THE SPECTRAL REPRESENTATION OF SINGULAR DISPERSIVE SYMMETRIC HYPERBOLIC SYSTEMS

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Abstract

The wave propagation problems of classical physics may be studied as symmetric hyperbolic systems, $-iE(x)\partial_t u = A(D)u + B(x)u + f(x,t)$. The matrix E is not assumed definite here and the spatial part may be nonselfadjoint. However a limiting absorption principle is valid and a local spectral representation may be developed. This type of system may be thought of as a limiting case for certain kinds of crystals.

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1 INTRODUCTION

We have discussed the system (see [2])

$$-iE(x)\partial_t u = A(D)u + B(x)u + f(x,t) \tag{1.1}$$

where $E(x)$ is positive definite and E and B satisfy conditions of the form $(|x| \to \infty, \epsilon > 0)$

$$|I - E| = O(|x|^{-1-\epsilon}),\ |B| = O(x^{-1-\epsilon}). \tag{1.2}$$

Here we will consider the case where E is only semidefinite in a certain sense. Several kinds of extensions to (1.1) have been discussed in the literature, for example the case when $E(x)$ is not a bounded function of x, or when it is time dependent, etc. The present paper extends (1.1) in a somewhat different way, allowing $E(x)$ to fail to be invertible. This has to be done with some restriction since (1.1) then fails to be well-posed in general. Our work here is a direct extension of previous work on transient waves in singular media [4]. The extension discussed here may be thought of as a limiting case for certain crystals.

The systems (1.1) describe the wave propagation problems of classical physics such as Maxwell's equations, the equations of acoustics, elasticity and other phenomena. We assume that $A(D)$ is strongly propagative here with additional assumptions reviewed below.

Spectral representation for the spatial part of (1.1) is defined by the following statements, with appropriate modifications as found in section 3.

Suppose A is a selfadjoint operator in a Hilbert space H. Suppose also that

a) There is a Borel measure μ whose support is the spectrum of A and

b) There is a unitary operator U taking H to $L_2(\mathbb{R}, \mu)$ such that

$$A_1 = UAU^{-1}$$

acts as the operator

$$A_1\psi(x) = x\psi(x),$$

with ψ in the domain of A_1. The domain of A_1 is defined in the usual way.

For nonselfadjoint operators related to (1.1) it is possible to make a similar definition since most of these operators have their essential spectrum on the real axis or perhaps on the real axis together with a set around the origin. Thus it may be possible to localize the problem to obtain a local representation.

In section 2 we review certain facts from [2]. In section 3 we construct a spectral representation. In section 3 a brief treatment using [1] is used to obtain the desired representation.

2 PRELIMINARY RESULTS

We state briefly some results of [2] modified to the present situation of limit crystals.

We assume $A(D)$ has the form: $(i = \sqrt{-1})$

(2.1)
$$iA(D) = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}$$

A_j are constant real symmetric $m \times m$ matrices. The symbol of $A(D)$ is the matrix

$$A(p) = \sum_{j=1}^{n} A_j p_j \tag{2.2}$$

with $p = (p_1, p_2, \ldots, p_n)$ a real n-dimensional vector. The propagation speeds are the solutions $\lambda(p)$ of the equation

$$\det(A(p) - \lambda(p)I) = 0 \tag{2.3}$$

I is identity matrix. If the number of solutions is even, $A(D)$ is called elliptic. For most of the equations of classical physics there are (counting multiplicities) an odd number of solutions and it is known that they are continuous functions of p. Moreover the associated orthoprojectors $P_j(p)$ for $\lambda_j(p)$ are measurable as functions of p [3]. We have $P_j(p)A(p) = A(p)P_j(p) = \lambda_j(p)P_j(p)$. For $\lambda = 0$, the null projector $P_0(p)$ determines an operator on L_2 by $P_0 = \Phi^* P_0(p)\Phi$. Φ^* is the (unitary) adjoint of Φ [3].

Define P as $I - P_0$ on L_2. The weighted spaces $L_{2\pm\alpha}$ are definied as the sets of measurable functions satisfying the condition

$$\int (|1 + |x|\,|^{\pm\alpha} |f(x)|)^2 dx < \infty$$

where the norm on such a space is defined in the obvious way. It is elementary to see that $\|Pf\|_{L_{2,\alpha}} \leq C\|f\|_{L_{2,\alpha}}$ for some constant C. Since $L_{2,\alpha}$ and $L_{2,-\alpha}$ are dual, the restriction of P to $L_{2,\alpha}$ has an adjoint P^* on $L_{2,-\alpha}$ which is a bounded operator on L_2. In fact,

$$(Pf, g)_{L_2} = (f, P^* g)L_2;\ f \text{ in } L_{2,\alpha},\ g \text{ in } L_{2,-\alpha}.$$

Therefore, P^* gives a bounded unique extension of P to $L_{2,-\alpha}$.

We require, as suggested in the introduction, the real symmetric matrix $E = E(x)$ satisfy the condition

$$(2.4) \qquad (Ev, v) \geq 0 \text{ for all complex } m\text{-vectors } v.$$

And that there is a constant matrix E_1 and positive scalars c and d with

$$(2.5) \qquad c(E_1 v, v) \leq (Ev, v) \leq d(E_1, v, v).$$

We require that the range and null space of E are preserved by A and the same is therefore true of E_1. The range and null space of $E(E_1)$ are orthogonal in L_2.

Notation:

1. $H = L_2(\mathbb{R}^n, \mathbb{C}^m)$ ($A(D)$ is selfadjoint on H.)
2. H_E is the seminorm space generated by

$$\|f\|_E = \left(\int (E(x)f(x), f(x))_{C^m} dx \right)^{\frac{1}{2}}.$$

This is a norm on range E. Furthermore, $A(D)$ must be selfadjoint on range E.

(2.5) implies that E has constant rank r. It may be shown that there is an $r \times m$ matrix of rull rank such that $E = C^*C$. Furthermore, $F(x) = C^*(CC^*)^{-1}$ is a right inverse for C. Moreover it is easily seen that F maps

$L_2(\mathbb{R}^n, \mathbb{C}^r)$ to range E as a unitary map when range E has the H_E norm. Define the domain of the operator $FF^*A(D)$ by

$$D(FF^*A(D)) = \{v \text{ in } H_E \,|\, v \text{ is in the domain of } A(D)\}. \tag{2.6}$$

It is now easy to see that $FF^*A(D)$ is selfadjoint on H_E. Note that $EFF^*(= FF^*E)$ is constant and equal to $E_1F_1F_1^*$. We assume that for some $\epsilon > 0$,

$$|E - E_1| = O(|x|^{-1-\epsilon}) \tag{2.7}$$

and

$$|B| = O(|x|^{-1-\epsilon}).$$

We also assume E and B are essentially bounded.

Upon replacing E by $F_1^*EF_1$ and $A(D)$ by $F_1^*A(D)F_1$, we may and shall suppose that $E_1 = EFF^* = I_{|H_E}$, the identity restricted to H_E. Note also that ($|x| \to \infty$)

$$|FFF^* - F_1F_1^*| = |F_1F_1^*(E_1 - E)F_1F_1^*| = O(|x|^{-1-\epsilon}). \tag{2.8}$$

Hence our results are applicable when $F_1^*A(D)F_1$ replaces $A(D)$.

The following result of [2] and [5] is needed:

THEOREM 2.1 *If $\alpha\ \beta > 1/2$, then $\lambda \to P(A(D) - \lambda I)^{-1}$ is a continuous operator-valued function in $\mathbb{C}^+ - \{0\}$ or $\mathbb{C}^- - \{0\}$ (the upper or lower half-planes including the real axis punctured at zero).*

The operators $P(A(D) - \lambda I)^{-1}$ are compact with domain $L_{2,\alpha}$ and range in $L_{2,-\beta}$.

COROLLARY 2.1 *$\lambda \to P(A(D) - \lambda I)^{-1}$ is holomorphic in $\mathbb{C}^+ - \mathbb{R}$ or $\mathbb{C}^- - \mathbb{R}$.*

Our equation in H_E is

$$-I_{|H_E} i\partial_t u = FF^* A(D)u + FF^* Bu - f'(x,t) \tag{2.9}$$

where $f' = FF^* f$.

If $f(x,t)$ is a separable sinusoidal disturbance, $f(x,t) = e^{-\lambda it} f(x)$, then (2.9) becomes

$$FF^* A(D)u + FF^* Bu - \lambda u(x,\lambda) = f \quad \text{(writing } f \text{ for } f'\text{)}. \tag{2.10}$$

Here λu means $\lambda I_{|H_E} u$ but our assumptions allow us to use the simpler notation. The limiting absorption principle for (2.9) is essentially a solution for (2.10).

Writing I_E for $I_{|H_E}$, we have the equivalent expression,

$$I_E A(D)u + I_E Bu - \lambda E u(x,\lambda) = Ef. \tag{2.11}$$

Thus

$$(A(D) - \lambda I)^{-1}[I_E A(D)u + I_E Bu - \lambda E u] = (A(D) - \lambda I)^{-1} Ef. \tag{2.12}$$

Hence we have

$$\begin{aligned} I_E u + \lambda I_E (A(D) - \lambda I_E)^{-1} u + I_E (A(D) - \lambda I_E)^{-1} Bu - \lambda (A(D) - \lambda I_E)^{-1} Eu \\ = (A(D) - \lambda I)^{-1} Ef \end{aligned}$$

and so

$$(2.13)\quad [I_E + \lambda(A(D) - \lambda I_E)^{-1}(I_E - E + I_E B\lambda^{-1})]u = (A(D) - \lambda I)^{-1}Ef.$$

It follows that

$$[I_E - I_E P_0(I_E - E + I_E B\lambda^{-1}) + I_E P\lambda(A(D) - \lambda I_E)^{-1}(I_E - E + I_E B\lambda^{-1})u$$
$$= (A(D) - \lambda I)^{-1}Ef.$$

Note that $I_E P_0 I_E = P_0 I_E$. At least formally then,

$$(2.14)\quad u(x,\lambda) = [I_E - I_E P_0(I_E - E + I_E B\lambda^{-1}) + I_E P\lambda(A(D) - \lambda I_E)^{-1}$$
$$(I_E - E + I_E B\lambda^{-1})]^{-1}(A(D) - \lambda I)^{-1}Ef.$$

It is shown in [2] that

$$J(\lambda) = [I_E - I_E P_0(I_E - E + I_E B\lambda^{-1}) + I_E P\lambda(A(D) - \lambda I_E)^{-1}(I_E - E + I_E B\lambda^{-1})]$$

has an inverse in the following sense.

THEOREM 2.2 $\lambda \to J(\lambda)^{-1}$ *is continuous as an operator from* $H_{E,\beta}$ *to itself provided* $\beta > 1/2$ *is sufficiently close to 1/2, where* λ *is restricted to the upper or lower complex plane with certain sets* $\Sigma_\pm$ *(depending on* B*) deleted from them. These sets may include a neighborhood of zero but elsewhere are discrete off the real axis and on the real axis they have measure zero and are nowhere dense. Generally, the values of* $J(\lambda)^{-1}$ *on the real axis of* $\mathbb{C}^+ - \Sigma_+$ *are different from its values on the real axis of* $\mathbb{C}^+ - \Sigma_-$.

The proof is essentially contained in [2].

The mentioned neighborhood of zero in Theorem 2.2 is contained in

$$|\lambda| \leq \frac{\|P_0 I_E B\|}{\|I_E - P_0(I_E - E)\|}.$$

In general it is not possible to eliminate this neighborhood.

Theorem 2.2, combined with (2.14) gives the required limiting absorption principle.

3 SPECTRAL REPRESENTATION

We add the additional assumption here that $EB = BE$.

Let $R(\lambda) = A(D) - \lambda I)^{-1}$ and this operator will be defined on various convenient spaces as necessary. Similarly, we write

$$R_{EB}(\lambda) = (FF^* A(D) + FF^* B - \lambda I)^{-1}.$$

These operators are members of $B(H_{E,\alpha'} H_{E,-\beta})$ $\alpha \neq \beta$, α, $\beta > 1/2$ but close enough to $1/2$ so Theorem 2.2 may be applied. In other words so that $R(\lambda)$ and $R_{EB}(\lambda)$ have boundary values in $B(H_{E,\alpha'} H_{E,-\beta})$ away from $\Sigma_{\pm}$.

Let ω stand for any unit vector in $\mathbb{R}^n$ and Ω the set of all such unit vectors with Lebsegue surface measure. M_δ will consist of all measurable square integrable $L_2(\Omega)$-valued functions defined on δ such that the closure of δ is in $\mathbb{R} - \Sigma_{\pm}$. As an operator on $H_{E,\alpha}$, define $F(\lambda) : H_{E,\alpha'} \to L_2^0(\Omega)$ by

(3.1) $$(F(\lambda)g)(\omega) = 2^{-\frac{1}{2}}(2\pi)^{\frac{\pi-1}{2}} \sum_{i\neq 0} \int_{R^n} \exp(-\sqrt{-1}\,\lambda\lambda_i(\omega)x\cdot\omega)P_i(\omega)f(x)dx$$

where $L_2^0(\Omega)$ is $P(\omega)L_2(\Omega)$.

It follows from [5] that for $\lambda \neq 0$, λ real, that

$$(F(\lambda)f, F(\lambda)g) = (2\pi i)^{-1}R^+(\lambda) - R^-(\lambda)) \tag{3.2}$$

$R^{\pm}(\lambda)$ refer to the boundary values of $R(\lambda)$ as noted above.

By Stone's representation theorem, writing $\pi(\Delta)$ for the spectral resolution associated with R, we have

$$(\pi(\Delta)f, g) = \int_\Delta (F(\lambda)f, F(\lambda)g_d\lambda.$$

We formulate a similar representation associated with the resolvent $R_{EB}(\lambda)$.

We assume that λ is outside of $\Sigma_\pm$ as appropriate.

By (2.14),

$$R_{EB}(\lambda) = R(\lambda) - \lambda R(\lambda)(I_E - E + B\lambda^{-1})R_{EB}(\lambda). \tag{3.3}$$

Following (3.2) we define

$$F_{EB}(\lambda) = F(\lambda)E - F(\lambda)(I_E - E + B\lambda^{-1})R^+_{EB}(\lambda). \tag{3.4}$$

We also define $R^*_{EB}(\lambda)$ by replacing B with B^* so that

$$R^*_{EB}(\lambda) = R(\lambda) - \lambda R(\lambda)(I_E - E + B^*\lambda^{-1})R^*_{EB}(\lambda) \tag{3.5}$$

$R^*_{EB}(\lambda)$ has boundary values on the real axis outside of sets $\Sigma^*_\pm$. And for λ outside of the union of $\Sigma_\pm$ and $\Sigma^*_\pm$, and f, g in $H_{E,\alpha'}$ we have

$$(R^+_{EB}(\lambda)f, g)_E = (f, R^*_{EB}(\lambda)g)_E. \tag{3.6}$$

(Here we have used the fact that E and B commute.)

Define

$$F^{*}_{EB}(\lambda) = F(\lambda)E - F(\lambda)(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda). \tag{3.7}$$

Then f, g and λ as in (3.6),

$$(F_{EB}(\lambda)f,\ F^{*}_{EB}(\lambda)g) = (F(\lambda)[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f,$$

$$F(\lambda)[E - \lambda(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda)]g)$$

$$= (2\pi i)^{-1}(R^{+}(\lambda)[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f$$

$$- R^{-}(\lambda)[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f,$$

$$[E - \lambda(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda)]g)$$

$$= (2\pi i)^{-1}(R^{+}(\lambda)[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f,$$

$$[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)g)$$

$$- (2\pi i)^{-1}(R^{-}(\lambda)[E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f,$$

$$[E - \lambda(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda)]g)$$

$$= -(2\pi i)^{-1}(R^{+}_{EB}(\lambda)f,\ [E - \lambda(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda)]g)$$

$$- (2\pi i)^{-1}(E - \lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)]f,\ R^{*+}_{EB}(\lambda)g)$$

$$= (2\pi i)^{-1}(R^{+}_{EB}(\lambda)f, E_g) + (2\pi i)^{-1}(R^{+}_{EB}(\lambda)f,$$

$$- \lambda(I_E - E + B^{*}\lambda^{-1})R^{*+}_{EB}(\lambda)g)$$

$$- (2\pi i)^{-1}(Ef, R^{*+}_{EB}(\lambda)g) - (2\pi i)^{-1}(-\lambda(I_E - E + B\lambda^{-1})R^{+}_{EB}(\lambda)f,$$

$$R^{*+}_{EB}(\lambda)g)$$

$$= (2\pi i)^{-1}(R^{+}_{EB}(\lambda)f, g)E - (2\pi i)^{-1}(f, R^{*+}_{EB}(\lambda)g)_E$$

$$= (2\pi i)^{-1}(R^{+}_{EB}(\lambda)f - R^{-}_{EB}(\lambda_f, g)_E.$$

We have established

$$(F_{EB}(\lambda)f,\ F^{*}_{EB}(\lambda)g) = (2\pi i)^{-1}(R^{+}_{EB}(\lambda)f - R^{-}_{EB}(\lambda)f, g)_E. \tag{3.8}$$

Following the proof of Theorem 2.5 of [1] can prove the existence of a spectral resolution $F(\delta)$ for R_{EB} defined on H_E and $F_{EB}(\lambda)$ may be extended to H_E to obtain the following

THEOREM 3.1 *For f, g in H_E there is a projection-valued measure $F(\cdot)$ such that*

$$(F(\delta)f, g)_E = \int_\delta (F_{EB}(\lambda)f,\ F^{*}_{EB}(\lambda)g)d\lambda. \tag{3.9}$$

If $B = B^*$, then δ can be any Borel set.

To construct the local analogue of wave operators we use the notation $F_E(\lambda)$ for $F_{EB}(\lambda)$ when $B = 0$ and define for f, g in $H_{E,\alpha'}$

$$(X_\pm(\delta)f, g)_E = \int_\delta (F_E(\lambda)f,\ F_E(\lambda)B^* R^{*\pm}_{EB}(\lambda)g)d\lambda \tag{3.10}$$

$$(Y_\pm(\delta)f, g)_E = \int_\delta (F_E(\lambda)BR^{\pm}_{EB}(\lambda)f,\ F_E(\lambda)g)d\lambda. \tag{3.11}$$

The operators X and Y extend to H_E since F_E does. Let F_E be the selfadjoint spectral resolution associated with $F_E(\lambda)$ and define $W_\pm(\delta) = F_E(\delta) - X_\pm(\delta)$, $Z_\pm(\delta) = F_E(\delta) - Y_\pm(\delta)$. It is easily checked that $F_E(\delta) = W_\pm(\delta)Z_\pm(\delta)$ and that $F_E(\delta) = Z_\pm(\delta)W_\pm(\delta)$ so that W and Z intertwine $R_E(\lambda)$ and $R_{EB}(\lambda)$ and in fact, $W_\pm^{-1} = Z_\pm$. To outline the proof we note the crucial steps: W and Z and one-to-one and onto maps. We do the computation for W only. Suppose there is f such that $W_\pm f = 0$. Then

$$\int_\delta (F_E(\lambda)[BR_{EB}^\pm(\lambda) - I]f,\ F_E(\lambda)g)d\lambda = 0 \tag{3.12}$$

for all g. But then (a.e.)

$$F_E(\lambda)[BR_{EB}^\pm(\lambda) - I]f = 0. \tag{3.13}$$

and so

$$[BR_{EB}^\pm(\lambda) - I]f = 0. \tag{3.14}$$

But this implies

$$R_{EB}^\pm(\lambda) = R_E^\pm(\lambda) - R_E^\pm(\lambda)BR_{EB}^\pm(\lambda). \tag{3.15}$$

Thus W is one-to-one. An entirely similar argument shows that W is onto. $W_\pm$ are the local wave operators for $L = FF^*A(D) + B$. From this we obtain the following result.

THEOREM 3.2 *Let $\alpha(\lambda) = \lambda$ for λ in δ and 0 otherwise. Then $\alpha(L)$ is defined and for f in H_E,*

$$F_{EB}(\alpha(L)f)(\lambda) = \alpha(\lambda)(F_{EB}(Lf)(\lambda). \tag{3.16}$$

Remark. We assumed that B and E commute for some of these results. This is a strong assumption and it would be useful to have a weaker hypothesis. However important problems such as acoustic waves in a rotating fluid satisfy this condition.

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