac{\pi}{2} \frac{\exp(-2 \operatorname{im} \xi |x-y|)}{|x-|^2} |k_{ij}^{1'}(x)|^2 |k_{kj}''(y)|^2 \, dx \, dy$$

$$\leqslant \|k_{kj}''\|_{L^2} \frac{\pi}{2} \max\{|\xi|^4, 1\} \sup_y \int \frac{\exp(-2 \operatorname{im} \xi |x-y|)}{|x-y|^2 (1+|x|)^{1+2\varepsilon}} \, dx$$

and a similar argument gives the result for a^+, etc.

The proof of Lemma 3.5 is left to the reader.

LEMMA 3.6. *Let z be fixed. Denote by* $s(z)$ *the operator generated by* $(s^{\pm} \otimes a^{\pm})(w, y, z)$. *Then for* γ *and* $\mathscr{K}$ *satisfying the hypotheses of Lemma* 3.5 *we have*

$$(\gamma I + \mathscr{K} + s(z))^{-1} = (\gamma + K)^{-1} - (\gamma I + K)^{-1} \{I + s(z)(\gamma I + K)^{-1}\}^{-1} \times s(z)(\gamma I + K)^{-1} \tag{3.13}$$

when the appropriate inverses exist.

Proof. Multiplication of the right side of (3.13) by $\gamma I + K + s(z)$ on either the left or right results in the identity.

LEMMA 3.7. $s(z)(\gamma I - K)^{-1}$ *is an analytic operator-valued function in z, continuous in the closed upper and lower half z planes.*

Proof.

$$(s(z_1) - s(z_2))/(z_1 - z_2) = -K_1^1 \otimes K_2^1 R_0(z_1) R_0(z_2)(K_1^1)^{-1} K_1 \otimes (K_2^1)^{-1} K_2$$

by the resolvent identity. Since

$$R_0(z_1) R_0(z_2) = \mathscr{K} - K_1' \otimes K_2' R_{0,0} R_{0,1}(z_1)(K_1')^{-1} K_1 \otimes (K_2')^{-1} K_2$$
$$- K_1' \otimes K_2' R_{0,1}(z_2) R_{0,0}(K_1^{-1}) K_1 \otimes (K_2')^{-1} K_2$$
$$- [K_1' \otimes K_2'] R_{0,1}(z_2)[(K_1'^{-1}) K_1 \otimes (K_2')^{-1} K_2]$$

and since K_1, K_2 are bounded, it suffices to show that $R_{0,1}(z)[K_1'^{-1}) K_1 \otimes (K_2')^{-1} K_2]$ is a continuous function of z when $\operatorname{im} z > 0$. Since the matrix portion of $R_{0,1}$ is certainly continuous in z it remains to show that the associated integral operator $R_{0,1,0}(k_1')^{-1}(k_1) \otimes (k_2')^{-1} k_2$ is a continuous function of z in the complement of the real axis. $R_{0,1,0}$ is of the form

$c[\exp(\pm iz\,|x-y|)\,|x-y|^{-1}$ the + or − is chosen according to whether z is above or below the real axis. Here we think of the real axis as having a top and bottom "edge." We define $R_{0,1,0}$ on the "top" edge using +, on the "bottom" edge using −. Let $|||\cdot|||$ denotes the Hilbert–Schmidt operator norm. We have ($\operatorname{im} z \geqslant 0$)

$$|||R_{0,1,0}(z_1)[(K_1)^{-1}K_1\otimes(K_2)^{-1}K_2]-R_{0,1,0}(z_2)[(K_1)^{-1}K_1\otimes(K_2)^{-1}K_2]|||$$

$$=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|[(K_1)^{-1}K_1\otimes(K_2)^{-1}K_2](x)|^2$$
$$\times|R_{0,1,0}(x,y,z_1)-R_{0,1,0}(x,y,z_2)|^2\,dx\,dy.$$

This integral may be decomposed over the four domains

$$D_1=(|x|\langle R,|y|<R),$$
$$D_2=(|x|>R,|y|<R),$$
$$D_3=(|x|>R,|y|>R),$$
$$D_4=(|x|<R,|y|>R).$$

Over D_1 we use the estimate

$$|\exp(iz,|x-y|)-\exp(iz_2|x-y|)|\leqslant|z_1-z_2|\,|x-y| \tag{3.14}$$

and for z_1, z_2 in a bounded set ($\operatorname{im} z_i>0$) the integrals in the other three domains may be made uniformly small by taking R large enough. To see that $s(z)$ is continuous down to the edge of the real axis, we have ($\operatorname{im} z_1\geqslant 0$, $\operatorname{im} z_2\geqslant 0$),

$$|||s(z_1)-s(z_2)|||$$
$$=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|K_1'\otimes K_2'(x)|^2\left|\delta(z_1)\,\frac{\exp(iz_1|x-y|)-\exp(iz_2|x-y|)}{|x-y|^2}\,\delta(z_2)\right|^2$$
$$\times|(K_1')^{-1}K_1\otimes(K_2')^{-1}K_2(y)|^2 \tag{3.15}$$

$$=\int_{|x|<R}\int_{|y|<R}+\int_{|x|>R}\int_{|y|<R}+\int_{|x|<R}\int_{|y|>R}+\int_{|x|>R}\int_{|y|<R}. \tag{3.16}$$

Since the last three integrals in (3.16) are uniformly small for z_1, z_2 in a compact set and for large R we consider the first integral in (3.16):

$$\int_{|x|<R}\int_{|y|<R}\leqslant\int_{|x|<R}\int_{|y|<R}\frac{|K_1'\otimes K_2'(x)|^2}{|x-y|^2}$$
$$\times|(K_1')^{-1}K_1\otimes(K_2')^{-1}K_2|^2\,B(x,y,z_1,z_2)\,dx\,dy, \tag{3.17}$$

where

$$B(x,y,z_1,z_2)=(|\delta(z_1)|\,|\exp(iz_1|x-y|)|-\exp(iz_2|x-y|)$$
$$-|\delta(z_1)-\delta(z_2)|\,|\exp iz_2|x-y||)^2 \tag{3.18}$$

and $\delta(z_1)$ is the tensor in (3.9). Using the estimate (3.14) and noting that $\delta(z)$ and $(\exp iz\,|x-y|)$ are continuous in z we obtain the required result.

LEMMA 3.8. *Suppose $\gamma\in\rho(\mathscr{K})$. There exists a closed nowhere-dense set of measure zero $S_{\mathbb{R}}$ on the real axis and a discrete set $S_{\mathbb{C}}$ ((perhaps unbounded) in $\mathbb{C}\setminus\mathbb{R}$ such that for z in $\mathbb{C}\setminus(S_{\mathbb{R}}\cup S_{\mathbb{C}})$, $(I+K+s(z))^{-1}$ exists as an operator-valued continuous function of z in the upper and lower closed half-planes.*

If $K_1\otimes K_2$ is very rapidly decreasing the function $s(z)$ may be extended analytically to the unphysical sheet in z. This means that $S_{\mathbb{R}}\cup S_{\mathbb{C}}$ is a discrete set.

Proof of Lemma 3.8. By Lemmas 3.6 and 3.7 it is sufficient to show that

$$(I+s(z)(\alpha I+\mathscr{K})^{-1})^{-1} \tag{3.19}$$

has the properties listed above. Since $s(z)(\alpha I+\mathscr{K})^{-1}$ is compact it may be approximated by finite rank operators in the uniform topology. Let z_0 be a point on the real axis by Lemma 3.7 there is a positive number r such that

$$\|s(z_0)(\alpha I+\mathscr{K})^{-1}-s(z)(\alpha I+\mathscr{K})^{-1}\|<\tfrac{1}{2} \tag{3.20}$$

when $|z-z_0|<r$, $\operatorname{im} z\geqslant 0$, say. There is a finite rank operator s such that

$$\|s(z)(\sigma I+\mathscr{K})^{-1}-s\|<\tfrac{1}{2}. \tag{3.21}$$

Inequalities (3.20) and (3.21) imply that

$$\|s(z)(\sigma I+\mathscr{K})^{-1}-s\|<1 \tag{3.22}$$

for $|z-z_0|<r$, $\operatorname{im} z\leqslant 0$. It follows that

$$(I+s(z)(\alpha I+\mathscr{K})^{-1}-s)^{-1} \tag{3.23}$$

has a Neumann expansion for $|z-z_0|<r$. We define

$$G(z)=s(I+(s(z)(\alpha I+\mathscr{K})^{-1}-s)^{-1}. \tag{3.24}$$

Then

$$(I+s(z)(\gamma I+\mathscr{K})^{-1})=(I+G(z))(I+s(z)(\gamma I+\mathscr{K})^{-1}-s). \tag{3.25}$$

By (3.24) and (3.25) the existence and continuity of (3.19) depends upon those same properties for

$$(I+G(z))^{-1}. \tag{3.26}$$

Since s has finite rank we may write

$$s(g)=\sum_{i=1}^{n}(g, g_i)f_i. \tag{3.27}$$

Let

$$g_k(z)=(I+s(z)(\gamma I+\mathscr{K})^{-1}-s)^{-1*}g_k \tag{3.28}$$

so (3.26) exists only if the determinant of

$$I_{n\times n}-(f_i, g_i(z))_{i,j} \tag{3.29}$$

is nonzero. We call this determinant $\Gamma(z)$. $\Gamma(z)$ is evidently analytic in $|z-z_0|<r$, $\operatorname{im} z>0$. Write $Z_+=\{z\mid |z-z_0|<r, \ \operatorname{im} z>0\}$ $Z=\{z\mid |z-z_0|<r, \ \operatorname{im} z\geqslant 0\}$. Set $\xi=r^{-1}(z-z_0)$, $w=(\xi^2+i\xi+1)(\xi^2-i\xi+1)^{-1}$; w takes Z to the unit disc D_w in the w-plane with $\operatorname{im} z=0$ going into ∂D_w. It is easily checked that $w(\cdot)$ preserves sets of measure zero. Define

$$M(w)=s(w^{-1}(w))=s(z). \tag{3.30}$$

Appealing to Theorem 15.19 of [7], we see that $s(z)$ vanishes on a set of linear measure zero on $\operatorname{im} z=0$. A similar argument holds in a neighborhood of any a_0 in $\mathbb{C}$. This completes the proof.

LEMMA 3.9. *Define for* $z\in\mathbb{C}\backslash(S_{\mathbb{C}}\cup\mathbb{R})$,

$$\begin{aligned}R(z)=R_0(z)-R_0(z)[(K_1')^{-1}K_1\otimes(K_2')^{-1}K_2]\\ \times(I+K+s(z))^{-1}K_1'\otimes K_2'R_0(z);\end{aligned} \tag{3.31}$$

then $R(z)$ is a bounded operator on

$$(H^1(\mathbb{R}^3,\mathbb{C}^2)\oplus L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes(H^1(\mathbb{R}^3,\mathbb{C})\oplus L^2(\mathbb{R}^3,\mathbb{C})).$$

Proof. By Lemma 2.1 the range of $R_0(z)$ on $L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes L^2(\mathbb{R}^3,\mathbb{C}^2)$ is *in* $(H^2(\mathbb{R}^3,\mathbb{C}^2)\oplus L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes(H^2(\mathbb{R}^3,\mathbb{C})\oplus L^2(\mathbb{R}^3,\mathbb{C}))$. An appeal to the Schwartz inequality and the hypothesis of Lemma 3.5 shows that the range of $R_0(z)(K_1')^{-1}K_1\otimes(K_2')^{-1}K_2$ on $(H^1(\mathbb{R}^3,\mathbb{C}^2)\oplus L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes(H^1(\mathbb{R}^3,\mathbb{C})\oplus L^2(\mathbb{R}^3,\mathbb{C}))$ lies in the space $(H^1(\mathbb{R}^3,\mathbb{C}^2)\oplus L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes$ $(H^1(\mathbb{R}^3,\mathbb{C})\oplus L^2(\mathbb{R}^3,\mathbb{C}))$. Since $K_1'\otimes K_2'$ is a bounded operator, the conclusion of the lemma follows.

COROLLARY 3.10. *For any* z *as in Lemma* 3.9, $R(z)$ *maps*

$$(\mathrm{BL}(\mathbb{R}^3,\mathbb{C}^2)\oplus L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes(\mathrm{BL}(\mathbb{R}^3,\mathbb{C})\oplus L^2(\mathbb{R}^3,\mathbb{C})$$

to itself.

THEOREM 3.11. $\sigma^*R(z)\,\sigma$ *is the resolvent of* T.

Proof. That $R(z)$ is a resolvent follows from the easily established fact that $R(z)$ is a pseudoresolvent with trivial null space. First, $R(z)$ is seen to satisfy the first resolvent equation since $A=K_1'\otimes K_2'$ $B=(K_1')^{-1}K_1\otimes(K_2')^{-1}K_2$ we obtain

$$\begin{aligned}R(z_1)\,R(z_2)=R_0(z_1)\,R_0(z_2)-R_0(z_1)\,R_0(z_2)\,B(I+s_1(z_2))^{-1}AR_0(z_2)\\ -R_0(z_1)\,B(I+s_1(z_1))^{-1}AR_0(z_1)\,R_0(z_2)\\ +R_0(z_1)\,B(I+s_1(z_1))^{-1}AR_0(z_1)\,R_0(z_2)\\ \times B(I+s_1(z_2))^{-1}BR_0(z_2).\end{aligned} \tag{3.32}$$

Now using the fact that $R_0(\cdot)$ is a resolvent we have

$$\begin{aligned}R(z_1)\,R(z_2)&=\frac{R_0(z_1)-R_0(z_2)}{z_2-z_1}\\ &\quad-\frac{R_0(z_1)-R_0(z_2)}{z_2-z_1}B(I+s_1(z_2))\,AR_0(z_2)\\ &\quad-R_0(z_1)\,G(I+s_1(z_1))^{-1}A\,\frac{R_0(z_1)-R_0(z_2)}{z_2-z_1}\\ &\quad+R_0(z_1(B(I+s_1(z_1))^{-1}\frac{s_1(z_1)+I)-(s_1(z_2)+I)}{z_2-z_1}\\ &\quad\times(I+s_1(z_2))\,AR_0(z_2)\\ &=\frac{R(z_1)-R(z_2)}{z_2-z_1}.\end{aligned} \tag{3.33}$$

Here $s_1(z)=\mathscr{K}+s(z)$. Second, following the standard reasoning (see Kato [5]), if $R(z)g=0$ then multiplying (3.30) on the left by A and using the definition of $s_1(z)$ we arrive at the fact that $AR_0(z)g=0$. By (3.31) this implies that $R_0(z)g=0$ and thus $g=0$. Since a similar result holds for the adjoint of $R(z)$, $R(z)$ has dense range. From Lemma 3.9 (Corollary 3.10) we see that $\sigma^* R(z)$ makes sense for $z \in \mathbb{C} \setminus (S_{\mathbb{C}} \cup \mathbb{R})$. Thus, if $(L - zI)^{-1} = R(z)$ then $T = \sigma^* L \sigma$ on $D(A(D))$.

COROLLARY 3.12. *The spectrum of T is either the entire complex plane or consists of isolated eigenvalues in the plane together with the real axis.*

Remark. The spectrum of T is of course not bounded in general, but further as is indicated by Lemma 3.3 and the remarks following assumption (4) in Section 1, the eigenvalues may form an unbounded set. From (2.17) we see that a portion of the difficulty in solving the steady state problem is to establish conditions under which $R(z)$ is defined. Therefore, Corollary 3.12 partially answers this question. In order to extract further information it will be necessary to consider the operator appearing formally in (2.17) namely

$$(I + (T - zI)^{-1} A)^{-1} \tag{3.34}$$

LEMMA 3.13. *The operator*

$$(I + (T - zI)^{-1} A)^{-1}$$

exists for all z outside of a discrete set $\tilde{S}_{\mathbb{C}} \subseteq \mathbb{C}$ *together with the real axis.*

Proof. An argument similar to that of the proof of Lemma 3.8 using Corollary 3.12 establishes the result. The set $\tilde{S}_{\mathbb{C}}$ is called the set of resonance points for equation (2.14). It is clear that $S_{\mathbb{C}} \subseteq \tilde{S}_{\mathbb{C}}$ generally.

THEOREM 3.14. *Equation* (2.14) *has a solution in H whenever* $\xi \in \mathbb{C} \setminus (\tilde{S}_{\mathbb{C}} \cup \mathbb{R}^1)$.

The natural generalization of the limiting absorption principle to (2.14) is the existence of limits of the form

$$\lim_{b \to 0} u(a + ib) \tag{3.35}$$

in some appropriate topology on the solution set of (2.14) where $a \notin \tilde{S}_{\mathbb{C}}$, $\xi = a + ib$.

It will be necessary to be precise information concerning the operator A of (2.14) and its effect on the resolvent $(T - zI)^{-1}$.

A in (2.14) is given by

$$A = \sum_{\substack{i \neq j \\ i \neq 0}} \sum_{l=1}^{\infty} \mu_l^{ij}(\cdot, \Psi_l^{ij}) \phi_l^{ij}. \tag{3.36}$$

If u is given by (2.17) using Lemma 3.13, then we may write

$$(T - \xi I) u = -Au + Pf \tag{3.37}$$

or

$$\begin{aligned} u(\xi, f) &= -(T - \xi I)^{-1} Au + (T - \xi I)^{-1} Pf \\ &= -\sum_{\substack{i \neq j \\ i \neq 0}} \sum_{l=1}^{\infty} \mu_l^{ij}(u, \Psi_l^{ij})(T - \xi I)^{-1} \phi_l^{ij} \\ &\quad + (T - \xi I)^{-1} Pf \end{aligned} \tag{3.38}$$

by Lemma 3.13, $\mu_l^i(u, \Psi_l^{ij}) = a_{ij}(\xi, f)_l$ is an analytic function of ξ and is continuous in f. We write K_ξ for $(T - \xi I)^{-1}$. Then

$$\begin{aligned} u(\xi, f) &= -\sum_{\substack{i \neq j \\ i \neq 0}} \sum_{l=1}^{\infty} a_{ij}(\xi, f)_l K_\xi \phi_l^{ij} + K_\xi Pf \\ &= -\sum_{l=1}^{\infty} \sum_{\substack{i \neq j \\ i \neq 0}} a_{ij}(\xi, f)_l K_\xi \phi_l^{ij} + K_\xi Pf. \end{aligned} \tag{3.39}$$

Now, taking inner products on both sides of (3.39) with $\sum_{i \neq j, i \neq 0} \Psi_h^{ij}$ we obtain

$$\begin{aligned} \sum_{\substack{i \neq j \\ i \neq 0}} a_{ij}(\xi, f)_h &= -\sum_{l=1}^{\infty} \sum_{\substack{i \neq j \\ i \neq 0}} a_{ij}(\xi, f)_l \left(K_\xi \phi_l^{in}, \sum_{\substack{n \neq m \\ n \neq 0}} \Psi_h^{nm} \right) \\ &\quad + \left(K_\xi Pf, \sum_{\substack{i \neq j \\ i \neq 0}} \Psi_h^{ij} \right). \end{aligned} \tag{3.40}$$

Each of the coefficients $a_{ij}(\xi, f)_h$ may be solved for in the form

$$\frac{C(\xi, f)}{D(\xi, f)}, \tag{3.41}$$

where

$$D(\xi, f) = \left(K_\xi \phi_h^{ij}, \sum_{\substack{n \neq m \\ n \neq 0}} \Psi_h^{nm} \right) + 1 \tag{3.42}$$

and $C(\xi, f)$ contains terms of the form $(K_\xi \phi_l^{ij}, \sum_{n \neq m, n \neq 0} \Psi_h^{nm})$ and

$(K_\xi Pf, \sum_{i\neq j, i\neq 0} \Psi_h^{ij})$. Writing $A=\text{diag}(a_{ij})_h$, $B=B_h(\xi)=\text{diag}(K_\xi \phi_h^{ij}, \Sigma\Psi^{nm})$ then

$$|\text{trace}(I+B)^{-1}| \leqslant |\text{trace}\, A|\; |\text{trace}\, A(I+B)|. \tag{3.43}$$

Thus, since trace$(A(I+B))$ is the right-hand side of (3.40) with the $l=h$ term missing, we see that values of ξ for which $(I+B_h(\xi))$ is a singular matrix correspond to points of $S_{\mathbb{C}}$. The inequality of (3.43) illustrates the fact that the singularities of the a_{ijl} may accumulate.

We turn to the existence of a spectral measure for L in the sense of [11].

THEOREM 3.15. *There exists a locally defined spectral measure μ for L. We refer the reader to* [11] *for the relevant concepts.*

To give the proof of Theorem 3.15 we shall need a series of lemmas beginning with

LEMMA 3.16. *The limit*

$$2\pi i\mu(\delta) f = \lim_{\varepsilon\to 0^+} \int_\delta R(\xi+i\varepsilon) f - R(\xi - i\varepsilon) f\, d\xi \tag{3.44}$$

exists for all bounded δ whose closure is contained in $\mathbb{R}\backslash S_{\mathbb{R}}$, $f\in L^2(\mathbb{R}^3, \mathbb{C}^4)\otimes L^2(\mathbb{R}^3, \mathbb{C}^2)$.

Proof.

$$\begin{aligned} &R(\xi+i\varepsilon) - R(\xi - i\varepsilon)\\ &\quad = R_0(\xi+i\varepsilon) - R_0(\xi-i\varepsilon) - R_0(\xi+i\varepsilon)\, K_1\otimes K_2 R(\xi+i\varepsilon)\\ &\qquad + R_0(\xi-i\varepsilon)\, K_1\otimes K_2 R(\xi - i\varepsilon). \end{aligned} \tag{3.45}$$

This follows from an argument like that in Lemma 3.6. Let us define

$$E(x) = |K_1\otimes K_2|^{1/2}\, I_{4\times 4}\otimes I_{2\times 2} = |K_1|^{1/2}\, I\otimes |K_2|^{1/2}\, I,$$

$$F(x) = |K_1 \otimes K_2|^{-1/2}\, K_1\otimes K_2 = \frac{K_1}{|K_1|^{1/2}} \otimes \frac{K_2}{|K_2|^{1/2}};$$

then

$$EF = FE = K_1\otimes K_2.$$

Note that $ER_{00}F = \mathscr{K}$ and by the proof of Lemma 3.1 of [4] we have that the limit (3.44) defines a bounded bilinear form $\mu(\delta, f, g) = (\mu(\delta) f, g)$, $\mu(\delta)$ being defined by the Riesz theorem. (There should be no confusion between the two uses of μ.) It is to be noted here that the results establishing Lemma 3.9 hold when $K_1\otimes K_2$ is replaced by EF. This follows from

$$\|ER_{0,1}F\| \leqslant C(\xi)\, \|K_1\otimes K_2\|_{L^{3/2}}$$

and

$$\|K_1\otimes K_2\|_{L^{3/2}}^{2/3} \leqslant \sum_i \left(\int_{\mathbb{R}^3} |K_i'|^6\, dx\right)^{1/4} \left(\int_{\mathbb{R}^3} |K_i'^{-1}|^2\, |K_i|^2\, dx\right)^{3/4} < \infty. \tag{3.46}$$

Here $C(\xi)$ is continuous. To see that μ is a locally defined spectral measure, it remains to establish that μ is countably additive and that it is multiplicative.

LEMMA 3.17. *μ is locally countably additive in the strong operator topology.*

Proof. Note that $\mu(\cdot, f, g)$ is absolutely continuous "inside" a δ whose closure does not intersect $S_{\mathbb{R}}$. Thus μ is weakly countably additive and by the Pettis theorem μ is countably additive on the Borel sets in δ and this gives the result.

LEMMA 3.18. *Let δ_1 and δ_2 be bounded Borel sets in $\mathbb{R}$ whose closures do not intersect $S_{\mathbb{R}}$. Then $\mu(\delta_1\cap\delta_2) = \mu(\delta_1)\,\mu(\delta_2)$.*

Proof. Suppose δ is an interval ($S_{\mathbb{R}}$ is closed and nowhere dense). It is sufficient to note that

$$f \to \frac{1}{2\pi i}\int_{c(\varepsilon)} f(\xi)\, R(\xi, L)\, d\xi$$

is an algebra map where $c(\varepsilon)$ is a piecewise-smooth closed curve of distance ε from the set δ except near the end points of δ and f is analytic in a neighborhood of δ vanishing at the endpoints of δ. By Lemma 3.9, the bilinear term $(u, R(z)\, v)$ has first order rate of growth near the real axis, i.e., as $\text{im}(z) \to 0^+$ or 0^-. It therefore follows that the mapping

$$f \to \frac{1}{2\pi i}\int_\delta f(\lambda)\{R(\xi + i0, L) - R(\xi - i0, L)\}\, d\xi \tag{3.47}$$

is an algebra map on the set of analytic functions defined above, the limit in (3.47) being taken in the weak sense. The Stone theorem then allows us to extend the multiplicative properties of (3.47) to all continuous functions vanishing at the endpoints of δ and thence by a standard argument to all characteristic functions of closed G_δ sets in d. This completes the proof by the countable additivity of μ. (See Proposition 1.4 of [10], for example.)

To complete the proof of Theorem 3.15 it is only required to construct the domain of the measure μ and prove that μ represents L. The domain of μ we take to be the set

$$D(\mu)=\bigcup_{\delta} \{\text{range } \mu(\delta)\}, \tag{3.48}$$

where δ ranges over all subsets of the type in Lemma 3.16. $D(\mu)$ is of course the analogue of the absolutely continuous subspace of L in $L^2(\mathbb{R}^3, \mathbb{C}^4)\otimes L^2(\mathbb{R}^3, \mathbb{C}^2)$.

Since $R(z)$ commutes with μ and $R(z)^{-1}$ we have

PROPOSITION 3.19. *Every* $\mu(\delta)$ *fixes*

$$\{H^2(\mathbb{R}^3, \mathbb{C}^2)\times L^2(\mathbb{R}^3, \mathbb{C}^2)\}\otimes\{H^2(\mathbb{R}^3, \mathbb{C})\times L^2(\mathbb{R}^3, \mathbb{C})\}.$$

THEOREM 3.20. *For* $x\in D(\mu)\cap D(L)$,

$$Lx=\int_{-\infty}^{\infty} \lambda\, d\mu(x). \tag{3.49}$$

Proof. In view of the proof of Lemma 3.18,

$$R(z)\,\mu(\delta)\lim_{\varepsilon\to 0}\int_{C(\varepsilon)} (\xi-z)\,R(\xi)\,d\xi$$

$$=\mu(\delta)\lim_{\varepsilon\to 0}\int_{C(\varepsilon)} R(\xi)-R(z)\,d\xi=\mu(\delta), \tag{3.50}$$

where we have assumed without loss of generality that δ is an interval.

Therefore if $x\in D(\mu)\cap D(L)$ then $x=\mu(\delta)\,x$ for some δ and for $z\in\rho(L)$,

$$(L-zI)\,x=\int_{\delta}^{*} (\xi-z)\,d\mu(\xi)(x) \tag{3.51}$$

and this gives (3.49).

It is now clear that for $\nu=\sigma^*\mu\sigma$,

$$T=\int \lambda\, d\nu \tag{3.52}$$

on $\sigma^*(D(\mu)\cap D(L))\subseteq D(T)$.

By Theorem (2.2) of [11] the support of ν is contained in $\sigma(T)$. (The following theorem is a corollary to a lemma [13], see also [1].)

PROPOSITION 3.19.

$$\sigma:\quad H^1_{2,\alpha}\oplus L_{2,\alpha}\to L_{2,\alpha}\oplus L_{2,\alpha}\qquad (\alpha>\tfrac{1}{2})$$

and

$$\sigma^*:\quad (H^2_{2,-\alpha}\oplus K_{2,-\alpha})\otimes(H^2_{2,-\alpha}\oplus L_{2,-\alpha})\to H^1_{2,-\alpha}\oplus L_{2,-\alpha}.$$

THEOREM 3.20 ([13, 1]).

$$R(z):\quad L_{2,\alpha}\otimes L_{2,\alpha}\to(H^2_{2,-\alpha}\oplus L_{2,-\alpha})\otimes(H^2_{2,-\alpha}\oplus L_{2,-\alpha}),$$

$\alpha>\frac{1}{2}$, *is bounded for* $z\in\mathbb{C}\setminus(S_{\mathbb{C}}\cup\{0\})$. *It is continuous in* z *on the closed upper half-plane (and closed lower half-plane) except for* $z\in S_{\mathbb{C}}\cup\{0\}$.

COROLLARY 3.21. *If* $f\in L_{2,\alpha}$ *and* $\Phi\phi_l^{i,j}\in H^1_{2,\alpha}$ (2.14) *gives a solution to the steady-state problem.*

4. CONCLUSION

This paper studies a medium which obeys the equations of elasticity in its unperturbed stated and has certain plasma-like properties in a perturbed state.

The reason for this study is simply to examine some of the interesting and rather unusual spectral properties of the resulting equations. The results are comparable to but are not the same as those for Schrödinger operators with complex potential and other near-spectral operators. We know of no empirical study which may indicate an example of such media, but perhaps the rather interesting problem is justification enough. The idea of studying the perturbation of the primary invariant subspaces of an operator of the type considered here may be a valuable concept in spectral theory. This is related to perturbation of spectra though we cannot claim originality here, just ignorance of related work.

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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **121**, 79–90 (1987)

Admissible Pairs and Integral Equations*

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Submitted by Ky Fan

Received June 5, 1985

The aim of this paper is to show how a general convergence principle of Bruck and Reich can be applied to the study of the asymptotic behavior of solutions to nonlinear Volterra integral equations.

1. INTRODUCTION

Our aim in this paper is to show how the general convergence principle presented in [2] can be applied to the study of the asymptotic behavior of solutions to nonlinear Volterra integral equations.

The formulation of this principle involves certain pairs of mappings said to be admissible. We begin the paper by recalling these pairs and providing some examples. We then review several facts concerning the existence and the properties of the solutions to the nonlinear integral equation (3.1).

In the fourth section we establish our main convergence theorem (Theorem 4.2). It is based on an analysis of the behavior of certain functionals defined on the trajectory of the solution to (3.1), rather than the trajectory itself. We then present an application of this result (Theorem 5.6). Another example is given in Section 6 (Theorem 6.1). We conclude the paper by pointing out that Theorem 6.1 can be applied to two problems of nonlinear heat flow in materials with memory. In connection with Theorems 5.6 and 6.1 we also mention two open problems.

2. ADMISSIBLE PAIRS

In the formulation of the convergence principle presented in [2] certain pairs of mappings were used. In this section we recall these admissible pairs and provide some examples.

* Partially supported by the Technion VPR Fund.

0022-247X/87 $3.00

409/121/1-6