IMA Journal of Applied Mathematics (1989) **43**, 47–69

Waves in a Perfectly Conducting Fluid Filling a Half-Space

WILLIAM V. SMITH

Mathematics Department, Brigham Young University, Provo, Utah 84602, *USA*

[Received 18 December 1987 and in revised form 21 September 1988]

The constant, maximal, energy preserving boundary conditions for the equations of magnetohydrodynamics in a perfectly conducting half-space give rise to two essentially different selfadjoint operators in the case when the external magnetic field is orthogonal to the boundary and exactly one such operator when the external field is parallel to the boundary. Neither of these problems admits surface waves. For a normalized external field, the generalized eigenfunction expansion is given below. It is shown that, in the second case, the modes are not coupled by the boundary, while for only one boundary condition for the orthogonal field is the wave motion essentially that of free space (in the sense that solutions are delivered by the group which determines solutions for the free space problem for special initial data). The Alvén wave in the parallel field case acts as a grazing wave. Asymptotic wave motion for perturbed problems (inhomogeneous media) is investigated as well as local decay of energy (this is *not* altogether trivial, since the operators involved are *never* coercive even *off* their null spaces).

1. Introduction

IN THIS paper, we study waves of finite energy propagating in a perfectly conducting compressible fluid-like medium filling a half-space, whose magnetic field, density, and fluid velocity are of relatively small variation, and so are governed by a linear first-order system of partial differential equations. The system we study, well known for many years, can be derived from the famous Maxwell system. We refer the reader to [14] for such a derivation. In order to place the present work in context with other wave motion problems, let us briefly consider the case of electromagnetic waves and elastic waves. The Maxwell equations which govern the propagation of electromagnetic waves are in the usual vector notation

$$\nabla \circ D = \rho_e$$
$$\nabla \circ B = 0$$
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \times H = J + \frac{\partial D}{\partial t}.$$

Assuming the medium is at rest, $D = \varepsilon E$ and $B = \mu H$, where ε and μ are, in general, 3×3 matrices representing the electric permittivity and magnetic permeability, respectively. The system we study here (but in linearized form) is

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B),$$

$$\rho \frac{\mathrm{D}V}{\mathrm{D}t} = -a^2 \nabla \rho + \mu (\nabla \times H) \times H,$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0,$$

$$\nabla \circ B = 0.$$

These are the so-called MHD fluid equations. For comparison purposes, let us also indicate the equations of elasticity. The linear equations describing the propagation of elastic waves in a source-free medium are (summation on repeated indices)

$$\rho \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k} \quad (i = 1, 2, 3),$$

where w is the displacement vector and $\sigma = (\sigma_{ik})$ is the stress tensor. Introducing the column vector $u = (u_1, u_2)$ with nine components, with u_1 the transpose of $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$ and u_2 the velocity vector, the elasticity equations may be written as

$$\frac{\partial u}{\partial t} = E^{-1}(x) A\left(\frac{\partial}{\partial x}\right) u = \begin{bmatrix} 0 & E_0^{-1}(x) M\left(\frac{\partial}{\partial x}\right) \\ \rho(x)^{-1\mathrm{t}} M\left(\frac{\partial}{\partial x}\right) & 0 \end{bmatrix} u.$$

Here the term involving u_1 is the divergence of the stress tensor and the term involving u_2 is a form of Hooke's law relating stress and strain. The point is that the hyperbolic part of each of these three problems, as well as many others, has the common formulation

$$\frac{\partial u}{\partial t} = E^{-1}(x) A\left(\frac{\partial}{\partial x}\right) u = E^{-1}(x) \sum_{j=1}^{3} A_j \frac{\partial u}{\partial x_j},$$

and the study of the (constant) matrices A_j lies at the heart of each problem. It is to be expected that methods useful in one of these problems will find use in each of the others. This of course is the case and has been shown in many contexts. The problem of particular interest here is the half-space problem. Many problems of irregular boundary may be reduced (at least locally) to plane boundary problems and this adds more physical interest to the present problem [2].

For each of the three systems (Maxwell, MHD, elastic) we may ask a number of interesting and important questions. For example, what boundary conditions imply that all energy is confined to the half-space? Can all such boundary conditions be characterized in a meaningful way? If the energy is conserved, how does the boundary interact with approaching waves? Does it couple them? In the case of our problem (MHD), does the direction and magnitude of the external magnetic field make a difference? Are the boundary conditions which preserve energy affected by the external field? Does the energy decay locally? What can be concluded when inhomogeneities are introduced? We shall attempt to answer these questions for the MHD problem. Some of these questions have been studied (again for the half-space) for electromagnetic waves in [7]. The *general* problem for elastic waves has not been studied directly to the author's knowledge. The general method used below appears to be applicable to this problem (see also [8]). Some contrasts between the Maxwell and elastic systems and the system studied here will be indicated shortly.

The problem studied here differs from those specific half-space wave propagation problems studied previously in several respects, two of which are that the system in not strongly propagative [1] and there are three distinct propagation speeds (see [10]–[12]). Here we shall investigate two cases which appear to be rather interesting and relevant. These are (1) where the external magnetic field is orthogonal to the boundary and (2) where the external magnetic field is parallel to the boundary plane. In case (1) there are two sets of (constant) one-parameter families of boundary conditions which preserve energy. In case (2) there is but a single (constant) energy preserving boundary condition. This second case has an important relation to the so-called 'MHD pinch' and its associated instability of confinement of conducting fluid (the reader will see from (1.11) that an external field parallel to the boundary is not enough: there is a relation between the fluid density and the external field strength which must be maintained). Furthermore, the Alfvén waves have nothing to do with this instability—see Theorem 3.6 below [9]. In linearized form, the set of equations we will consider are

$$\left.\begin{aligned}\rho_0\frac{\partial u}{\partial t}&=-a^2\nabla\rho+\nabla\times H\times\mu^{-1}H_0,\\ \frac{\partial H}{\partial t}&=\nabla\times(u\times H_0),\\ \frac{\partial \rho}{\partial t}&=-\rho_0\nabla\circ u.\end{aligned}\right\}\tag{1.1}$$

Here u, H, and ρ are, respectively, the fluid velocity, magnetic field, and density which depend on the four variables (x_1, x_2, x_3, t). The operator ∇ is given by $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and ρ_0 and H_0 are the equilibrium values of ρ and H. The coordinate system is chosen so the boundary coincides with the plane $(x_1, x_2, 0)$. By a straightforward change of variables, we may suppress the constants μ_0, a, and ρ_0. This gives the system

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=-\nabla\rho+\nabla\times H\times H_0,\\ \frac{\partial H}{\partial t}&=\nabla\times(u\times H_0),\\ \frac{\partial \rho}{\partial t}&=-\nabla\circ u.\end{aligned}\right\}\tag{1.2}$$

If we write $H_0=(h_1, h_2, h_3)$, then equation (1.2) may be written as

$$-\mathrm{i}\frac{\partial v}{\partial t}=\sum_{j=1}^{3}-A_j\mathrm{i}\frac{\partial v}{\partial x_j} \quad (\mathrm{i}=\sqrt{-1}), \tag{1.3}$$

where i is introduced for later convenience and

$$A_1=\begin{bmatrix} 0 & 0 & 0 & 0 & -h_2 & -h_3 & -1 \\ 0 & 0 & 0 & 0 & h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h_2 & h_1 & 0 & 0 & 0 & 0 & 0 \\ -h_3 & 0 & h_1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.4}$$

$$A_2=\begin{bmatrix} 0 & 0 & 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & 0 & -h_3 & -1 \\ 0 & 0 & 0 & 0 & 0 & h_2 & 0 \\ h_2 & -h_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_3 & h_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.5}$$

$$A_3=\begin{bmatrix} 0 & 0 & 0 & h_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & -h_2 & 0 & -1 \\ h_3 & 0 & -h_1 & 0 & 0 & 0 & 0 \\ 0 & h_3 & -h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.6}$$

For the right-hand side of (1.3) we shall write $A(D)$.

The principle result of [9], as already noted, was the existence of two sets of essentially different boundary conditions for (1.3) which preserve energy when $H_0=(0, 0, h_3)$ and a single boundary condition when $H_0=(0, h_2, 0)$. It was further indicated in [9] that for $H_0=(h_1, 0, h_3)$ $(h_1\neq 0, h_3\neq 0)$ there is again but a single, constant, energy preserving boundary condition. This covers all cases except as noted in [9], when $H_0=(0, 0, 1)$. This case admits many more boundary conditions in addition to those valid for $H_0=(0, 0, h_3)$. This is perhaps significant in light of the change of variables we have performed above, but we shall not pursue this at present. However, as far as the analysis of the two cases $h_3=1$ and $h_3\neq 1$ (or $h_2=1$ and $h_2\neq 1$ for that matter) is concerned, the only difference both in result and technique is that $h_3 \neq 1$ is much more cumbersome. So, while the case $h_3 \neq 1$ has been worked out, only the details for $h_3 = 1$ are presented here.

A (to us) somewhat startling fact elucidated below is that none of the boundary conditions considered above admits surface waves which decay away from the boundary, i.e. all waves of finite energy propagate away from the boundary with a certain exception noted below and are represented as the superposition of plane waves without exception. This is in stark contrast to the case of electromagnetic waves and elastic waves, both of which admit surface waves decaying exponentially away from the boundary for certain energy preserving boundary conditions (see [7], [8], and [14], for example). It will further be shown below that in case (2) the boundary does not couple the plane wave modes associated with (1.3). This can also happen in case (1) for exactly one boundary condition. We shall study the problem of asymptotic solutions of systems with non-constant coefficients related to (1.3) by the formula

$$-\mathrm{i}\frac{\partial v}{\partial t} = E(x)^{-1}\sum_{j=1}^{3} -A_j\mathrm{i}\frac{\partial v}{\partial x_j} = E(x)^{-1}A(D)$$

where $E(x)$ is a uniformly positive definite Hermitian matrix. An example of a physically meaningful E is one of the form diag $(\mu_1(x),\ \mu_2(x),\ \mu_3(x),\ \rho(x),\ \rho(x),\ \rho(x),\ c(x)^{-2}\rho(x)^{-1})$ where 'diag' means the square matrix whose main diagonal elements are those indicated and all other entries are zero. The cases of $h_3 \neq 1$ and $h_2 \neq 1$ can be reduced essentially to the cases we study here using this same pattern.

We use the following standard notation: $\mathbb{R}^3_+$ for $\{(x_1, x_2, x_3) \mid x_i$ real and $x_3 > 0\}$, $\bar{X}$ for the closure of X, $\mathbb{C}^m$ for an m-dimensional complex space with the usual inner product, $L^2(\boldsymbol{X}, \mathbb{C}^m)$ for the Lebesgue measurable, square integrable $\mathbb{C}^m$-valued functions on $\boldsymbol{X} \subseteq \mathbb{R}^3$, $\mathscr{D}(\boldsymbol{X}, \mathbb{C}^m)$ for the infinitely differentiable functions on $\boldsymbol{X}$ with values in $\mathbb{C}^m$ having compact support if $\boldsymbol{X}$ is open and bounded support if $\boldsymbol{X}$ is closed. We shall consider the boundary operators

$$B^3_{\lambda,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.7}$$

$$B^3_{\infty,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.8}$$

$$B^3_{\lambda,2} = \begin{bmatrix} 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 \end{bmatrix}, \tag{1.9}$$

$$B^3_{\infty,2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{1.10}$$

$$B^2 = [0, 0, 0, 0, h_2, 0, 1]. \tag{1.11}$$

We define the operators $A^3_{\lambda,1}$, $A^3_{\lambda,2}$, and A^2 as follows.

$A^3_{\lambda,i}$ is the closure of the symmetric operator $A(D)$ with domain $\{u \mid u \in \mathscr{D}(\bar{\mathbb{R}}^3_+, \mathbb{C}^7),\ B^3_{\lambda,i}u(x_1, x_2, 0) = 0\}$, where, for u in this set and $H_0 = (0, 0, h_3)$, $A^3_{\lambda,i}u = A(D)u$.

A^2 is defined in exactly the same way but with B^2 and $H_0 = (0, h_2, 0)$. That these give selfadjoint operators follows from c.f. definition 1.7 of [9]. We gratefully acknowledge the use of ideas from [7] and [15] below.

2. Resolvents

The basis of our analysis is Stone's theorem on the construction of the spectral family of a selfadjoint operator A in a Hilbert space $\mathscr{H}$ with inner product $(,)$. Let $R(\lambda) = (A - \lambda I)^{-1}$ and let $E(\lambda)$ be the (right continuous) spectral family of A. Then for (a, b) a finite interval and for f,g in $\mathscr{H}$,

$$(\tfrac{1}{2}(E(b) + E(b-))f - \tfrac{1}{2}(E(a) + E(a-))f, g)$$
$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi \mathrm{i}} \int_a^b ((R(k + \mathrm{i}\varepsilon) - R(k - \mathrm{i}\varepsilon))f, g)\, \mathrm{d}k. \tag{2.1}$$

Using the well-known relations

$$R^*(\lambda) = R(\bar{\lambda}), \qquad R(\lambda_1) - R(\lambda_2) = (\lambda_1 - \lambda_2)R(\lambda_1)R(\lambda_2), \tag{2.2}$$

the right-hand side for (2.1) may be rewritten as

$$\lim_{\varepsilon\to 0+} \frac{\varepsilon}{\pi} \int_a^b (R(k - \mathrm{i}\varepsilon)f, R(k - \mathrm{i}\varepsilon)g)\, \mathrm{d}k. \tag{2.3}$$

Taking $f = g$ we have

$$(\tfrac{1}{2}(E(b) + E(b-))f - \tfrac{1}{2}(E(a) + E(a-))f, f) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi} \int_a^b |R(k - \mathrm{i}\varepsilon)f|^2\, \mathrm{d}k, \tag{2.4}$$

with $|\cdot|$ representing the norm in $\mathscr{H}$. Equation (2.4) gives (2.1) upon polarization. Therefore, we seek to compute (2.4) for $A^3_{\lambda,i}$ and A^2. When it is not necessary to distinguish these operators we simply write A.

We shall have need of the Fourier transform. On $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^m)$, the space of smooth rapidly decreasing $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$, the Fourier transform is defined as ($x \circ y = \Sigma x_i y_i$)

$$\boldsymbol{\Phi}_n f(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\circ p} f(x)\, \mathrm{d}x, \tag{2.5}$$

with $\boldsymbol{\Phi}_n^{-1} = \boldsymbol{\Phi}_n^*$ defined by

$$(\boldsymbol{\Phi}_n^{-1} f)(p) = (\boldsymbol{\Phi}_n f)(-p). \tag{2.6}$$

$\boldsymbol{\Phi}_n$ is an isomorphism on $\mathscr{S}$ which extends by duality to $\mathscr{S}'$, the continuous dual of $\mathscr{S}$, and by continuity to $L^2(\mathbb{R}^n, \mathbb{C}^m)$ (see [6], for example). We shall employ the notation $\mathscr{H}$ for $L^2(\mathbb{R}^3_+, \mathbb{C}^7)$. Now, using Parseval's formula in the case of $\boldsymbol{\Phi}_3$, (2.4) may be written as (here and below, χ_c is the characteristic function of the set c)

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi}\int_a^b |\boldsymbol{\Phi}_3(\chi_{\mathbb{R}^3_+}R(k-\mathrm{i}\varepsilon)f)|^2\,\mathrm{d}k, \tag{2.7}$$

which equals

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^3}\frac{\varepsilon}{\pi}\int_a^b |(\boldsymbol{\Phi}_3\chi_{\mathbb{R}^3_+}R(k-\mathrm{i}\varepsilon)f)(p)|^2\,\mathrm{d}k\,\mathrm{d}p. \tag{2.8}$$

We shall first wish to obtain

$$\frac{\varepsilon}{\pi}(\boldsymbol{\Phi}_3\chi_{\mathbb{R}^3}R(k-\mathrm{i}\varepsilon)f)(p) \tag{2.9}$$

in a form which can be studied as $\varepsilon\to 0+$. To this end, we need to compute the 'resolvent kernel' of $R(\lambda)$. This is a function $R(x, y; z)$ such that for f in $\mathscr{H}$,

$$R(z)f(x)=\int_{\mathbb{R}^3_+}R(x, y; z)f(y)\,\mathrm{d}y. \tag{2.10}$$

The idea is to seek $R(x, y; z)$ in the form (see [5])

$$\mathcal{E}(x-y; z)-F(x, y; z), \tag{2.11}$$

where $\mathcal{E}(x-y; z)$ is a solution in $\mathscr{S}'$ of

$$(A(D)-zI)\mathcal{E}(x; z)=\delta(x)I_{7\times 7} \tag{2.12}$$

and F satisfies the three conditions:

(1) $(A(D)-zI)F(x, y; z)=0, \quad x,y\in\mathbb{R}^3_+$ (differentiation on x), (2.13)

(2) $B^i_{\lambda,j}F(x_1, x_2, 0, y; z)=B^i_{\lambda,j}\mathcal{E}(x-y; z)|_{x_3=0}, \quad y\in\mathbb{R}^3_+$, (2.14)

(3) $\int_{\mathbb{R}^3_+}F(x, y; z)f(y)\,\mathrm{d}y$ is in $\mathscr{H}$ for f in $\mathscr{H}$. (2.15)

Let us define $A(p)$ to be $\Sigma A_j p_j$ for all non-zero p in $\mathbb{R}^3$. Then it is clear from our definition of $\boldsymbol{\Phi}_3$ that, in $\mathscr{S}'$,

$$\mathcal{E}(\bullet; z)=(2\pi)^{-3/2}\boldsymbol{\Phi}_3^*(A(p)-zI)^{-1}\boldsymbol{\Phi}_3. \tag{2.16}$$

Taking the Fourier transform $\boldsymbol{\Phi}_2$ (on (x_1, x_2)) of (2.13) and (2.14) results in a first order initial value problem in x_3. To solve this, it is necessary to compute $\boldsymbol{\Phi}_2 B^i_{\lambda,j}\mathcal{E}(x-y; z)|_{x_3=0}$. It is evident that we will need $\boldsymbol{\Phi}_2\mathcal{E}(x-y; z)|_{x_3=0}$ explicitly. In $\mathscr{S}'$, this means the evaluation of the integral ($\operatorname{Im} z\neq 0$)

$$(2\pi)^{-2}\mathrm{e}^{-\mathrm{i}y_3p_3}[A(n, p_3)-z]^{-1}\,\mathrm{d}p_3, \tag{2.17}$$

where we have used the notation $n=(p_1, p_2)$ and $y'=(y_1, y_2)$. This will be done by means of the residue theorem through deforming the integration into the lower half-plane. It is therefore necessary to consider the integrand as being extended as a function of p_3 into $\mathbb{C}$; n and z are not zero. We write $\tau=p_3+\mathrm{i}\alpha$. We must consider the zeros of

$$\det([A(n, \tau)-z]^{-1}) \tag{2.18}$$

in τ. These occur in the upper and lower half-plane at values $\tau_\pm$. We must consider the cases $A=A^3$ and $A=A^2$ separately now. The roots of $\det(A^3(p)-$

zI) are ${}_3\lambda_0$, ${}_3\lambda_{\pm 1}$, ${}_3\lambda_{\pm 2}$, ${}_3\lambda_{\pm 3}$, where

$$\left.\begin{aligned} {}_3\lambda_0 &\equiv 0,\\ {}_3\lambda_{\pm 1}(p) &\equiv \pm h_3 p_3,\\ {}_3\lambda_{\pm 2}(p) &\equiv \pm\left(\frac{(1+h_3^2)\,|p|^2 - |p|\,((1+h_3^2)^2\,|p|^2 - 4h_3^2p_3^2)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}},\\ {}_3\lambda_{\pm 3}(p) &\equiv \pm\left(\frac{(1+h_3^2)\,|p|^2 + |p|\,((1+h_3^2)^2\,|p|^2 - 4h_3^2p_3^2)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.19}$$

and, for A^2,

$$\left.\begin{aligned} {}_2\lambda_0 &\equiv 0,\\ {}_2\lambda_{\pm 1}(p) &\equiv \pm h_2 p_2,\\ {}_2\lambda_{\pm 2}(p) &\equiv \pm\left(\frac{(1+h_2^2)\,|p|^2 - |p|\,((1+h_2^2)^2\,|p|^2 - 4h_2^2p_2^2)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}},\\ {}_2\lambda_{\pm 3}(p) &\equiv \pm\left(\frac{(1+h_2^2)\,|p|^2 + |p|\,((1+h_2^2)^2\,|p|^2 - 4h_2^2p_2^2)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.20}$$

where $|p| = (p_1^2 + p_2^2 + p_3^2)^{\frac{1}{2}}$.

In order to simplify the notation and without essential loss as we have indicated already, we assume $h_3 = 1$ and $h_2 = 1$ in (2.19) and (2.20), respectively. There are other ways to achieve the same result, but this is the simplest. Thus we have

$$\left.\begin{aligned} {}_3\lambda_0 &\equiv 0,\\ {}_3\lambda_{\pm 1} &\equiv \pm p_3,\\ {}_3\lambda_{\pm 2} &\equiv \pm(|p|^2 - |n|\,|p|)^{\frac{1}{2}},\\ {}_3\lambda_{\pm 3} &\equiv \pm(|p|^2 + |n|\,|p|)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.21}$$

and

$$\left.\begin{aligned} {}_2\lambda_0 &\equiv 0,\\ {}_2\lambda_{\pm 1} &\equiv \pm p_2,\\ {}_2\lambda_{\pm 2} &\equiv \pm(|p|^2 - |n_1|\,|p|)^{\frac{1}{2}},\\ {}_2\lambda_{\pm 3} &\equiv \pm(|p|^2 + |n_1|\,|p|)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.22}$$

where $|n_1|^2 = p_1^2 + p_2^3$. It is interesting to consider the constant speed surfaces ('slowness surfaces' or 'normal surfaces' (see [2] and [16])), i.e. the surfaces $(j>0)\ {}_i\lambda_j(p) = 1$. For $i = 2$ or 3 and $j = 1$, these are just planes which are orthogonal to, or parallel to, the dual boundary—always orthogonal to the external field (the Alfvén waves are just one-dimensional waves propagating in the direction of the external field).

These move toward or away from the origin as the external field intensity increases of decreases. For $j = 2$ the same movement is observed, but the slowness surface consists of two sheets which may or may not intersect the single sheet of $j = 3$ depending on the external field intensity (this sole dependence on field intensity is a result of our change of variable to get (1.2)). The centre sheet

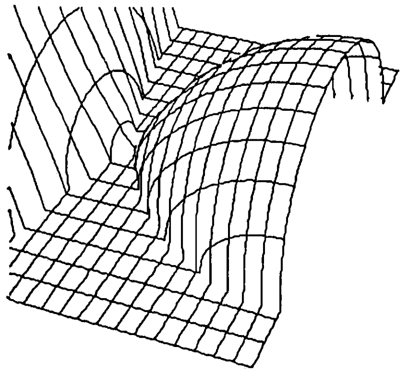

FIG. 1

($j=3$) for $h_2=1$ (see Fig. 1) moves faster than the two unbounded sheets ($j=2$). For $i=3$ and $j=2$, the surface consists again of only two sheets but roughly parallel to the dual boundary (which bisects the line connecting the two cusps). In Fig. 1 the planar surface is the p_1p_2 plane for landmark purposes. For $i=2$, 3 and $j=3$ the surfaces are bounded. We show portions in $\mathbb{R}^3_+ \cap \{p_2 \leq 0\}$ (Fig. 1) for $i=2$ and $j=2, 3$ (the ellipsoid is the $j=3$ sheet), or $\mathbb{R}^3$ (Fig. 2) for $i=3$ and $j=2$.

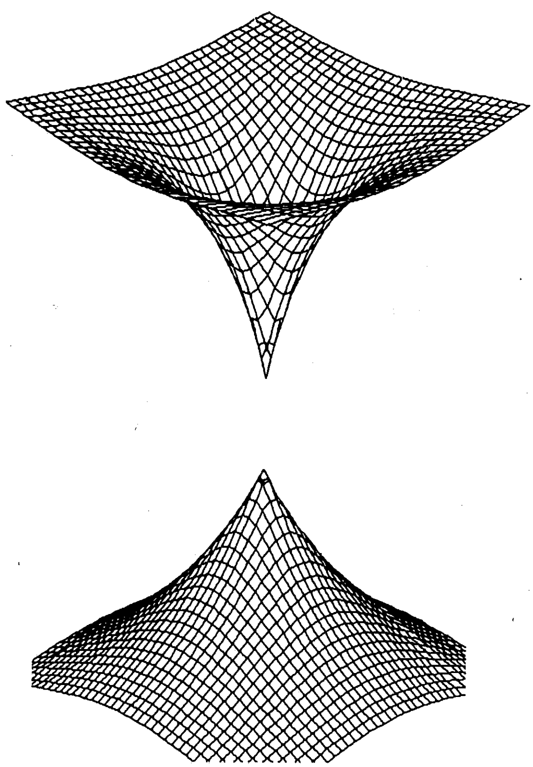

FIG. 2

For $i=2, 3$ and $j=0, 1, 2, 3$ let ${}_iP_{\pm j}(p)$ be the associated eigenprojectors on $\mathbb{C}^7$ of $A^i(p)$. By the spectral theorem,

$$[A^i(p)-z]^{-1}=\sum_{j=-3}^{3}({}_i\lambda_j(p)-z)^{-1}{}_iP_j(p). \tag{2.23}$$

We wish to extend (in single-valued fashion) ${}_i\lambda_j(n, p_3)$ to ${}_i\lambda_j(n, \tau)$ and likewise ${}_iP_j(n, p_3)$ to ${}_iP_j(n, \tau)$ so that (2.23) remains valid, with all poles determined by the coefficients $({}_i\lambda_j(n, \tau)-z)^{-1}$. This may be done easily for λ_0, ${}_i\lambda_{\pm 1}$. For ${}_3\lambda_{\pm 2}$ and ${}_3\lambda_{\pm 3}$ we will make branch cuts in the τ-plane along the intervals $(\mathrm{i}\,|n|, \infty)$ and $(-\infty, -\mathrm{i}\,|n|)$. For ${}_2\lambda_{\pm 2,3}$ branch cuts are made on the intervals $(-\infty, -\mathrm{i}\,|p_1|)$ and $(\mathrm{i}\,|p_1|, \infty)$. Generally, we will interpret the outer square roots to have positive imaginary part. The solutions to the equations $(j>0)$

$${}_i\lambda_j(n, t)-z=0 \tag{2.24}$$

are $(\operatorname{Re} z>0)$

$$\left.\begin{aligned}{}_3\tau_{\pm 1}&=\pm z,\\ {}_3\tau_{\pm 2}&=\pm\tfrac{1}{2}(2z^2-|n|^2+|n|\,(4z^2+|n|^2)^{\frac{1}{2}})^{\frac{1}{2}},\\ {}_3\tau_{\pm 3}&=\pm\tfrac{1}{2}(2z^2-|n|^2-|n|\,(4z^2+|n|^2)^{\frac{1}{2}})^{\frac{1}{2}},\end{aligned}\right\} \tag{2.25}$$

and

$$\left.\begin{aligned}{}_2\tau_{\pm 2}&=\pm\left(\frac{(|n|^2-z^2)^2-|n|^2p_1^2}{2z^2-p_2^2}\right)^{\frac{1}{2}} \quad (|n|>|z|),\\ {}_2\tau_{\pm 3}&=\pm\left(\frac{(z^2-|n|^2)^2-|n|^2p_1^2}{2z^2-p_2^2}\right)^{\frac{1}{2}} \quad (|z|>|n|).\end{aligned}\right\} \tag{2.26}$$

Note that $|z|=|n|$ only when $|n|=0$ or $|n_1|=0$. These possibilities are eliminated by considerations shortly to be made. For ${}_3\tau_{\pm 3}$ branch cuts are made in the z-plane along $(-\infty, -(2)^{\frac{1}{2}}\,|n|)$ and $((2)^{\frac{1}{2}}\,|n|, \infty)$. For ${}_3\tau_{\pm 2}$ a single branch cut along $(-\infty, 0)$ is made. For ${}_2\tau_{\pm 2}$ a branch cut is made along $(-(|n|^2-|n|\,|p1|)^{\frac{1}{2}}, (|n|^2-|n|\,|p1|)^{\frac{1}{2}})$ and for ${}_2\tau_{\pm 3}$ along $(-\infty, -(|n|^2+|n|\,|p1|)^{\frac{1}{2}})$ and $((|n|^2+|n|\,|p1|)^{\frac{1}{2}}, \infty)$. Of course there are other choices that can be made; these are given for convenience. In order to make use of (2.23), we need the projectors ${}_iP_j$. These may be obtained in a variety of ways. They are relatively simple to compute since their ranges are almost everywhere one dimensional. In any case they can be written as follows (in the following matrices, we abuse the notation and write p for $|p|$ and n for $|n|$):

$${}_3P_{\pm 1}=\frac{1}{2\,|n|^2}\begin{bmatrix} p_2^2 & -p_1p_2 & 0 & \pm p_2^2 & \mp p_1p_2 & 0 & 0\\ -p_1p_2 & p_1^2 & 0 & \mp p_1p_2 & \pm p_1^2 & 0 & 0\\ 0&0&0&0&0&0&0\\ \pm p_2^2 & \mp p_1p_2 & 0 & p_2^2 & -p_1p_2 & 0 & 0\\ \mp p_1p_2 & \pm p_1^2 & 0 & -p_1p_2 & p_1^2 & 0 & 0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\end{bmatrix}, \tag{2.27}$$

$$
{}_3P_{\pm 2}(p)=\frac{1}{4\,|n|^2\,|p|^2}\begin{bmatrix}
\frac{p_1^2({}_3\lambda_2^2-2n^2)^2}{{}_3\lambda_2^2} & \frac{p_1p_2({}_3\lambda_2^2-2n^2)^2}{{}_3\lambda_2^2} & \frac{p_3p_1np({}_3\lambda_2^2-2n^2)}{{}_3\lambda_2^2} & \frac{p_1^2p_3^2({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{p_1p_2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{n^2p_1({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{npp_1({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} \\
\frac{p_1p_2({}_3\lambda_2-2n^2)^2}{{}_3\lambda_2^2} & \frac{p_2^2({}_3\lambda_2^2-2n^2)^2}{{}_3\lambda_2^2} & \frac{-p_3p_2np({}_3\lambda_2^2-2n^2)}{{}_3\lambda_2^2} & \frac{p_1p_2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{p_2^2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{n^2p_2({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{npp_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} \\
\frac{p_3p_1np({}_3\lambda_2^2-2n^2)}{{}_3\lambda_2^2} & \frac{-p_3p_2np({}_3\lambda_2^2-2n^2)}{{}_3\lambda_2^2} & n^2p^2p_3^2 & \frac{-npp_1p_3^2}{{}_3\lambda_{\pm 2}} & \frac{-npp_2p_3^2}{{}_3\lambda_{\pm 2}} & \frac{-n^3pp_3}{{}_3\lambda_{\pm 2}} & \frac{-n^2p^2p_3}{{}_3\lambda_{\pm 2}} \\
\frac{p_1^2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{p_1p_2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{npp_1p_3^2}{{}_3\lambda_{\pm 2}} & p_1^2p_3^2 & p_1p_2p_3^2 & n^2p_1p_3 & npp_1p_3 \\
\frac{p_1p_2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{p_2^2p_3({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{-npp_2p_3^2}{{}_3\lambda_{\pm 2}} & p_1p_2p_3^2 & p_2^2p_3^2 & n^2p_2p_3 & npp_2p_3 \\
\frac{n^2p_1({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{n^2p_2({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{-n^3pp_3}{{}_3\lambda_{\pm 2}} & n^2p_1p_3 & n^2p_2p_3 & n^4 & n^3p \\
\frac{npp_1({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{npp_2({}_3\lambda_2^2-2n^2)}{{}_3\lambda_{\pm 2}} & \frac{-n^2p^2p_3}{{}_3\lambda_{\pm 2}} & npp_1p_3 & npp_2p_3 & n^3p & n^2p^2
\end{bmatrix}, \tag{2.28}
$$

$$
{}_3P_{\pm3}(p)=\frac{1}{4\,|p|^2\,|n|^2}
\begin{bmatrix}
p_1^2\,{}_3\lambda_3^2 & p_1p_2\,{}_3\lambda_3^2 & p_3npp_1 & p_1^2p_3\,{}_3\lambda_{\pm3} & p_1p_2p_3\,{}_3\lambda_{\pm3} & -n^2p_1\,{}_3\lambda_{\pm3} & -npp_1\,{}_3\lambda_{\pm3} \\
p_1p_2\,{}_3\lambda_3^2 & p_2^2\,{}_3\lambda_3^2 & p_2p_3np & p_1p_2p_3\,{}_3\lambda_{\pm3} & p_2^2p_3\,{}_3\lambda_{\pm3} & -n^2p_2\,{}_3\lambda_{\pm3} & -npp_2\,{}_3\lambda_{\pm3} \\
npp_1p_3 & npp_2p_3 & \dfrac{n^2p^2p_3^2}{{}_3\lambda_3^2} & \dfrac{npp_1p_3^2}{{}_3\lambda_{\pm3}} & \dfrac{npp_2p_3^2}{{}_3\lambda_{\pm3}} & \dfrac{-n^3pp_3}{{}_3\lambda_{\pm3}} & \dfrac{-n^2p^2p_3}{{}_3\lambda_{\pm3}} \\
p_1^2p_3\,{}_3\lambda_{\pm3} & p_1p_2p_3\,{}_3\lambda_{\pm3} & \dfrac{npp_1p_3^2}{{}_3\lambda_{\pm3}} & p_1^2p_3^2 & p_1p_2p_3^2 & -n^2p_1p_3 & -npp_1p_3 \\
p_1p_2p_3\,{}_3\lambda_{\pm3} & p_2^2p_3\,{}_3\lambda_{\pm3} & \dfrac{npp_2p_3^2}{{}_3\lambda_{\pm3}} & p_1p_2p_3^2 & p_2^2p_3^2 & -np_2p_3 & -npp_2p_3 \\
-n^2p_1\,{}_3\lambda_{\pm3} & -n^2p_2\,{}_3\lambda_{\pm3} & \dfrac{-n^3pp_3}{{}_3\lambda_{\pm3}} & -n^2p_1p_3 & -np_2p_3 & n^4 & n^3p \\
-npp_1\,{}_3\lambda_{\pm3} & -npp_2\,{}_3\lambda_{\pm3} & \dfrac{-n^2p^2p_3}{{}_3\lambda_{\pm3}} & -npp_1p_3 & -npp_2p_3 & n^3p & n^2p^2
\end{bmatrix},
\tag{2.29}
$$

$$
{}_2P_{\pm1}(p)=\frac{1}{4\,|n_1|^2\,|p|^2}
\begin{bmatrix}
p_3^2 & 0 & -p_1p_3 & \pm p_3^2 & 0 & \mp p_1p_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p_1p_3 & 0 & p_1^2 & \mp p_1p_3 & 0 & \pm p_1^2 & 0 \\
\pm p_3^2 & 0 & \mp p_1p_3 & p_3^2 & 0 & -p_1p_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\pm p_1p_3 & 0 & \pm p_1^2 & -p_1p_3 & 0 & p_1^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\tag{2.30}
$$

$$
{}_2P_{\pm2}(p)=\frac{1}{4\,|n_1|^2\,|p|^2}\begin{bmatrix}
p_1^2\,{}_2\lambda_2^2 & -p_2p_1pn_1 & p_1p_3\,{}_2\lambda_2^2 & p_1^2p_2\,{}_2\lambda_{\pm2} & -n_1^2p_1\,{}_2\lambda_{\pm2} & p_1p_2p_3\,{}_2\lambda_{\pm2} & n_1pp_1\,{}_2\lambda_{\pm2} \\
-n_1pp_2p_1 & \dfrac{n_1^2p^2p_2^2}{{}_2\lambda_2^2} & -n_1pp_2p_3 & -\dfrac{n_1pp_1p_2^2}{{}_2\lambda_{\pm2}} & \dfrac{n_1^3pp_2}{{}_2\lambda_{\pm2}} & -\dfrac{n_1pp_2^2p_3}{{}_2\lambda_{\pm2}} & -\dfrac{n_1^2p^2p_2}{{}_2\lambda_{\pm2}} \\
p_1p_3\,{}_2\lambda_2^2 & -n_1pp_2p_3 & p_3^2\,{}_2\lambda_2^2 & p_1p_2p_3\,{}_2\lambda_{\pm2} & -n_1^2p_3\,{}_2\lambda_{\pm2} & p_3^2p_2\,{}_2\lambda_{\pm2} & n_1pp_3\,{}_2\lambda_{\pm2} \\
p_1^2p_2\,{}_2\lambda_{\pm2} & \dfrac{-n_1pp_1p_2^2}{{}_2\lambda_{\pm2}} & p_1p_2p_3\,{}_2\lambda_{\pm2} & p_1^2p_2^2 & -n_1^2p_1p_2 & p_1p_2^2p_3 & n_1pp_1p_2 \\
-n_1^2p_1\,{}_2\lambda_{\pm2} & \dfrac{n_1^3pp_2}{{}_2\lambda_{\pm2}} & -n_1^2p_3\,{}_2\lambda_{\pm2} & -n_1^2p_1p_2n_1^4 & n_1^4 & -n_1^2p_2p_3 & -n_1^3p \\
p_1p_2p_3\,{}_2\lambda_{\pm2} & \dfrac{-n_1pp_2^2p_3}{{}_2\lambda_{\pm2}} & p_3^2p_2\,{}_2\lambda_{\pm2} & p_1p_2^2p_3 & -n_1^2p_2p_3 & p_2^2p_3^2 & n_1pp_2p_3 \\
n_1pp_1\,{}_2\lambda_{\pm2} & \dfrac{-n_1^2p^2p_2}{{}_2\lambda_{\pm3}} & n_1pp_3\,{}_2\lambda_{\pm2} & n_1pp_1p_2 & -n_1^3p & n_1pp_2p_3 & n_1^2p^2
\end{bmatrix},
\tag{2.31}
$$

$$
{}_2P_{\pm3}(p)=\frac{1}{4\,|n_1|^2\,|p|^2}\begin{bmatrix}
p_1^2\,{}_2\lambda_3^2 & n_1pp_1p_2 & p_1p_3\,{}_2\lambda_3^2 & p_1^2p_2\,{}_3\lambda_{\pm3} & -n_1^2p_1\,{}_2\lambda_{\pm3} & p_1p_2p_3\,{}_2\lambda_{\pm3} & -n_1pp_1\,{}_2\lambda_{\pm3} \\
n_1pp_2p_1 & \dfrac{n_1^2p^2p_2}{{}_2\lambda_3^2} & p_2n_1pp_3 & \dfrac{n_1pp_1p_2^2}{{}_2\lambda_{\pm3}} & \dfrac{-n_1^3pp_2}{{}_2\lambda_{\pm3}} & \dfrac{n_1pp_2^2p_3}{{}_2\lambda_{\pm3}} & \dfrac{n_1^2p^2p_2}{{}_2\lambda_{\pm3}} \\
p_1p_3\,{}_2\lambda_3^2 & p_2n_1pp_3 & p_3^2\,{}_2\lambda_3^2 & p_1p_2p_3\,{}_2\lambda_{\pm3} & -n_1^2p_3\,{}_2\lambda_{\pm3} & p_3p_2\,{}_2\lambda_{\pm3} & -n_1pp_3\,{}_2\lambda_{\pm3} \\
p_1^2p_2\,{}_2\lambda_{\pm3} & \dfrac{n_1pp_1p_2^2}{{}_2\lambda_{\pm3}} & p_1p_2p_3\,{}_2\lambda_{\pm3} & p_1^2p_2^2 & -n_1^2p_1p_2 & p_1p_2^2p_3 & -n_1pp_1p_2 \\
-n_1^2p_1\,{}_2\lambda_{\pm3} & \dfrac{-n_1^3pp_2}{{}_2\lambda_{\pm3}} & -n_1^2p_3\,{}_2\lambda_{\pm3} & -n_1^2p_1p_2 & n_1^4 & -n_1^2p_2p_3 & n_1^3p \\
p_1p_2p_3\,{}_2\lambda_{\pm3} & \dfrac{n_1pp_2^2p_3}{{}_2\lambda_{\pm3}} & p_2p_3^2\,{}_2\lambda_{\pm3} & p_1p_2^2p_3 & -n_1^2p_2p_3 & p_2^2p_3^2 & -n_1pp_1p_3 \\
-n_1pp_1\,{}_2\lambda_{\pm3} & -\dfrac{n_1^2p^2p_2}{{}_2\lambda_{\pm3}} & -n_1pp_3\,{}_2\lambda_{\pm3} & -n_1pp_1p^2 & n_1^3p & -n_1pp_2p_3 & n_1^2p^2
\end{bmatrix}.
\tag{2.32}
$$

Finally, ${}_2P_0 = {}_3P_0$ and their common value is given by

$$ {}_iP_0(p) = \frac{1}{|p|^2}\,{}^t(0, 0, 0, p_1, p_2, p_3, 0) \otimes {}^t(0, 0, 0, p_1, p_2, p_3, 0). \tag{2.33}$$

Recalling that n_1, p, and ${}_i\lambda_j$ contain p_3 (with noted exceptions) we substitute τ for p_3 in (2.28)–(2.33) to obtain

$$ [A(n, \tau) - zI]^{-1} = \sum_{j=3}^{3} (\lambda_j(n, \tau) - z)^{-1} P_j(n, \tau). \tag{2.34}$$

Using (2.34), (2.17), and the residue theorem, we obtain via l'Hôpital's rule the following expression for (2.17) in the case of A^3 (note that $\tau_- = -\tau_+$):

$$ -(2\pi i)^{-1} e^{-y'\circ n} \sum_{j=1}^{3} e^{iy_3\tau_j} {}_3c_j\, {}_3P_j(n, -\tau_j, z); \tag{2.35}$$

where ${}_3P_j(n, -\tau_j, z)$ is obtained from (2.27)–(2.29) substituting z for λ and $-{}_3\tau_j$ for p_3. Here

$$ -{}_3c_1 = \left(\frac{\partial_3\lambda_1}{\partial\tau}\right)^{-1}\Big|_{\tau=-{}_3\tau_1} = 1, \tag{2.36}$$

$$ {}_3c_2 = \left(\frac{\partial_3\lambda_2}{\partial\tau}\right)^{-1}\Big|_{\tau=-{}_3\tau_2} = \frac{z}{-{}_3\tau_2(1 - n/2(n^2 + {}_3\tau_2^2)^{\frac{1}{2}})}, \tag{2.37}$$

$$ {}_3c_3 = \left(\frac{\partial_3\lambda_3}{\partial\tau}\right)^{-1}\Big|_{\tau=-{}_3\tau_3} = \frac{z}{-{}_3\tau_3(1 + n/2(n^2 + {}_3\tau_3^2)^{\frac{1}{2}})}. \tag{2.38}$$

For A^2 we have the following expression of (2.17):

$$ -(2\pi i)^{-1} e^{-iy'\circ n} \sum_{j=2,3} e^{iy_3{}_2\tau_j} \left(\frac{\partial_2\lambda_j}{\partial\tau}\right)^{-1}\Big|_{\tau={}_2\tau_j} {}_2P_j(n, -{}_2\tau_j, z), \tag{2.39}$$

with

$$ \left(\frac{\partial_2\lambda_2}{\partial\tau}\right)^{-1}\Big|_{\tau=-{}_2\tau_2} = \frac{z}{-{}_2\tau_2\left(1 - \dfrac{n^2 + 2{}_2\tau_2^2 + p_1^2}{2(n^2 + {}_2\tau_2^2)^{\frac{1}{2}}(p_1^2 + {}_2\tau_2^2)^{\frac{1}{2}}}\right)} \tag{2.40}$$

$$ \left(\frac{\partial_2\lambda_3}{\partial\tau}\right)^{-1}\Big|_{\tau=-{}_2\tau_3} = \frac{z}{-{}_2\tau_3\left(1 + \dfrac{n^2 + 2{}_2\tau_3^2 + p_1^2}{2(n^2 + {}_2\tau_3^2)^{\frac{1}{2}}(p_1^2 + {}_2\tau_3^2)^{\frac{1}{2}}}\right)}. \tag{2.41}$$

Note that only one term in (2.39) occurs depending on the modulus of z relative to $|n|$. (Again we remind the reader that n is often employed in numerical expressions instead of $|n|$ for the sake of brevity even though strictly speaking $n = (p_1, p_2)$.)

It is now possible to solve (2.13). The general solution could be written as $\Sigma\, e^{m_jx_3} K_j(n, y; z)$, for example. In order that (see (2.15)) this be bounded for

$x_3 > 0$ we see that

$$(A(n, D_3)-z)\sum e^{m_j x_3}K=(A(n, D_3)-z)\Big(\sum \alpha_j e^{m_j x_3}P_j\Big)M_j$$

$$=\sum (\alpha_j(A_3 m_j e^{m_j x_3}+A_1 p_1 e^{m_j x_3}+A_2 p_2 e^{m_j x_3})P_j - z\alpha_j e^{m_j x_3}P_j)M_j$$

$$=\sum \alpha_j e^{m_j x_3}[(A(n, \mathrm{i}m_j)-z)P_j]M_j,$$

so that $m_j = -\mathrm{i}_i\, \tau_j$ and therefore $(\boldsymbol{\Phi}_2 F)(n, x_3, y; z)$ is given by

$$\sum \alpha_j(n, y; z)e^{-\mathrm{i}_i\tau_j x_3}P_j(n, {}_i\tau_j; z)M_j(n, {}_i\tau_j, z). \tag{2.42}$$

Imposing the initial condition (2.14) we observe that, depending on z, only one of ${}_2P_{2,3}$ is present in general for A^2. We see also that α_j may be chosen as

$$-(2\pi\mathrm{i})^{-1}e^{-\mathrm{i}y'\circ n}e^{\mathrm{i}y_{3\,i}\tau_j}{}_ic_j, \tag{2.43}$$

where ${}_ic_j$ is $(\partial_i\lambda_j/\partial t)^{-1}|_{\tau={}_i\tau_j}$ except for $i=3, j=1$ when we choose it to be -1. Then ${}_iM_j$ is selected so that (2.14) is satisfied. Generally there are many possible choices for the ${}_iM_j$, a fact which is useful below. The idea is to select the simplest among these for each of the boundary conditions. We note that for $h_3 \neq 1$ or $h_2 \neq 1$, the development above is completely parallel except for the explicit formulae of the ${}_ic_j$ and ${}_iP_j$. As we have stated, the ${}_iP_j$ are rather cumbersome to write down especially for $j=2$, 3 and $H_0=(h_1, 0, h_3)$. For this H_0, the ${}_i\tau_j$ are very complex, almost beyond usefulness. They take many pages to record in full detail.

Definition 2.1 For $k \neq j \neq 0$, let $\boldsymbol{\beta}$ be the set of points in p space where any ${}_i\lambda_j$ coincides with another ${}_i\lambda_k$. It is easy to see that this is a set of measure zero in p space. For example, ${}_3\lambda_2(p)=0$ when $p_3=0$.

We may now write down $\boldsymbol{\Phi}_3\chi_{\mathbb{R}^3_+}R(p, y, z)$ by applying $\boldsymbol{\Phi}_1$ in x_3 to (2.42) using (2.11) to get

$$(2\pi)^{-\frac{3}{2}}e^{-\mathrm{i}y'\circ n}\left(e^{-\mathrm{i}y_3p_3}[A^3(p)-zI]^{-1}+\sum_{j=1}^{3}\frac{e^{\mathrm{i}y_{3\,3}\tau_j}}{({}_3\tau_j+p_3)}{}_3c_j\,{}_3P_j(n, -{}_3\tau_j, z)\right)$$
$$+(2\pi)^{-\frac{3}{2}}e^{-\mathrm{i}y'\circ n}\sum_{j=1}^{3}\left(\frac{{}_3c_j}{{}_3\tau_j-p_3}\right){}_3P_j(n, {}_3\tau_j, z)_3M_j, \tag{2.44}$$

with a completely analogous expression for A^2.

3. Eigenfunction expansions and perturbations

We make the following definition.

Definition 3.1

$${}_i\psi_j^* = ({}_i\lambda_j(p)-\bar{z})\chi_{\mathbb{R}^3_+\setminus\boldsymbol{\beta}}(p)\boldsymbol{\Phi}_3\chi_{\mathbb{R}^3_+}R(p, y, \bar{z}), \tag{3.1}$$

$${}_i\hat{f}_j(p, z)=\int_{\mathbb{R}^3_+}{}_i\psi_j^*(p, y, z)f(y)\,\mathrm{d}y, \tag{3.2}$$

where f is smooth and has bounded support.

Set

$$ {}_i\psi^*_{\pm j}(p, y) = \lim_{z \to {}_i\lambda_{\pm j}(p)+\mathrm{i}0} {}_i\psi^*_j(p, y, z) \tag{3.3}$$

$$ = (2\pi)^{-\frac{3}{2}}\chi_{\mathbb{R}^3\setminus\beta}(p)\chi_{\mathbb{R}\mp\mathrm{sig}(j)}(p_3)_iP_j(p)\{\mathrm{e}^{\mathrm{i}y\circ p}I - {}_iM_j(n, {}_i\lambda_{\pm j}(p) - \mathrm{i}0, y_3)\mathrm{e}^{-\mathrm{i}y'\circ n}\}. \tag{3.4}$$

LEMMA 3.1 *If f is smooth and has compact support, then*

$$\hat{f}_{i\pm j}(p) = \lim_{z \to {}_i\lambda_{\pm j}(p)+\mathrm{i}0} {}_i\hat{f}_j(p, z) = \int_{\mathbb{R}^3_+} {}_i\psi^*_j(p, y) f(y)\, \mathrm{d}y$$

defines a function which is smooth and rapidly decreasing almost everywhere.

Proof. Equations (3.1) and (3.4) show that the function on the left-hand side of (3.2) in this case converges by the definition of ${}_iM_j$ and the dominated convergence theorem as indicated. The fact that the Fourier transform of f is smooth and rapidly decreasing together with (3.4) gives the result.

LEMMA 3.2

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \frac{\varepsilon}{\pi} \int_a^b |\Phi_3 \chi_{\mathbb{R}^3_+} R(k - \mathrm{i}\varepsilon) f(p)|^2 \,\mathrm{d}k\,\mathrm{d}p$$

$$= \int_{\mathbb{R}^3} \lim_{\varepsilon\to 0+} \frac{\varepsilon}{\pi} \int_a^b |\Phi_3 \chi_{\mathbb{R}^3_+} R(k - \mathrm{i}\varepsilon) f(p)|^2 \,\mathrm{d}k\,\mathrm{d}p. \tag{3.5}$$

There is no problem in switching the order of integration in (2.7) *to get the left-hand side of the equation above for positive ε.* (*See* (2.8) *above where this was done without comment. Since the integrand is continuous in k and measurable in p and non-negative, this is justified.*) *The proof of this lemma is tedious but a straightforward model of that found in the appendix of* [7].

THEOREM 3.3

$$\lim_{\varepsilon\to 0+} \frac{\varepsilon}{\pi} \int_a^b |\Phi_3 \chi_{\mathbb{R}^3_+} R(k - \mathrm{i}\varepsilon) f(p)|^2 \,\mathrm{d}k = \sum_{j\neq 0} \chi_{\lambda_j(p)\in(a,b)}(p)\, |{}_i\hat{f}_j(p)|^2 \tag{3.6}$$

for all f in the orthogonal complement of the null space of A.

Before giving the proof we make some observations. First, the matrices ${}_iM_j(n, {}_i\tau_j, z)$ have determinants which are non-zero as $z \to \mathbb{R}$, no matter which of the boundary conditions are applied. In fact, the only boundary condition for A^3 which exhibits such a possibility (of zero determinant) is $B^3_{\infty,1}$. This is because the so-called Lopatinski determinant has an apparent zero at $z = \pm(2)^{\frac{1}{2}}|n|$. But in fact for this case the condition (2.14) admits the following trivial solution for the ${}_3M_j$:

$$\left.\begin{aligned} {}_3M_3 &= \mathrm{e}^{\mathrm{i}\,{}_3\tau_3 y_3}\,\mathrm{diag}\,(1, 1, -1, 1, 1, -1, 1),\\ {}_3M_2 &= \mathrm{e}^{\mathrm{i}\,{}_3\tau_2 y_3}\,\mathrm{diag}\,(1, 1, -1, 1, 1, -1, 1),\\ {}_3M_1 &= -\mathrm{e}^{\mathrm{i}\,{}_3\tau_1 y_3} I_{7\times 7} \end{aligned}\right\} \tag{3.7}$$

(In fact, we may substitute the diagonal matrix in the first two equations for the identity matrix in the last equation by (2.27).) This shows that the boundary condition $B^3_{\infty,1}$ does not couple the modes of propagation for A^3. It is a matter of some computational complexity to show that none of the other boundary conditions $B^3_{\lambda,i}$ uncouples the modes, but it is merely a matter of checking the appropriate signs. The fact that the Lopatinski determinant has no real zeros for $B^3_{\lambda,i}$ looks more formidable and involves the computation of around one thousand 3×3 symbolic determinants. This sounds tedious but in fact all that needs to be done is to locate one non-zero determinant in each group of 343 per boundary condition ($\lambda = \infty$ is best done separately). We state this as the following theorem.

THEOREM 3.4 *A^3 admits no surface waves for any boundary condition $B^3_{\lambda,i}$ and the boundary condition $B^3_{\infty,1}$ is the only one which uncouples the modes, i.e. an incident ${}_i\lambda_j$ wave begets a reflected ${}_i\lambda_j$ wave.*

We can say more.

THEOREM 3.5 *The ${}_3\lambda_1$ wave is never coupled to the other modes by any of the boundary conditions $B^3_{\lambda,i}$.*

The proof is an easy computation based on (2.14) (${}_3M_1$ is always independent of the others and may be taken the same regardless of the boundary condition). The ${}_i\lambda_1$ mode is of course just the Alfvén wave.

In physical terms, this says that when energy is preserved in the half-space, Alfvén waves never give rise to the magnetosonic waves by interaction with the boundary for A^3.

For A^2 the situation is even simpler.

THEOREM 3.6 *The modes are uncoupled for A^2.*

Proof. All we must do is exhibit the ${}_2M_j$ so that (2.14) is satisfied in such a way that each jth term is independent of the others. These are

$$\left.\begin{aligned} {}_2M_2 &= \mathrm{e}^{\mathrm{i}\,{}_2\tau_2 y_3}\operatorname{diag}(-1,-1,1,-1,-1,1,-1), \\ {}_2M_3 &= \mathrm{e}^{\mathrm{i}\,{}_2\tau_3 y_3}\operatorname{diag}(-1,-1,1,-1,-1,1,-1). \end{aligned}\right\} \tag{3.8}$$

Theorems 3.6 and 3.4 will be useful in studying the effect of perturbations.

Proof of Theorem 3.3. We apply the classical elementary result

$$\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\pi}\int_a^b \frac{f(x)}{(k-x)^2+\varepsilon^2}\,\mathrm{d}x = \chi_{(a,b)}(x)f(x)$$

for any continuous f. Set δ sufficiently small so that ($p \notin \boldsymbol{\beta}$) the sets

$$\Delta_j = (a, b) \cap ({}_i\lambda_j(p) - \delta,\ {}_i\lambda_j(p) + \delta)$$

are pairwise disjoint. Then (3.5) is equal to

$$\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{\Delta_j}\frac{\varepsilon}{({}_i\lambda_j(p)-k)^2+\varepsilon^2}\,|({}_i\lambda_j(p)-\bar{z})\boldsymbol{\Phi}_3\chi_{\mathbb{R}^3_+}R(k+\mathrm{i}\varepsilon)f(p)|^2\,\mathrm{d}k.$$

Using (3.8) and Definition 3.1 we obtain the expression (3.6).

In any case it is evident from (3.3) that (the second $*$ means adjoint)

$$A^i(p)\,{}_i\psi_j^{**} = {}_i\lambda_j(p)\,{}_i\psi_j^{**} \tag{3.9}$$

and that the columns of ${}_i\psi_j^{**}$ satisfy the appropriate boundary condition by definition of the ${}_iM_j$. Furthermore, the entries ${}^t(e_4, e_5, e_6)$ in each column are divergence free. Let P_0^i be the projection onto the null space of $A_{\lambda,k}^i$. Then for f in $(I-P_0^i)\mathscr{H}$, and extended by zero to $\mathbb{R}^3$, smooth with compact support

$$(f, g)_{\mathscr{H}} = \sum_{j\neq 0}\int_{\mathbb{R}^3} |{}_i\hat{f}_j(p)|^2\, \mathrm{d}p. \tag{3.10}$$

This follows from Theorem 3.3 and the well-known facts about spectral measures.

Let us define for $g \in \mathscr{D}(\mathbb{R}_+^3)$

$$\boldsymbol{\Phi}_j g(p) = \hat{g}_j(p). \tag{3.11}$$

This is not to be confused with our notation for the Fourier transform! We may, by (3.10) extend $\boldsymbol{\Phi}_j$ to all of $\mathscr{H}$. We shall use the same notation for this extension. Notice that the integral in (3.6) may actually extend over an unbounded set. This causes no difficulty as in Lemma 3.1. By theorems 1.8 and 1.9 of [9] the set of functions in $\mathscr{D}(\mathbb{R}_+^3)$ satisfying one of the appropriate boundary conditions is a core of A^3, A^2 and for f in $\mathscr{D}(A^i)$ there exists $f_n \in \mathscr{D}(\mathbb{R}_+^3)$, $f_n \to f$ in graph norm ($f_n \to f$ and $Af_n \to Af$ in $\mathscr{H}$).

THEOREM 3.7 *Let $g,h \in \mathscr{N}(A^i)^\perp$; then for any interval $I \subseteq \mathbb{R}$,*

$$(E^i(I)g, h) = \sum_{j\neq 0}\int_{\mathbb{R}^3} \chi_{\{{}_i\lambda_j(p)\in I\}}(p)\hat{g}_j(p)^*\hat{h}_j(p)\, \mathrm{d}p. \tag{3.12}$$

Proof. This follows from Lemma 3.3, (3.6), and the polarization identity, in a Hilbert space with inner product (,),

$$4(x, y) = (x+y, x+y) - (x-y, x-y) + \mathrm{i}(x+\mathrm{i}y, x+\mathrm{i}y) - \mathrm{i}(x-\mathrm{i}y, x-\mathrm{i}y).$$

The adjoints of the maps $\boldsymbol{\Phi}_j$ are given by

$$\boldsymbol{\Phi}_j^* f(x) = \int_{\mathbb{R}^3} \psi_j^{**}(p, x) f(p)\, \mathrm{d}p, \tag{3.13}$$

for if $h \in \mathscr{D}(\mathbb{R}_+^3)$ and $f \in \mathscr{D}(\mathbb{R}^3)$ then

$$(h, \boldsymbol{\Phi}_j^* f) = (\boldsymbol{\Phi}_j h, f) = \int_{\mathbb{R}^3}\left\{\int h^*(x)\psi_j^{**}(p, x)\, \mathrm{d}x\right\} f(p)\, \mathrm{d}p. \tag{3.14}$$

Now, $h^*\psi_j^{**}$ is integrable and bounded in p so (3.14) makes sense. Thus we may interchange the order of integration to obtain (3.13). The maps ${}_i\boldsymbol{\Phi}_j$ and ${}_i\boldsymbol{\Phi}_j^*$ yield the reduction of the unitary groups $\mathrm{e}^{-\mathrm{i}tA^i}$. To check that they are orthogonal in the sense that the range of $\boldsymbol{\Phi}_j^*$ is in the null space of $\boldsymbol{\Phi}_k$, $k \neq j$, we suppose f is smooth and rapidly decreasing. Then the expression

$$\begin{aligned}\boldsymbol{\Phi}_j f(r) &= \int_{\mathbb{R}_+^3} (\psi_j^{**}(x, r) f(x)\, \mathrm{d}x \\ &= (2\pi)^{-\frac{3}{2}}\chi_{\mathbb{R}^3\setminus\beta}\chi_{\mathbb{R}_-}(r_3)P_j(r)\int_{\mathbb{R}_+^3}\{\mathrm{e}^{-\mathrm{i}x\circ r}I - M_j^*(r)\mathrm{e}^{\mathrm{i}(x'\circ r' - x_3 r_3)}\} f(x)\, \mathrm{d}x\end{aligned}$$

makes sense pointwise, and further (we assume $j,k>0$)

$$\boldsymbol{\Phi}_k^* g(x) = \int_{\mathbb{R}^3} \psi_k^{**}(x, s) g(s)\,\mathrm{d}s$$

$$= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \chi_{\mathbb{R}_-}(s_3) \chi_{\mathbb{R}^3 \setminus \boldsymbol{\beta}}(p) \{\mathrm{e}^{\mathrm{i}x_3 s_3} I - \mathrm{e}^{-\mathrm{i}x_3 s_3}\} M_k^*(s) P_k(s) g(s)\,\mathrm{d}s$$

is a smooth rapidly decreasing function if g is, and g vanishes in a neighbourhood of $\boldsymbol{\beta}$ and for a fixed p on a neighbourhood of the set of s such that $\lambda_k(s) = \lambda_j(p)$. If F satisfies this condition then $g(s) = F(s)/(\lambda_j(p) - \lambda_k(s))$ also satisfies the same condition. Thus for such a g, $\boldsymbol{\Phi}_k^* g$ is smooth and rapidly decreasing and satisfies the boundary conditions and so is in the domain of A. Let $\mathcal{D}_1 = \{f \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^7) \mid \boldsymbol{\beta} \cap \operatorname{supp}(f) = \varnothing\}$; Fix $F \in \mathcal{D}_1$ and $p \in \mathbb{R}^3$, and set $g(s)$ as above. $g \in \mathcal{D}_1$. Then $\boldsymbol{\Phi}_k^* g$ is smooth, rapidly decreasing and satisfies the boundary conditions and so is in the domain of A, and $A\boldsymbol{\Phi}_k^* g = \boldsymbol{\Phi}_k^* \lambda_k(\bullet) g$. Hence $\boldsymbol{\Phi}_j A \boldsymbol{\Phi}_k^* g = \boldsymbol{\Phi}_j \boldsymbol{\Phi}_k^* \lambda_k(\bullet) g$. But also $\boldsymbol{\Phi}_j A \boldsymbol{\Phi}_k^* g = \lambda_j(p) \boldsymbol{\Phi}_j \boldsymbol{\Phi}_k^* g(p)$. Subtracting, we obtain $\boldsymbol{\Phi}_j \boldsymbol{\Phi}_k^* F(p) = 0$. Since p is arbitrary and $\mathcal{D}_1$ is dense, this proves the required relation. We may show in similar fashion that for f in $\mathcal{N}(A^i)^{\perp}$,

$$\mathrm{e}^{-\mathrm{i}tA^i} f = \sum_{j \neq 0} \boldsymbol{\Phi}_j^* \mathrm{e}^{-\mathrm{i}|p|t} \boldsymbol{\Phi}_j f.$$

In light of the preceding, it is of some interest to consider the rudiments of potential scattering in a half-space. For the boundary condition $B^3_{\infty,1}$, we obtain the following theorem.

THEOREM 3.8 *Let $E(x)$ be an almost everywhere uniformly positive definite bounded Lebesgue measurable 7×7 matrix-valued function defined on the closure of $\mathbb{R}^3_+$. Suppose also that $|E(x) - I| = O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$. Define $A_E = E(x)^{-1} A^3$ on the Hilbert space $\mathcal{H}_E = \mathcal{H}$ with the equivalent inner product*

$$(f, g)_E = \int_{\mathbb{R}^3_+} (f(x), E(x) g(x))_{\mathbb{C}^7}\,\mathrm{d}x$$

and with the boundary condition $B^3_{\infty,1}$. Then A_E is essentially selfadjoint on $\mathcal{H}_E$ and for f in $(I - P_0)\mathcal{H}$, and the map $J : \mathcal{H} \to \mathcal{H}_E$ defined by $Jf = f$, the strong limits

$$W_{\pm} f = \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tA_E} J \mathrm{e}^{-\mathrm{i}tA^3 f} \tag{3.15}$$

exist and define isometries. The same result holds for A^2 substituted for A^3 above.

COROLLARY 3.9 *The solutions $u_{\pm}(x, t)$ and $u(x, t)$ to $\mathrm{i}\,\partial t u_{\pm} = A_E(D) u_{\pm}$ and $\mathrm{i}\,\partial t u = A^3(D) u$ for initial data $f_{\pm}$ and f, respectively (where $f_{\pm} = W_{\pm} f$), are equal at $t = \pm\infty$, i.e.*

$$\lim_{t \to \pm\infty} \|u_{\pm}(\bullet, t) - Ju(\bullet, t)\|_E = 0. \tag{3.16}$$

A similar result holds for A^2. For the other boundary conditions, the following result holds.

THEOREM 3.10 *Let* $|E(x)-I| = O(|x|^{-2.5-\varepsilon})$ *then for any of the boundary conditions, the solutions of the two equations of Corollary* 3.9 *satisfy similar conditions.*

It would be desirable to weaken the hypothesis of Theorem 3.10.

Proof of Theorem 3.8. It is known that this result holds in the $\mathbb{R}^3$ (no boundary) case by, say, [13]. Consider any $f \in L^2(\mathbb{R}^3_+, \mathbb{C}^7)$ and define $G(x) = f(x)$ in $\mathbb{R}^3_+$ and as $\operatorname{diag}(1, 1, -1, 1, 1, -1, 1) f(x_1, x_2, -x_3)$ for $x_3 < 0$. Then

$$\begin{aligned}
\boldsymbol{\Phi}_j^* \boldsymbol{\Phi}_j f(x) &= \int_{\mathbb{R}^3} \psi_j(p, x) \boldsymbol{\Phi}_j f(p)\, dp \\
&= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3_-} [e^{ix\circ p} P_j(p) \boldsymbol{\Phi}_3 G(p) - e^{in\circ x' - ip_3 x_3} P_j(n, -p_3) M_j \boldsymbol{\Phi}_3 G(p)]\, dn\, dp_3 \\
&= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3_-} e^{ix\circ p} P_j(p) \boldsymbol{\Phi}_3 G(p)\, dp \\
&\quad - (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3_+} e^{ix\circ p} P_j(p) M_j \boldsymbol{\Phi}_3 G(n, -p_3)\, dn\, dp_3 \\
&= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ip\circ x} P_j(p) \boldsymbol{\Phi}_3 G(p)\, dp = \boldsymbol{\Phi}_3^* P_j \boldsymbol{\Phi}_3 G(x),
\end{aligned}$$

where $M_j = \operatorname{diag}(1, 1, -1, 1, 1, -1, 1)$ for $j = 1, 2, 3$. Therefore, the group generated by A^3 is given by $\sum_{j\neq 0} \boldsymbol{\Phi}_3 e^{i\lambda_j(\cdot)} P_j(\bullet) \boldsymbol{\Phi}_3 G$ for all f. For this group, call it $U_0(t)$, $A^3 U_0(t)$ is defined by $\sum_{j\neq 0} \boldsymbol{\Phi}_3 \lambda_j(\bullet) e^{i\lambda_j(\cdot)} P_j(\bullet) \boldsymbol{\Phi}_3 G$ which makes sense for *all* smooth rapidly decreasing functions G. There is a set S_0 of smooth rapidly decreasing functions dense in $L^2(\mathbb{R}^3, \mathbb{C}^7)$ such that Huygen's principle holds, i.e. $U_0(t)S_0 = S_0$ and for some R and a depending on f in S_0 (see [1]) $U_0(t)f = 0$ in the set $\{(x, t) : |x| < a\,|t| - R,\ |t| > R/a\}$. For f smooth rapidly decreasing and satisfying the boundary condition,

$$U(-t)\chi_{\mathbb{R}^3_+} U_0(t)f - f = \int_0^t U(-s)(E-I)A^3 U_0(s)f\, ds,$$

so

$$U(-t)\chi_{\mathbb{R}^3_+} U_0(t)f - f = \int_0^t U(-s)(E-I)A^3 U_0(s)f\, ds, \tag{3.17}$$

where $U(t)$ is the group on $L^2(\mathbb{R}_3^+, \mathbb{C}^7)$ generated by A_E. If we extend E to all of $\mathbb{R}^3$ by defining it as I in $\mathbb{R}^3_-$, then both sides of (3.17) make sense for all G which are smooth and rapidly decreasing on $\mathbb{R}^3$ and equality still holds. Now the left side of (3.17) is uniformly bounded in t for all G in $L^2(\mathbb{R}^3, \mathbb{C}^7)$. It is a simple matter to see that for G in S_0, the integral on the right of (3.17) is absolutely convergent as $t \to \pm\infty$ in the sense of norm in $\mathscr{H}_E$, and therefore that the limits $W_\pm$ exist for all f in the domain of A^3 (extend such f to $\mathbb{R}^3$ by zero say) and hence on all of $(I - P_0)\mathscr{H}$ (where they are isometric).

The corollary follows from this (see [12], for example). The same calculation holds for A^2 upon substitution of $\operatorname{diag}(-1, -1, 1, -1, -1, 1, -1)$ above in defining G, etc. We follow the same route to prove Theorem 3.10.

Proof of Theorem 3.10
We may still write

$$U(-t)\chi_{\mathbb{R}^3_+}U_0(t)f - f = \int_0^t U(-s)(E-I)A^3U_0(s)f \, \mathrm{d}s \tag{3.18}$$

for any f which is smooth and rapidly decreasing in $\mathbb{R}^3_+$ and satisfies the boundary conditions. The characteristic function on the left of (3.18) may be dropped. It is necessary to check that

$$\|(E^{-1}-I)A^i\mathrm{e}^{\mathrm{i}tA^i}f\| = O(t^{-1-\delta}) \quad \text{as } t \to \infty. \tag{3.19}$$

This is equivalent to seeing that

$$\int_{\mathbb{R}^3_+} |(E^{-1}-I)|^2 \, |A^i\mathrm{e}^{\mathrm{i}tA^i}f|_{\mathbb{C}^7} \, \mathrm{d}x = O(t^{-2-\delta}). \tag{3.20}$$

Equation (3.20) may be bounded by

$$\int_{\mathbb{R}^3_+} (|x|+1)^{-5-2\varepsilon} |A^i\mathrm{e}^{\mathrm{i}tA^i}f|^2 \, \mathrm{d}x = \int_{\mathbb{R}^3_+} (|x|+1)^{-5-2\varepsilon} \left| \sum_{j\neq 0} \boldsymbol{\Phi}^*_{ji}\lambda_j(\bullet)\mathrm{e}^{\mathrm{i}t_i\lambda_j(\bullet)}\boldsymbol{\Phi}_j f(x) \right|^2 \mathrm{d}x$$

$$= C\int_{\mathbb{R}^3_+} (|x|+1)^{-5-2\varepsilon} \sum_{j\neq 0} |\boldsymbol{\Phi}^*_{ji}\lambda_j(\bullet)\mathrm{e}^{\mathrm{i}t_i\lambda_j(\bullet)}\boldsymbol{\Phi}_j f(x)|^2 \, \mathrm{d}x. \tag{3.21}$$

Now $\boldsymbol{\Phi}^*_j\mathrm{e}^{\mathrm{i}t_i\lambda_j(p)}\boldsymbol{\Phi}_j f$ is equal to

$$\int_{\mathbb{R}^3} {}_i\psi^{**}_j(p,x)\mathrm{e}^{\mathrm{i}t_i\lambda_j(p)}\boldsymbol{\Phi}_j f \, \mathrm{d}p. \tag{3.22}$$

Choose f smooth and rapidly decreasing such that $\boldsymbol{\Phi}_j f = f_j(p)$ is supported in $0<a<|p|<b<\infty$ in the complement of $\boldsymbol{\beta}$. Changing to polar coordinates $\rho = |p|$ and $\omega = p/|p|$ in (3.22) and using the fact that ${}_iM_j$ and $P_j(p)$ are bounded and smooth in the support of f_j, we integrate by parts twice in ρ to obtain that (3.22) is bounded above by $K(|b|^2+1)/(t^2+1)$. This is sufficient to obtain the convergence necessary in (3.20). Since such f are clearly dense in the domain of A, this concludes the proof.

An interesting question arises in the present context: when are solutions of the various mixed problems for A^3 asymptotically equal as in the statement of the conclusion of Corollary 3.9? A sufficient condition is that at least one set of ψ_j satisfies both of the relevant boundary conditions.

A remark concerning A^2 is now in order. It will be observed that the Alfvén wave for A^2 (the ${}_2\lambda_1$ mode) acts as a kind of surface wave—a grazing wave. It propagates parallel to the boundary. In fact the Alfvén wave is a solution to the classical one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_2^2}.$$

This wave does not interact with the boundary.

Local decay of energy and partition of energy is assured in the case of $B^3_{\infty,1}$ and B^2. For the other boundary conditions local decay of energy is observed in the proof of Theorem 3.10. It therefore holds for the perturbed problems as well.

4. Conclusions

As we remarked in [9], the present system has been shown to have some interesting and unusual properties. It may be possible to extend the method used to slab type geometries instead of the half-space. Relaxation of the perfect conductor status of the medium may also be considered. Perturbations in the form of boundary layers remain as well. The relevant computations promise to be more complex than those encountered here. Finally, the asymptotic conditions of the hypothesis of Theorem 3.10 should be weakened.

The absence of surface waves when energy is preserved is interesting but to be expected, since they would perhaps involve drag at the boundary if density and velocity changes were involved. If these are excluded, this leaves only the Alfvén wave and this propagates parallel to the boundary in the case of A^2. As we stated in the introduction, this contrasts with the case of elastic waves where (finite energy) Rayleigh waves can exist for certain physical energy preserving boundary conditions [8].

Acknowledgements

This work was partially supported by a faculty research grant from Brigham Young University and a scientific investigation grant from CTRC, Orem, Utah.

References

1. Avila, G., & Costa, D. 1980 Asymptotic properties of general symmetric hyperbolic systems. *J. Funct. Anal.* **55,** 41–63.
2. Courant, R., & Hilbert, D. 1963 *Methods of Mathematical Physics,* Vol. 2. New York: Interscience.
3. Friedlander, F. G. 1959 Sound pulses in a conducting medium. *Proc. Camb. Phil. Soc.* **55,** 341–67.
4. Lighthill, M. J. 1960 Studies on magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc.* A **252,** 397–430.
5. Matsumura, M. 1970 Comportement asymptotique de solutions de certains problems mixtres pour des systemes hyperboliques symetrique a coefficients constants. *Publ. RIMS, Kyoto Univ.* **5,** 301–60.
6. Rudin, W. 1973 *Functional Analysis.* New York: McGraw-Hill.
7. Schulenberger, J. 1978 Boundary waves on perfect conductors. *J. Math. Anal. Appl.* **66,** 514–49.
8. Schulenberger, J. 1978 Elastic waves in the half space $\mathbb{R}^2_+$. *J. Diff. Eqns.* **29,** 405–38.
9. Smith, W. Energy preserving boundary conditions for plasmas in a half-space. Preprint.
10. Smith, W. 1987 Strongly propagative systems, generalized nonselfadjoint wave equations and their steady-state solutions. *J. Math. Anal. Appl.* **127,** 246–60.
11. Smith, W. 1987 Perturbation of the invariant subspaces of the equations of elasticity: Spectral theory. *J. Math. Anal. Appl.* **121,** 57–78.

12. Smith, W. 1986 A local limiting absorption theorem in a singular dispersive medium. *Q. J. Mech. Appl. Math.* **39**, 453–66.
13. Tamura, H. 1986 Spectral and scattering theory for symmetric systems of non-constant deficit. *J. Funct. Anal.* **67**, 73–104.
14. van Kampen, N. G. 1967 *Theoretical Methods in Plasma Physics*. Amsterdam: North-Holland.
15. Wakabayashi, S. 1975 Eigenfunction expansions for symmetric systems of first order in the half-space $\mathbb{R}^n_+$. *Publ. RIMS, Kyoto Univ.* **11**, 67–147.
16. Wilcox, C. 1966 Wave operators and asymptotic solutions of wave propagation problems of classical physics. *Arch. Rat. Mech. Anal.* **22**, 37–78.