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ENERGY PRESERVING BOUNDARY CONDITIONS FOR PLASMA IN A HALF SPACE

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It is the case that for most wave propagation problems in classical physics, the time evolving part of the solution satisfies a first order constant deficit hyperbolic system. The spatial part of these systems is therefore never elliptic. In the case of plasmas, this is a complication of the second order precisely because the relevant system has non-constant deficit. The ideas of [3] are used freely below.

The boundary conditions which preserve energy for conducting fluids like MHD plasmas are of obvious importance. We shall classify all such constant boundary conditions in two important subcases where the external magnetic field is either 1) orthogonal or 2) parallel to the boundary. The boundary conditions studied here are of the usual type for such systems: the real and imaginary parts of the solution are required to lie in a prescribed subspace at the boundary.

It seems likely that the operators studied here will exhibit some unusual behavior. Some questions that remain to be studied include the existence of wave operators for the two boundary orientations of A^3 (see below for definition), the existence and nature of surface waves, the representation of the spectral families in terms of the modes, the asymptotics of potential scattering in a half space, the effect of boundary layers, etc. These questions are complicated because of the somewhat complex and unusual behavior of the plane wave speeds. Other questions concern energy distributions and partition of energy. We hope to study some of these questions elsewhere (see [4], [5]).

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The magnetohydrodynamics equations may be written in linearized from as [1]

$$
\begin{aligned}
\rho_0 \frac{\partial u}{\partial t} &= -c^2 \nabla \rho + \nabla \times H \times \mu H \\
\frac{\partial H}{\partial t} &= \nabla \times (u \times H_0) \\
\frac{\partial \rho}{\partial t} &= -\rho_0 \nabla \circ u \,.
\end{aligned}
\tag{1.1}
$$

Here, ρ_0 is the equilibrium density, $H_0 = (h_1, h_2, h_3)$ is the external magnetic field, c is the equilibrium speed of sound, μ is the magnetic permeability, H is the magnetic field, u is the velocity field and ρ is the density.

We may write (1.1) in matrix form for appropriate A_j as

$$
i\frac{\partial u}{\partial t} = \sum_{j=1}^{3} A_j i \frac{\partial u}{\partial x_i}, \quad i = \sqrt{-1} \tag{1.2}
$$

For the right hand side of (1.2), we write $A(D)$. By an appropriate change of variables, we may assume that ρ_0, μ and c are 1.

As noted, we shall consider the case of a half-space where the vector H_0 is given by either $(0, 0, h_3)$ or $(0, h_2, 0)$. When it is necessary to distinguish between these two cases, we shall do so by using a superscript as $A^3(D)$ and $A^2(D)$. By $\mathbf{n}$, we mean the inward unit normal vector to ∂G (=boundary of G) where G is some domain in $\mathbf{R}^3$. In all applications here,

$$
G = \mathbf{R}^3_+ = \{x \mid x = (x_1, x_2, x_3), \quad x_3 > 0\}.
$$

Definition 1.1. [2] *A subspace $S(\mathbf{n})$ of $\mathbf{R}^7$ is a maximal conservative boundary space for $A(D)$ in G if and only if $\zeta \circ A(\mathbf{n})\zeta = 0$ for all ζ in $S(\mathbf{n})$ and $S(\mathbf{n})$ is maximal with respect to this property (note $A(\mathbf{n}) = A_3$ from (1.2)).*

By the positive and negative eigenvectors we shall mean those corresponding to the positive and negative eigenvalues, respectively. Their number depends on H_0 and p.

Lemma 1.2. [2] *Let $N(A(\mathbf{n}))$, $X(\mathbf{n})$, $Y(\mathbf{n})$ denote, respectively, the null space of $A(\mathbf{n})$, the subspace spanned by the positive eigenvectors of $A(\mathbf{n})$, and the subspace spanned by the negative eigenvectors of $A(\mathbf{n})$. Let ζ_j be any orthonormal base of $N(A(\mathbf{n}))$ (for $A^3(D)$ $j = 1$, for $A^2(D)$ $j = 1, 2, 3, 4, 5$). Let ξ_j be any base of $X(\mathbf{n})$ which is orthonormal with respect to $A(\mathbf{n})$; i.e., $\xi_i \circ A(\mathbf{n})\xi_j = \delta_{ij}$ (for A^3 $j = 1, 2$ for A^2, $j = 1$) and η_j be any base of $Y(\mathbf{n})$ orthonormal with respect to $-A(\mathbf{n})$ (j as for $X(\mathbf{n})$). Suppose $S^{3,2}(\mathbf{n})$ is the subspace of $\mathbf{R}^7$ spanned by $\{\zeta_j,\ \xi_i + \eta_i\ (\text{all } i, j)\}$. Then, $S^{3,2}(\mathbf{n})$ is a maximal conservative boundary space for $A(D)$ and and any such boundary space may be constructed in this way (in general, we have $S^H \circ (\mathbf{n})$).*

The lemma is obvious when the eigenvalues of $A(\mathbf{n})$ are computed (here and below $\mathbf{n} = (0, 0, 1)$). Under the assumptions about μ, ρ_0, for A^3 the eigenvalues of

$A(\mathbf{n})$ are $\lambda = 0$ (multiplicity 1), $\lambda_{\pm 1} = \lambda_{\pm 3} = \pm 1$ (multiplicity 1) and $\lambda_{\pm 2} = \pm h_3$ (multiplicity 2). For A^2 they are $\lambda_\pm = \pm(1 + h_2^2)^{1/2}$ (multiplicity 1) and $\lambda = 0$ (multiplicity 5). For A^2 then, a boundary space has dimension 6 while for A^3, it has dimension 4.

To classify such spaces, consider any basis of $X \oplus y$ say for A^3. We have ξ_i, ξ_2, ξ_3 and η_1, η_2, η_3 with $\lambda_2 \leftrightarrow \xi_1$, $\lambda_2 \leftrightarrow \xi_2$, $\lambda_1 \leftrightarrow \xi_3$, etc. Let e_2^1, e_2^2, e_1^3, e_{-2}^1, e_{-2}^2, e_{-1}^3 be any such fixed basis. Then, we have

$$\begin{aligned} \eta_i &= d_{i1} e_{-1}^1 + d_{i2} e_{-1}^2 \quad (i = 1, 2) \\ \eta_3 &= d_3 e_{-1}^3 \\ \xi_i &= c_{i1} e_1^1 + c_{i2} e_1^2 \quad (i = 1, 2) \\ \xi_3 &= c_3 e_1^3. \end{aligned} \tag{1.5}$$

In order that the orthonormality conditions be satisfied, it must be that $c_{i1}c_{j1} + c_{i2}c_{j2} = \delta_{ij}$, and thus the matrix $[c_{ij}]$ must be orthogonal and the same is true of $[d_{ij}]$. The constants d_3 and c_3 must have the value 1. Thus, by letting $C = [c_{ij}]$ and $D = [d_{ij}]$ run through all possible such matrices, we obtain every basis of a maximal conservative boundary space for A^3. There are, therefore, just two possible orientations for such a boundary space. The orientation of the $\{e_{\pm i}^j\}$ (in which case the determinant of $CD = 1$) and the opposite orientation ($\det(CD) = -1$). We shall restate this in slightly different terms:

Theorem 1.3. *Suppose k_0, k_1^+, k_2^+, k_3^+, k_1^-, k_2^-, k_3^- are orthonormal (in the usual $\mathbf{R}^7$ sense) eigenvectors of $A^3(\mathbf{n})$ spanning the null space of A^3, X and Y, respectively. Then the subspace of $\mathbf{R}^7$ given by $S(\mathbf{n}) = \text{span}\{k_0, \xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3\}$ where*

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} C_{2\times 2} & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} k_1^+ \\ k_2^+ \\ k_3^+ \end{bmatrix} \tag{1.6}$$

with a similar statement for D, etc., and $\sqrt{h_3}C$, $\sqrt{h_3}D$ belong to the orthogonal group of dimension 2 ($\mathcal{O}(2)$), is a maximal conservative boundary space for A^3 and every such boundary space is obtained by letting $\sqrt{h_3}C$ and $\sqrt{h_3}D$ run through $\mathcal{O}(2)$.

There is an interesting difference in this analysis for the special case where $h_3 = 1$. Then $\lambda_{\pm 3} = \lambda_{\pm 2} = \lambda_{\pm 1}$. We will not consider this here.

For A^2, the statement is

Theorem 1.4. *Suppose k_{01}, k_{02}, k_{03}, k_{04}, k_{05} and k_1^+, k_1^- are orthonormal eigenvectors of $A^2(\mathbf{n})$ spanning the null space, X and Y, respectively. The subspace $S^2(\mathbf{n}) = \text{span}\{k_{01}, k_{02}, k_{03}, k_{04}, k_{05}, \xi_1 + \eta_1\}$ where $\xi_1 = (h_2^2 + 1)^{-1/4} k_1^+$ and $\eta_1 = (h_2^2 + 1)^{-1/4} k_1^-$ is a maximal conservative boundary space for $A^2(D)$ and every such boundary space is obtained in this way.*

Corollary 1.5. *The boundary space $S^2(\mathbf{n})$ is determined by the boundary condition* $\{\xi = (\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7) \in \mathsf{R}^7 \mid h_2\xi_5 + \xi_7 = 0\}$.

The associated boundary operator is

$$B^2 = [0, 0, 0, 0, h_2, 0, 1]. \tag{1.10}$$

Note that the boundary condition depends on the intensity of the external field.

Let us now consider the operator $A^3(D)$. The following two one parameter classes of boundary operators are obtained:

$$B^+_{\lambda,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^+_{\infty,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.19}$$

$$B^+_{\lambda,2} = \begin{bmatrix} 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 \end{bmatrix}$$

$$B^+_{\infty,2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{1.20}$$

Note that the boundary operators are independent of the external field strength.

In the general case where H_0 is neither parallel nor orthogonal to the boundary plane, we may, by choosing the coordinate system in a judicious way, take $H_0 = (h_1, 0, h_3)$. The previous results in the case of A^3 essentially apply to this more general situation. The two sets of boundary conditions given for A^3 above reduce to one in this case. It is seen that $S^{H_0}(\mathbf{n})$ is of dimension four. We could now examine the operators A^3 and A^2 in $L_2(\mathsf{R}^2 \times \mathsf{R}_+, \mathsf{C}^7)$ to determine that they are selfadjoint there. We will consider this elsewhere [5].

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