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A mathematical analysis of a fish school model

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Abstract

A model proposed in the literature for fish schools of relatively large size is studied for mathematical and qualitative properties. Existence, uniqueness and positivity of solutions are established and bifurcation properties relative to diffusion and alignment parameters are studied.

Keywords: Schooling; Alignment; Diffusion; Stability; Bifurcation; Variance; Operator semigroups

1. Introduction

In this paper, we perform the analysis of a model of animal orientation. The model is close to one discussed by Grunbaum [4]. It represents the arrangement of a large group of individuals, a fish school for example, according to a structuring variable which is the angle made by the oriented axis associated to any given individual (from tail to head), supposedly lying in horizontal position, with a fixed horizontal oriented axis. The fixed oriented axis may be, for example, the direction of the gradient of temperature, or generally it is a direction which a single individual would tend to follow, when looking for a more favorable environment. The problem is that the individual is not alone, it is surrounded by many others, and may have a lower perception of the environmental cues; this is the price to be paid for being in a group. But, it counts on the group to help it find its way towards a better environment. A standing hypothesis of the model is that the group is very big and, homogeneous at

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some scale so that the state variable will be the proportion of individuals, per volume unit, having a certain angle orientation. The scale defined by the volume unit is a piece of volume within which all individuals can see each other's orientation. To be more specific, some definitions and notations have to be introduced. Throughout the paper, the variable θ represents the angle made by the tail-to-head orientation with the fixed environmental gradient, and the population is described and structured by the proportion density function at time t, $u(\theta, t)$: that is to say, for any θ_1, θ_2, with $0 \leqslant \theta_2 - \theta_1 < 2\pi$,

$$\int_{\theta_1}^{\theta_2} u(\theta, t)\, d\theta$$

is the proportion of the population (per volume unit) which, at time t, points in one of the directions θ of the interval $[\theta_1, \theta_2]$. Clearly, $u(\theta, t)$ must be periodic in θ, with period 2π. The problem to be investigated in Section 2 of the paper, designated as (CP) (for Cauchy problem), is made up of the three Eqs. (1)–(3) stated next

$$\frac{\partial}{\partial t}u(\theta, t) = \frac{\partial^2}{\partial \theta^2}(D(\theta)u(\theta, t)) \\ - \beta\frac{\partial}{\partial \theta}\left(u(\theta, t)\left[\int_{\theta}^{\theta+\pi} F(\theta')u(\theta', t)\, d\theta' - \int_{\theta-\pi}^{\theta} F(\theta')u(\theta', t)\, d\theta'\right]\right), \quad (1)$$

for $(\theta, t) \in R \times (0, \infty)$. The functions $D(\theta)$ and $F(\theta)$ are a.e. positive and periodic with period 2π. Further, properties of these functions will be stated and discussed next. The following equations specify the periodicity in θ:

$$u(\theta, t) = u(\theta + 2\pi, t), \ \frac{\partial}{\partial \theta}u(\theta, t) = \frac{\partial}{\partial \theta}u(\theta + 2\pi, t) \quad \text{on } (0, \infty), \qquad (2)$$

and the initial condition

$$u(\theta, 0) = u_0(\theta) \quad \text{in } (0, 2\pi). \qquad (3)$$

The above conditions together with the periodicity of the coefficients allow us to restrict the study of the problem on the interval $[0, 2\pi]$ and extend the solution to the whole real axis by periodicity.

Let us now discuss the model. In order to understand the rationale underlying such a model, it is necessary to keep in mind that the population is so crowded that any individual movement, even turning around its center of gravity, may impact on other individuals around. In such a world, it is best to stay "parallel". But being parallel to a bunch of individuals around means following these individuals, and there is a risk associated with this. An orientation analog of the avoidance mechanism leads then to a Fickian dispersion or repulsion mechanism contributing to the flux by a quantity proportional to the gradient of concentration of the population. Opposed to this repulsive effect are two other mechanisms: the first one is entailed by some perception of the environment, it adds up to the flux in proportion to some gradient of favorability; the second one is the analog of a gregarious effect, individuals tend to adopt the dominant orientation. It is modeled by the second member of the right-hand side of Eq. (1). We will come back to it after we discuss in a little more detail the first two effects. These are accounted for in the first member of the right-hand side of Eq. (1). To see this, let us drop for a moment the time dependence in this member. We may write it as

$$\frac{d^2}{d\theta^2}(D(\theta)u(\theta)) = \frac{d}{d\theta}\left(D(\theta)\frac{d}{d\theta}u(\theta)\right) + \frac{d}{d\theta}(D'(\theta)u(\theta)).$$

Therein, $\frac{d}{d\theta}(D(\theta)\frac{d}{d\theta}u(\theta))$ is the (Fickian) dispersive term and $-\frac{d}{d\theta}(D'(\theta)u(\theta))$ is the environmental-induced advection. While there is no fundamental reason for these two distinct processes to be modeled by a single function $D(\theta)$, it may be the case that it is so: for example, if the Fickian coefficient is approximately constant, large compared to the environment coefficient, D may be defined as the sum of both. This is the view taken here, $D(\theta)$ will be assumed in the form

$$D(\theta) = D_0 + D_1 f(\theta),$$

with D_0 and D_1, positive constants, D_1 suitably smaller than D_0, f positive, bounded and 2π-periodic, so that D be far from 0 and ∞. For simplicity, we also assume that f is in $W^{2,\infty}(0, 2\pi)$, or equivalently, the same property for D. On occasion, we will use such a weighted scalar product with D as a weight

$$\langle \varphi, \psi \rangle_D = \int_0^{2\pi} \varphi(\theta)\psi(\theta)D(\theta)\, d\theta. \tag{4}$$

The notations $\langle .,. \rangle_D$ and $||.||_D$ will be used accordingly. With the assumptions on D, the underlying Hilbert space is the standard space $L^2(0, 2\pi) =_{\text{def}} X$. The notations $\langle .,. \rangle$ and $||.||$ correspond to the usual scalar product and its associated norm in X. We now turn to the second term on the right-hand side of Eq. (1). For notational purposes, we define the operator g_F as follows:

$$g_F(\varphi)(\theta) = \int_\theta^{\theta+\pi} F(\theta')\varphi(\theta')\, d\theta' - \int_{\theta-\pi}^{\theta} F(\theta')\varphi(\theta')\, d\theta', \tag{5}$$

g_F is well defined as soon as F is a measurable, bounded function. We also assume that F is continuous. It is then straightforward to check that g_F sends 2π-periodic functions into themselves. One can also easily check that in terms of the weighted scalar product $\langle .,. \rangle_F$, it holds that

$$\langle g_F(\varphi), \psi \rangle_F = -\langle \varphi, g_F(\psi) \rangle_F \tag{6}$$

that is, g_F is antisymmetric in the space of 2π-periodic functions endowed with this weighted (not necessarily definite) scalar product. If, in particular, we take $F = 1$, $\langle , \rangle_1$ is just the usual scalar product $\langle , \rangle$ and we have

$$(g_1)^* = -g_1. \tag{7}$$

Evidently, $g_1(1) = 0$, so using (6) with $\psi = 1$, we have, for each 2π-periodic function φ,

$$\int_0^{2\pi} g_1(\varphi)(\theta)\, d\theta = 0. \tag{8}$$

The interpretation of g_F is the following: for each θ, the first integral on the right-hand side of (5) is the weighted integral of the population (in the volume unit) whose orientation is on the 'left' of θ, and the second one is, accordingly, the weighted integral of the population whose orientation is on the 'right' of θ. So, g_F gives the sign of the rate of change of the orientation as a result of gregarious behavior, combined with an environment-induced preference modeled by $F(\theta)$. If $F = 1$, the integrals are just evaluating the proportions of individuals whose orientation is on the 'left' of θ and on the 'right' of θ, and the effect modeled by g_1 is that individuals will tend to turn 'right' or 'left' dependent upon whether $g_1 < 0$ or > 0. The parameter β in front of the integral can be viewed as an intensity factor which sets up the relative importance of the gregarious behavior compared to the other factors.

Our goal in this paper is to perform an analytic study of (CP). Two aspects have to be considered:

(1) Existence of solutions, that is, the Cauchy problem associated to the equation. This is a quasilinear problem with nonlinearities in the first-order term, which can be represented as an abstract Cauchy problem,

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + B[D'u(t) - \beta u(t) g_F(u(t))], \\ u(0) = u_0 \in X, \end{cases} \tag{9}$$

with $u(t)$ used for $u(.,t)$. The operator $A : \mathscr{D}(A) \subset X \to X$ is defined by

$$Aw = \frac{d}{d\theta}\left(D(.)\frac{d}{d\theta}\right)w,$$

$$\mathscr{D}(A) = \{w \in H^2(0, 2\pi):\ w(0) = w(2\pi) \text{ and } w'(0) = w'(2\pi)\}, \tag{10}$$

and the operator $B : \mathscr{D}(B) \subset X \to X$ by

$$Bw = \frac{d}{d\theta} w,$$

$$\mathscr{D}(B) = \{w \in H^1(0, 2\pi):\ w(0) = w(2\pi)\}. \tag{11}$$

$H^1(0, 2\pi)$ and $H^2(0, 2\pi)$ denote usual Sobolev functions spaces [5]. $\mathscr{D}(B)$ (resp. $\mathscr{D}(A)$) can be identified to the subspace, in $H^1_{loc}(\mathbb{R})$, of the 2π-periodic functions, resp. the subspace, in $H^2_{loc}(\mathbb{R})$, of the 2π-periodic functions. Throughout the paper, we will use the Banach space structure endowed to $\mathscr{D}(B)$ by the graph norm

$$|\phi|_{\mathscr{D}(B)} = ||\phi|| + \left|\left|\frac{d\phi}{d\theta}\right|\right|.$$

We will also use the well-known fact that $\mathscr{D}(B)$ imbeds continuously in the space of continuous functions, with

$$|v|_\infty \leqslant \tfrac{4}{3}\pi^{3/2}|v|_{\mathscr{D}(B)}, \quad \forall v \in \mathscr{D}(B). \tag{12}$$

We point out that the operator A does not commute with B (unless D is constant); this complicates the treatment of the problem (see also remark at the end of Section 2). However, the following easily derived formula relating A and B will be useful:

$$\langle Bu, DBu \rangle = -\langle u, Au \rangle, \quad \forall u \in \mathscr{D}(A). \tag{13}$$

The main existence result will be derived using successive approximations in a space of continuous functions from some suitable interval $[0, t_0]$ (where $t_0 > 0$ will be chosen later on) into $\mathscr{D}(B)$. On occasion, we will use the notation $Y = C([0, t_0], \mathscr{D}(B))$. A crucial, while obvious, fact, when handling the nonlinearity, is that $g_F : \mathscr{D}(B) \to \mathscr{D}(B)$, continuously, so there exists a constant δ, so that

$$|g_F(\phi)|_{\mathscr{D}(B)} \leqslant \delta |\phi|_{\mathscr{D}(B)}, \quad \forall \phi \in \mathscr{D}(B). \tag{14}$$

We will also prove a regularity result, namely if $u_0 \in \mathscr{D}(A)$, then the solution is classical. Global existence fails in many nonlinear situations. It holds here for nonnegative solutions. The fact, mentioned earlier in the introduction, that $u(\theta, t)$ is indeed a proportion, namely, that

$$u(\tilde{\theta}, t) \geqslant 0 \quad \text{and} \quad \int_{\tilde{\theta}}^{\tilde{\theta}+2\pi} u(\theta, t)\, d\theta = 1, \quad \forall \tilde{\theta},\ \forall t \geqslant 0 \tag{15}$$

will be shown to hold when the same assumptions are made on u_0: $u_0(\theta) \geqslant 0$ and $\int_0^{2\pi} u_0(\theta)\, d\theta = 1$, $\forall \theta$ (Theorems 2.1 and 2.3).

(2) Asymptotic behavior and existence of a stable steady state. We envision the situation as follows: Below a certain threshold value of the gregarious behavior intensity β, the dispersion dominates and the population organizes itself asymptotically as if there were no gregarism. The solution then should tend to a limit, a steady state. For example, if we assume that $D(\theta) = \bar{D}$ (i.e., the mean value of $D(.)$, that is $\bar{D} = \frac{1}{2\pi}\int_0^{2\pi} D(\theta)\, d\theta$) and $F(\theta) = 1$, $\bar{U}(\theta) = \frac{1}{2\pi}$ is the only trivial steady state, that is, an individual's orientation is equally distributed in all directions. One can see that this solution exists for all values of β, and is stable for β small enough, but one suspects that past some β, stability is lost, and other steady states arise. This is when the action of the repulsion moderated by the interplay of environment and gregarism produces another structure. Although the program seems reasonable, it involves a number of technical steps which make it difficult to complete in the most general situation. As an illustration of the plausibility of the above-mentioned scenario, a particular example has been considered. In order to describe what is going on in more detail, we first restrict ourselves to the case when $D(\theta) = \bar{D} =$ constant. We then perform the study in terms of the parameter $\lambda = \frac{\beta}{\bar{D}}$. We show that the trivial equilibrium $\bar{U} = \frac{1}{2\pi}$ is asymptotically stable if $\lambda < \frac{\pi}{2}$ and unstable if $\lambda > \frac{\pi}{2}$. So, the question arises: How is stability or instability affected at $\lambda = \frac{\pi}{2} = \lambda_0$? Using a bifurcation theorem (see [1, Theorem 1.7]), we prove that $(\lambda_0, \bar{U})$ is an odd type [1] bifurcation point. Therefore, a branch of nontrivial steady-state (λ, U) branches off from this point. We prove that the branch is supercritical and the solutions on the branch near $\lambda = \lambda_0$ are stable. Near $\lambda = \lambda_0$, the bifurcated solutions read

$$U(\theta) = \frac{1}{2\pi} + C\cos\theta + o(C),$$

for some constant $C = C(\lambda) > 0$. In terms of the model, this means that when the ratio λ exceeds some threshold value λ_0, then the group starts to acquire a distinctive shape with one dominant direction. We then briefly justify the fact that the local branch can be extended to a larger branch which is unbounded in λ, so that is, for each $\lambda > \lambda_0$, there exists a nontrivial steady state.

The organization of the paper is as follows: Section 2 deals with existence, uniqueness and positivity; Section 3 is devoted to the study of the stability of the trivial steady state as a function of the parameter β and the onset of a branch of nontrivial steady states, as well as the computation of some quantities (the average angular orientation and the dispersion about it). Section 4 is the conclusion which, in particular, has some comments about the interpretation of the result in terms of spatial location and pattern: it may be useful to anticipate these and warn the reader that there is no connection between space and pattern; more precisely, space has not been accounted for in the model, thus, unsurprisingly, is not playing any role in our results.

Finally, a number of notations and assumptions that have been stated in the introduction will be used throughout the text without further notice.

2. Existence, uniqueness and positivity

2.1. Preliminaries

System (1)–(3), (15) will be studied via the theory of operator semigroups. For that, let us recall standard definitions and some relevant results of the semigroup theory. We refer to Pazy [7], Engel and Nagel [2], Henry [5] or Friedman [3] for further information on this subject.

Let X be a Banach space and let $\mathscr{A}$ be a closed linear operator, with a dense domain $\mathscr{D}(\mathscr{A})$. We consider the inhomogeneous initial value problem

$$\begin{cases} \frac{d}{dt}u(t) = \mathscr{A}u(t) + f(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{16}$$

with $u_0 \in X$ and $f \in L^1(0, T; X)$.

Definition 2.1 (Dazy [7]). Suppose $\mathscr{A}$ is the infinitesimal generator of a C^0-semigroup $T(t)$. Then

(1) $u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds$ is called a mild solution of the Cauchy problem (16).

(2) u is called a classical solution of (16) if u is continuous on $[0, \infty)$, continuously differentiable on $]0, \infty[$, $u(t) \in \mathscr{D}(\mathscr{A})$, $t > 0$ and u satisfies Eq. (16).

We now turn back to (CP).

Proposition 2.1. *The operator A defined by* (10) *is the generator of an analytic semigroup of contractions in X, $(T(t))_{t \geqslant 0}$, compact for $t > 0$. The restrictions $T(t)_{/\mathscr{D}(B)}$ send $\mathscr{D}(B)$ into itself and are uniformly bounded in $\mathscr{D}(B)$ (that is, there exists $C_1 \geqslant 0$, such that, $|T(t)_{/\mathscr{D}(B)}|_{D(B)} \leqslant C_1$, for $t \geqslant 0$).*

Proof. The first part is standard. We include a proof for completeness. A has a dense domain, since $C_c^\infty([0, 2\pi]) \subset \mathscr{D}(A)$ and is dense in X. It is clear that if u and v are in $\mathscr{D}(A)$, $\langle Au, v \rangle = \langle u, Av \rangle$, via integration by parts. Hence A is symmetric. One can also easily show, using a standard argument [2] that $R(I + A) = X$. So, A is symmetric, maximal and has a dense domain, which implies that A is self-adjoint. Thus A is the generator of an analytic semigroup $(T(t))_{t \geqslant 0}$. So $T(t)$ is continuous in the uniform operator topology for $t > 0$. In order to show that $T(t)$ is compact for $t > 0$, it is thus enough, by virtue of Theorem 2.3.1 [7], to establish that A has a compact resolvent, which follows from both $\mathcal{D}(A) \subset H^2(0, 2\pi)$ and the compactness of the canonical injection $H^2(0, 2\pi) \hookrightarrow X$.

Let us now prove the second part of the proposition. From $\langle Au, u \rangle \leqslant 0$, $\forall u \in \mathcal{D}(A)$, we conclude that $T(t)$ is a family of contractions on the space X. So,

$$||T(t)|| \leqslant 1, \quad \forall t \geqslant 0. \tag{17}$$

To show boundedness in $\mathcal{D}(B)$, it is convenient to represent $T(t)$ in terms of the eigenvalues of A, which we denote $-\omega_j^2$. To each eigenvalue $-\omega_j^2$, an eigenvector ϕ_j is associated and it holds that

$$A\phi = \sum_{j=1}^{\infty} -\omega_j^2 \langle \phi, \phi_j \rangle \phi_j, \quad \phi \in \mathcal{D}(A) \tag{18}$$

and

$$T(t)\phi = \sum_{j=1}^{\infty} e^{-\omega_j^2 t} \langle \phi, \phi_j \rangle \phi_j, \quad \phi \in X.$$

Using the weighted inner product, defined by formula (4), one can easily check

$$\begin{aligned} ||BT(t)\phi||_D^2 &= \sum_{j=1}^{\infty} e^{-2\omega_j^2 t} |\langle \phi, \phi_j \rangle|^2 ||B\phi_j||_D^2 \\ &\leqslant \sum_{j=1}^{\infty} |\langle \phi, \phi_j \rangle|^2 ||B\phi_j||_D^2 \\ &= ||B\phi||_D^2, \quad \forall \phi \in \mathcal{D}(B), \end{aligned}$$

which, leads to

$$||BT(t)\phi|| \leqslant C_0 |\phi|_{\mathcal{D}(B)}, \quad \forall \phi \in \mathcal{D}(B)$$

(for some constant C_0 independent on ϕ and t). This together with inequality (17), leads to

$$|T(t)\phi|_{\mathcal{D}(B)} \leqslant C_1 |\phi|_{\mathcal{D}(B)}, \quad \forall \phi \in \mathcal{D}(B) \tag{19}$$

(for some constant C_1 independent on ϕ and t). This completes the proof of the proposition. □

We now turn to (CP). The solving of this problem involves two steps: first, one deals with local existence, which is shown to hold under very mild regularity assumptions on the nonlinearity; next, a noncontinuation principle will be established which will ensure solutions exist on as long a time interval as desired.

To prove local existence of solutions for problem (9), we write it in integral form by using the variation of constants formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)B[D'u(s) - \beta u(s)g_F(u(s))]\, ds. \tag{20}$$

2.2. Local existence of solutions

This subsection is concerned with local existence of solutions to the integral equation (20). For this purpose, we start by establishing some useful estimates.

Lemma 2.1. (1) *There exists a constant M, such that, for all u, $v \in \mathscr{D}(B)$, we have*

$$||B[ug_F(u)] - B[vg_F(v)]|| \leqslant M \max(|u|_{\mathscr{D}(B)}, |v|_{\mathscr{D}(B)})|u-v|_{\mathscr{D}(B)}.$$

(2) *There exists a positive constant Q, such that, for all $u \in \mathscr{D}(B)$, it holds that*

$$||B[ug_F(u)]|| \leqslant Q|u|_{\mathscr{D}(B)}||u||.$$

(3) *There exists a positive constant C, such that, for all $u \in X$, it holds that*

$$||BT(t)u|| \leqslant \frac{C}{\sqrt{t}}||u||, \quad \forall t > 0. \tag{21}$$

Proof. Let u, $v \in \mathscr{D}(B)$. We may write the expression $B[ug_F(u)] - B[vg_F(v)]$ as

$$\begin{aligned} B[ug_F(u)] - B[vg_F(v)] &= B[(u-v)g_F u] + B[vg_F(u-v)] \\ &= B(u-v)g_F u + (u-v)Bg_F u \\ &\quad + Bvg_F(u-v) + vBg_F(u-v). \end{aligned}$$

Then

$$\begin{aligned} ||B[ug_F(u)] - B[vg_F(v)]|| \leqslant &\ \{||B(u-v)|| + ||u-v||\}\max(|g_F u|_\infty; |Bg_F u|_\infty) \\ &+ \{||Bv|| + ||v||\}\max(|g_F(u-v)|_\infty; |Bg_F(u-v)|_\infty). \end{aligned} \tag{22}$$

We now estimate each of the terms of $\max(|g_F u|_\infty; |Bg_F u|_\infty)$ and $\max(|g_F(u-v)|_\infty; |Bg_F(u-v)|_\infty)$ of the right-hand side of (22) separately.

As a result of Hölder's inequality, we get

$$|g_F u|_\infty \leqslant \sqrt{2\pi}|F|_\infty ||u|| \quad \text{and} \quad |g_F(u-v)|_\infty \leqslant \sqrt{2\pi}|F|_\infty ||u-v||. \tag{23}$$

On the other hand, we have, via the inequality

$$|Bg_F u|_\infty \leqslant 4|F|_\infty |u|_\infty,$$

combined with (12) that

$$|Bg_F u|_\infty \leqslant \tfrac{16}{3}\pi^{3/2}|F|_\infty |u|_{\mathscr{D}(B)} \quad \text{and} \quad |Bg_F(u-v)|_\infty \leqslant \tfrac{16}{3}\pi^{3/2}|F|_\infty |u-v|_{\mathscr{D}(B)}. \tag{24}$$

Then, inequality (22) becomes

$$||B[ug_F(u)] - B[vg_F(v)]|| \leqslant M \max(|u|_{\mathscr{D}(B)}, |v|_{\mathscr{D}(B)})|u-v|_{\mathscr{D}(B)}$$

(for some positive constant M independent on u and on v), which is the desired inequality.

(2) To achieve the desired inequality (which is a slight improvement, compared to the one just obtained), we write the expression $B[ug_F(u)]$ as

$$B[ug_F(u)] = (Bu)g_F(u) + uBg_F(u),$$

which immediately yields

$$||B[ug_F(u)]|| \leqslant ||Bu|| |g_F u|_\infty + ||u|| |Bg_F u|_\infty.$$

From inequalities (23) and (24), we conclude that there exists a positive constant Q, such that

$$||B[ug_F(u)]|| \leqslant Q|u|_{\mathscr{D}(B)} ||u||, \quad \forall u \in \mathscr{D}(B).$$

(3) In view of (13), we have

$$||Bu||^2 \leqslant C_1 ||Au|| ||u||, \quad \forall u \in \mathscr{D}(A) \tag{25}$$

(for some constant C_1 independent on u). Analyticity of the semigroup $T(t)$ implies existence of a constant C_2 such that, for $t > 0$

$$||AT(t)|| \leqslant \frac{C_2}{t}. \tag{26}$$

Then, it follows via (25), combined with (26) that

$$||BT(t)u||^2 \leqslant C_1 \frac{C_2}{t} ||T(t)u|| ||u||, \quad \forall u \in X, \ \forall t > 0.$$

This together with inequality (17), leads to

$$||BT(t)u|| \leqslant \frac{C}{\sqrt{t}} ||u||, \quad \forall u \in X, \ \forall t > 0 \tag{27}$$

(for some constant C independent on u and t). This completes the proof of the lemma. □

Theorem 2.1. *For every $R>0$, there exists $t_0>0$, $t_0=t_0(R)$, such that, for each $u_0\in\mathscr{B}_{\mathscr{D}(B)}(R)$ (i.e., the ball of radius R centered at 0 of $\mathscr{D}(B)$), the Cauchy problem* (9) *has a unique mild solution u defined on the interval $[0,t_0]$. Moreover, the map $u_0\to u$ is Lipschitz continuous from $\mathscr{B}_{\mathscr{D}(B)}(R)$ into Y. Finally, $\int_0^{2\pi}u(\theta,t)\,d\theta=\int_0^{2\pi}u_0(\theta)\,d\theta$ for all $t\geqslant 0$.*

Proof. The approach to be taken in the proof is based on the method of successive approximations. Let $u_0\in\mathscr{D}(B)$ and define a sequence $(u_n)_{n\geqslant 1}$ by

$$u_{n+1}(t)=T(t)u_0+\int_0^t T(t-s)B[D'u_n(s)-\beta u_n(s)g_F(u_n(s))]\,ds. \tag{28}$$

Assume that the sequence u_n is bounded in Y, namely, that there exist $t_0>0$ and γ to be determined later on, such that

$$|u_n(t)|_{\mathscr{D}(B)}\leqslant\gamma R,\quad \forall n\geqslant 1 \text{ and } \forall t\in[0,t_0].$$

We have from (28), combined with (17), that

$$\begin{aligned}||u_{n+1}(t)-u_n(t)||\leqslant t\sup_{0\leqslant s\leqslant t}\ &[||B(D'u_n(s)-D'u_{n-1}(s))||\\ &+||B(\beta u_n(s)g_F(u_n(s)-\beta u_{n-1}(s)g_F(u_{n-1}(s)))||],\quad \forall t\geqslant 0 \text{ and } n\geqslant 1.\end{aligned}$$

Denote $L=\max(|D'|_\infty,|D''|_\infty)$, which by the assumptions made in the introduction is finite.

Then, according to part (1) of Lemma 2.1, we have

$$||u_{n+1}(t)-u_n(t)||\leqslant t(L+\beta M\gamma R)\sup_{0\leqslant s\leqslant t}|u_n(s)-u_{n-1}(s)|_{\mathscr{D}(B)},\quad \forall t\geqslant 0 \text{ and } n\geqslant 1.$$

We also have

$$\begin{aligned}||B(u_{n+1}(t)-u_n(t))||\leqslant\bigg|\bigg|\int_0^t &BT(t-s)B[D'u_n(s)-\beta u_n(s)g_F(u_n(s))\\ &-D'u_{n-1}(s)+\beta u_{n-1}(s)g_F(u_{n-1}(s))]ds\bigg|\bigg|,\quad \forall t\geqslant 0.\end{aligned}$$

which gives, in view of (21),

$$\begin{aligned}||B(u_{n+1}(t)-u_n(t))||\leqslant\int_0^t\frac{C}{\sqrt{t-s}}\sup_{0\leqslant s\leqslant t}\ &[||B(D'u_n(s)-D'u_{n-1}(s))||\\ &+||B(\beta u_n(s)g_F(u_n(s)-\beta u_{n-1}(s)g_F(u_{n-1}(s)))||]\,ds,\quad \forall t\geqslant 0.\end{aligned}$$

Using part (1) of Lemma 2.1 again, we arrive at

$$\|B(u_{n+1}(t)-u_n(t))\| \leqslant (\beta M\gamma R+L))2C\sqrt{t}\sup_{0\leqslant s\leqslant t}|u_n(s)-u_{n-1}(s)|_{\mathscr{D}(B)}.$$

Combining the above two inequalities, we obtain

$$|u_{n+1}(t)-u_n(t)|_{\mathscr{D}(B)} \leqslant [(\beta M\gamma R+L))(2C\sqrt{t}+t)]\sup_{0\leqslant s\leqslant t}|u_n(s)-u_{n-1}(s)|_{\mathscr{D}(B)}, \quad \forall t\geqslant 0 \text{ and } n\geqslant 1.$$

Then, by choosing $t_0>0$ small enough so that

$$[(\beta M\gamma R+L))(2C\sqrt{t_0}+t_0)]<\tfrac{1}{2}, \tag{29}$$

the sequence $(u_{n+1}(t)-u_n(t))$ is the general term of an absolutely convergent series in Y. To complete the proof of the theorem, we have to show that the sequence $u_n(t)$ remains bounded in $\mathscr{D}(B)$.

From (28), one has

$$|u_{n+1}(t)|_{\mathscr{D}(B)} \leqslant \sup_{0\leqslant s\leqslant t}|T(s)u_0|_{\mathscr{D}(B)} + \left\|\int_0^t T(t-s)B[D'u_n(s)-\beta u_n(s)g_F(u_n(s))]\,ds\right\| + \left\|B\int_0^t T(t-s)B[D'u_n(s)-\beta u_n(s)g_F(u_n(s))]\,ds\right\|, \quad \forall t\geqslant 0.$$

Using this inequality, we will show that one can find $\gamma>0$ and $t_0>0$, so that $|u_n(t)|_{\mathscr{D}(B)}\leqslant\gamma R$, $\forall n\geqslant 1$ and $\forall t\in[0,t_0]$. In fact, let us assume that $|u_n(t)|_{\mathscr{D}(B)}\leqslant\gamma R$, $\forall t\in[0,t_0]$. Then, in view of (21), combined with (19) and (17), we obtain

$$|u_{n+1}(t)|_{\mathscr{D}(B)} \leqslant C_1R+[t+2\sqrt{t}C](\beta\max\{|g_Fu_n(s)|_\infty,|Bg_Fu_n(s)|_\infty\}+L) \sup_{0\leqslant s\leqslant t}|u_n(s)|_{\mathscr{D}(B)}, \quad \forall t\geqslant 0.$$

Applying (23) and (24), we get

$$|u_{n+1}(t)|_{\mathscr{D}(B)} \leqslant C_1R+\gamma R[t_0+2\sqrt{t_0}C](\beta\tfrac{16}{3}\pi^{3/2}|F|_\infty\gamma R+L), \quad \forall t\in[0,t_0].$$

Therefore, by choosing $t_0>0$ so that (29) holds and

$$[t_0+2\sqrt{t_0}C](\beta\tfrac{16}{3}\pi^{3/2}|F|_\infty\gamma R+L)<\tfrac{1}{2}, \tag{30}$$

we arrive at

$$|u_{n+1}(t)|_{\mathscr{D}(B)} \leqslant C_1R+\tfrac{1}{2}\gamma R, \quad \text{for } t\leqslant t_0.$$

The inequality will be extended to $n+1$ if we can choose both γ large enough for

$$C_1 R + \tfrac{1}{2}\gamma R \leqslant \gamma R \quad \text{(which holds as soon as } \gamma \geqslant 2C_1\text{)}$$

and $t_0 > 0$ such that (29) and (30) hold which, once γ has been chosen, can always be accomplished. So, assuming γ and t_0 have been chosen as indicated, the first claim of the proof is proved.

Now, let v be another mild solution of (9) on $[0, t_0]$ with the initial value $v_0 \in \mathscr{B}_{\mathscr{D}(B)}(R)$.

From (28), combined with (17), we have

$$\|u(t) - v(t)\| \leqslant \|u_0 - v_0\| + t_0 \sup_{0 \leqslant s \leqslant t_0} [\|B(D'u(s) - D'v(s))\| + \|B(\beta u(s) g_F(u(s) - \beta v(s) g_F(v(s)))\|], \quad \forall t \in [0, t_0]$$

which gives, in view of part (1) of Lemma 2.1

$$\|u(t) - v(t)\| \leqslant \|u_0 - v_0\| + t_0(L + \beta M \gamma R) \sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)}, \quad \forall t \in [0, t_0].$$

Using (21), we obtain

$$\|B(u(t) - v(t))\| \leqslant \|B(u_0 - v_0)\| + (\beta M \gamma R + L) 2C\sqrt{t_0} \sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)}, \quad \forall t \in [0, t_0].$$

Therefore,

$$|u(t) - v(t)|_{\mathscr{D}(B)} \leqslant |u_0 - v_0|_{\mathscr{D}(B)} + [(\beta M \gamma R + L)(2C\sqrt{t_0} + t_0)] \sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)}, \quad \forall t \in [0, t_0],$$

which immediately yields

$$\sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)} \leqslant |u_0 - v_0|_{\mathscr{D}(B)} + [(\beta M \gamma R + L)(2C\sqrt{t_0} + t_0)] \sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)}.$$

Then, for $\gamma > 0$ and $t_0 > 0$ chosen as above, we have

$$|u(t) - v(t)|_{\mathscr{D}(B)} \leqslant |u_0 - v_0|_{\mathscr{D}(B)} + \tfrac{1}{2} \sup_{0 \leqslant s \leqslant t_0} |u(s) - v(s)|_{\mathscr{D}(B)}, \quad \forall t \in [0, t_0].$$

So,

$$|u(t) - v(t)|_{\mathscr{D}(B)} \leqslant 2|u_0 - v_0|_{\mathscr{D}(B)}, \quad \forall t \in [0, t_0],$$

which yields both the uniqueness of u and the Lipschitz continuity of the map $u_0 \to u$ in the ball $\mathscr{B}_{\mathscr{D}(B)}(R)$.

Finally, by integration of Eq. (1) on both sides, from 0 to 2π, one can see that

$$\frac{d}{dt}\int_0^{2\pi} u(\theta, t)\, d\theta = 0,$$

that is,

$$\int_0^{2\pi} u(\theta, t)\, d\theta = \int_0^{2\pi} u_0(\theta)\, d\theta, \quad \text{for all } t \geqslant 0.$$

This completes the proof of the theorem. □

2.3. *Continuation of solutions*

This subsection is concerned with the extension of solutions to the integral equation (20). Our first result in this direction is the following theorem.

Theorem 2.2. *For every $u_0 \in \mathscr{D}(B)$, the abstract Cauchy problem* (9) *has a unique mild solution on a maximal interval of existence* $[0, t_{\max}[$.

If $t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}} \sup |u(t)|_{\mathscr{D}(B)} = \infty.$$

Proof. First, we note that a mild solution of Eq. (9) defined on a closed interval $[0, \tau]$ can be extended to a larger interval $[0, \tau + \delta]$, with $\delta > 0$, by defining $u(t)$ on $[\tau, \tau + \delta]$, as $u(t) = w(t)$, where $w(t)$ is the solution of the integral equation

$$w(t) = T(t - \tau)u(\tau) + \int_\tau^t T(t - s)B[D'w(s) - \beta w(s) g_F(w(s))]\, ds.$$

Existence and uniqueness of solutions on a maximal interval of existence follow from the noncontinuation principle; namely letting $[0, t_{\max})$ be the maximal interval of existence to which the mild solution $u(.)$ of Cauchy problem (9) can be extended, we have the following alternative: either $t_{\max} = \infty$ or $t_{\max} < \infty$ and

$$\lim_{t \to t_{\max}} \sup |u(t)|_{\mathscr{D}(B)} = \infty.$$

We will prove the second part of the alternative

$$\text{if } t_{\max} < \infty \text{ then } \lim_{t\to t_{\max}} \sup |u(t)|_{\mathscr{D}(B)} = \infty .$$

The proof is done by contradiction. Indeed, if $t_{\max} < \infty$, and $\lim_{t\to t_{\max}} \sup |u(t)|_{\mathscr{D}(B)} < \infty$, then $|u(t)|_{\mathscr{D}(B)}$ would be uniformly bounded on $[0, t_{\max}[$.

Let $N = \sup_{0 \leqslant s \leqslant t_{\max}} |D'u(s) - \beta u(s) g_F(u(s))|_{\mathscr{D}(B)}$.

Given any $\rho > 0$, so that $0 < \rho < t < t' < t_{\max}$, using standard algebra, we obtain the following inequality:

$$\begin{aligned} |u(t) - u(t')|_{\mathscr{D}(B)} \leqslant & \ |T(t)u_0 - T(t')u_0|_{\mathscr{D}(B)} + |(T(\rho) - T(t' - t + \rho)) \\ & \times \int_0^{t-\rho} T(t-s-\rho) B[D'u(s) - \beta u(s) g_F(u(s))] ds \Big|_{\mathscr{D}(B)} \\ & + \left| \int_{t-\rho}^{t} (T(t-s) - T(t'-s)) B[D'u(s) - \beta u(s) g_F(u(s))] ds \right|_{\mathscr{D}(B)} \\ & + \left| \int_t^{t'} T(t'-s) B[D'u(s) - \beta u(s) g_F(u(s))] ds \right|_{\mathscr{D}(B)} . \end{aligned} \tag{31}$$

We now estimate each of the terms of the right-hand side of (31) separately. In view of (21), there exists a constant $C > 0$, such that $|T(t)u|_{\mathscr{D}(B)} \leqslant \frac{C}{\sqrt{t}} |u|_{\mathscr{D}(B)}$, $0 < t < t_{\max}$. Therefore,

$$\begin{aligned} & \left| (T(\rho) - T(t' - t + \rho)) \int_0^{t-\rho} T(t-s-\rho) B[D'u(s) - \beta u(s) g_F(u(s))] \, ds \right|_{\mathscr{D}(B)} \\ & \leqslant \sqrt{t-\rho} CN |(T(\rho) - T(t' - t + \rho))|_{\mathscr{D}(B)}, \end{aligned}$$

$$\left| \int_{t-\rho}^{t} (T(t-s) - T(t'-s)) B[D'u(s) - \beta u(s) g_F(u(s))] \, ds \right|_{\mathscr{D}(B)} \leqslant 2\sqrt{\rho} CN$$

and

$$\left| \int_t^{t'} T(t'-s) B[D'u(s) - \beta u(s) g_F(u(s))] \, ds \right|_{\mathscr{D}(B)} \leqslant 2NC\sqrt{t'-t}.$$

Then, inequality (31) becomes

$$|u(t) - u(t')|_{\mathscr{D}(B)} \leqslant |T(t)u_0 - T(t')u_0|_{\mathscr{D}(B)} + 2\sqrt{\rho}CN + \sqrt{t-\rho}MN|(T(\rho) - T(t'-t+\rho))|_{\mathscr{D}(B)} + 2NC\sqrt{t'-t}. \quad (32)$$

Analyticity of $T(t)$ entails that $t \to T(t)$ is continuous in the uniform operator topology from $]0, +\infty]$ into $\mathscr{D}(A)$. So, $|T(t)u_0 - T(t')u_0|_{\mathscr{D}(B)} \to 0$, and $|(T(\rho) - T(t'-t+\rho))|_{\mathscr{D}(B)} \to 0$ as $t, t' \to t_{\max}$. Since $0 < \rho < t$ is arbitrary, the right-hand side of (32) tends to zero as t, t' tend to $t_{\max}$. Therefore

$$\lim_{t \to t_{\max}} u(t) \text{ exists (limit in } \mathscr{D}(B)),$$

which, according to the remark made in the beginning of the proof, would entail that the solution can be extended to the right of $t_{\max}$, in contradiction with the definition of $t_{\max}$. This completes the proof of the theorem. □

Global existence (i.e., the fact that the solutions are defined on the whole of $t > 0$) is established for positive solutions. For that, we will show, the boundedness of the solution $u(t)$ in the $\mathscr{D}(B)$ norm. This property, together with Theorem 2.2, implies that $t_{\max} = \infty$. Prior to this, we will prove that (CP) preserves positiveness, which will be needed in the a priori estimates of the solutions.

Theorem 2.3. (*CP*) *preserves positiveness, that is*: $u_0 \geqslant 0$ *implies that* $u(.,t) \geqslant 0$ *for all* $t \geqslant 0$.

Proof. Let $u(.,t)$ be a solution of (CP) with initial value $u_0 \geqslant 0$; u^- (resp. u^+) denoting the negative (resp. positive) part of u.

From

$$\int_0^{2\pi} u(\theta, t)\, d\theta = 1,$$

one has

$$\int_0^{2\pi} u^+(\theta, t)\, d\theta - \int_0^{2\pi} u^-(\theta, t)\, d\theta = 1,$$

that is,

$$\left| \int_\theta^{\theta+\pi} F(\theta')u(\theta', t)\, d\theta' - \int_{\theta-\pi}^{\theta} F(\theta')u(\theta, t)\, d\theta' \right| \leqslant 2|F(.)|_\infty (1 + 2\sqrt{2\pi}\|u^-(.,t)\|). \quad (33)$$

Now multiplying (using inner products) both sides of (1) by $u^-(\theta,t)$, we have

$$\int_0^{2\pi} u^-(\theta,t)\frac{\partial}{\partial t}u(\theta,t)\,d\theta$$
$$= \int_0^{2\pi} u^-(\theta,t)\frac{\partial^2}{\partial\theta^2}(D(\theta)u(\theta,t))\,d\theta$$
$$- \beta\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(u(\theta,t)\left[\int_\theta^{\theta+\pi}F(\theta')u(\theta',t)\,d\theta' - \int_{\theta-\pi}^{\theta}F(\theta')u(\theta',t)\,d\theta'\right]\right)u^-(\theta,t)\,d\theta.$$

Using the obvious identity

$$-|u^-(\theta,t)|^2 = u^-(\theta,t)u(\theta,t),$$

and integrating from 0 to 2π, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}|u^-(\theta,t)|^2\,d\theta$$
$$= \int_0^{2\pi} D(\theta)\left[\frac{\partial}{\partial\theta}u^-(\theta,t)\right]^2 d\theta$$
$$- \int_0^{2\pi} D'(\theta)u(\theta,t)\left[\frac{\partial}{\partial\theta}u^-(\theta,t)\right] d\theta$$
$$+ \beta\int_0^{2\pi} u(\theta,t)\left[\int_\theta^{\theta+\pi}F(\theta')u(\theta',t)\,d\theta' - \int_{\theta-\pi}^{\theta}F(\theta')u(\theta',t)\,d\theta'\right]\frac{\partial}{\partial\theta}u^-(\theta,t)\,d\theta.$$

We now have, via the identity,

$$ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \tag{34}$$

combined with (33) that

$$\int_0^{2\pi}\left(u(\theta,t)\left[\int_\theta^{\theta+\pi}F(\theta')u(\theta',t)\,d\theta' - \int_{\theta-\pi}^{\theta}F(\theta')u(\theta',t)\,d\theta'\right]\frac{\partial}{\partial\theta}u^-(\theta,t)\right)d\theta$$
$$\leqslant \frac{4}{\varepsilon}|F(.)|_\infty^2\,||u^-(.,t)||^2(1+2\sqrt{2\pi}||u^-(.,t)||)^2 + \varepsilon\left|\left|\frac{\partial}{\partial\theta}u^-(.,t)\right|\right|^2.$$

Moreover, one has via (34) and the fact that the functions $D(.)$ and $D'(.)$ in $[0,2\pi]$ are bounded that

$$\int_0^{2\pi} D'(\theta)u(\theta,t)\left[\frac{\partial}{\partial\theta}u^-(\theta,t)\right] d\theta \leqslant |D'(.)|_\infty^2\frac{1}{\varepsilon}||u^-(.,t)||^2 + \varepsilon\left|\left|\frac{\partial}{\partial\theta}u^-(.,t)\right|\right|^2.$$

Therefore,

$$\frac{d}{dt}\int_0^{2\pi} |u^-(\theta,t)|^2 d\theta \leqslant -2\int_0^{2\pi} D(\theta)\left[\frac{\partial}{\partial\theta}u^-(\theta,t)\right]^2 d\theta$$
$$+ |D'(.)|_\infty^2 \frac{1}{\varepsilon}||u^-(.,t)||^2 + \varepsilon\left|\left|\frac{\partial}{\partial\theta}u^-(.,t)\right|\right|^2$$
$$+ 4\frac{\beta}{\varepsilon}|F(.)|_\infty^2 (1+2\sqrt{2\pi}||u^-(.,t)||)^2||u^-(.,t)||^2$$
$$+ \varepsilon\beta\left|\left|\frac{\partial}{\partial\theta}u^-(.,t)\right|\right|^2,$$

which immediately yields

$$\frac{d}{dt}\int_0^{2\pi} |u^-(\theta,t)|^2 d\theta \leqslant (-2D_0+\varepsilon\beta+\varepsilon)\left|\left|\frac{\partial}{\partial\theta}u^-(.,t)\right|\right|^2$$
$$+ |D'(.)|_\infty^2 \frac{1}{\varepsilon}||u^-(.,t)||^2$$
$$+ 4\frac{\beta}{\varepsilon}|F(.)|_\infty^2 (1+2\sqrt{2\pi}||u^-(.,t)||)^2||u^-(.,t)||^2. \quad (35)$$

By choosing $\varepsilon > 0$ small enough, we can eliminate the term of the right-hand side of (35) containing $\frac{\partial}{\partial\theta}u^-(\theta,t)$. We arrive at the differential inequality

$$\frac{d}{dt}||u^-(.,t)||^2 \leqslant L_0||u^-(.,t)||^2(1+||u^-(.,t)||^2) \quad (36)$$

(for some constant L_0, independent on $u^-(.,t)$). Let

$$v(t) = ||u^-(.,t)||^2.$$

Then, inequality (36) becomes

$$v'(t) \leqslant L_0 v(t)(1+v(t)),$$

so by integration, we get

$$\frac{v(t)}{1+v(t)} \leqslant \frac{v(0)}{1+v(0)}e^{L_0 t},$$

which, with

$$v(0) = 0,$$

yields that

$$v(t) \leqslant 0, \quad \text{for all } t > 0.$$

Therefore

$$||u^-(.,t)|| = 0, \quad \text{for all } t \geqslant 0,$$

that is, $u(.,t) \geqslant 0$, for all $t \geqslant 0$, which completes the proof of the theorem. □

Using the above result, we can give an estimate of the solutions as follows:

Proposition 2.2. *There exists a function* $K: \mathbf{R}^+ \to]0, +\infty]$, *nonincreasing, such that, if* $u(.,t)$ *is a solution of (CP) with* $u_0 \in \mathscr{D}(B)$ *and* $u_0 \geqslant 0$, *then it holds that* $|u(t)|_{\mathscr{D}(B)} \leqslant K_1(||u_0||)|u_0|_{\mathscr{D}(B)}$, *for all* $t \in [0, K(||u_0||)]$, *where* $K_1(x) = 2[C_1 + \exp(L_1 K(x))]$ (Moreover, $\|u(.,t)\|$ is uniformly bounded on bounded time intervals.)

Proof. The proof is done in two steps. First, we show that the solution is bounded in X norm on each bounded time interval. Then, we prove boundedness in the $\mathscr{D}(B)$ norm, using the result for the X norm.

Multiplying by $u(\theta, t)$ both sides of Eq. (1) and integrating from 0 to 2π, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2(\theta,t)\,d\theta = & \int_0^{2\pi} u(\theta,t)\frac{\partial}{\partial\theta}\left(D(\theta)\frac{\partial}{\partial\theta}u(\theta,t)\right)d\theta \\ & + \int_0^{2\pi} u(\theta,t)\frac{\partial}{\partial\theta}(D'(\theta)u(\theta,t))\,d\theta \\ & + \beta\int_0^{2\pi} u(\theta,t)\left[\int_\theta^{\theta+\pi} F(\sigma)u(\sigma,t)d\sigma\right. \\ & \left. - \int_{\theta-\pi}^{\theta} F(\sigma)u(\sigma,t)\,d\sigma\right]\frac{\partial}{\partial\theta}u(\theta,t)\,d\theta. \end{aligned} \tag{37}$$

Integrating by parts on the right-hand side of (37), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2(\theta,t)d\theta = & -\int_0^{2\pi} D(\theta)\left(\frac{\partial}{\partial\theta}u(\theta,t)\right)^2 d\theta + \int_0^{2\pi} u(\theta,t)\frac{\partial}{\partial\theta}(D'(\theta)u(\theta,t))\,d\theta \\ & + \beta\int_0^{2\pi} u(\theta,t)\left[\int_\theta^{\theta+\pi} F(\sigma)u(\sigma,t)d\sigma\right. \\ & \left. - \int_{\theta-\pi}^{\theta} F(\sigma)u(\sigma,t)d\sigma\right]\frac{\partial}{\partial\theta}u(\theta,t)\,d\theta. \end{aligned}$$

Using a similar argument to the one given in the proof of Theorem 2.3, we arrive at the following inequality:

$$\int_0^{2\pi} u(\theta,t)\frac{\partial}{\partial\theta}(D'(\theta)u(\theta,t))d\theta \leqslant \frac{1}{\varepsilon}\int_0^{2\pi} u^2(\theta,t)d\theta + \varepsilon|D'(.)|_\infty^2 \int_0^{2\pi}\left(\frac{\partial}{\partial\theta}u(\theta,t)\right)^2 d\theta,$$

and

$$\beta\int_0^{2\pi} u(\theta,t)\left[\int_\theta^{\theta+\pi} F(\sigma)u(\sigma,t)d\sigma - \int_{\theta-\pi}^{\theta} F(\sigma)u(\sigma,t)d\sigma\right]\frac{\partial}{\partial\theta}u(\theta,t)\,d\theta$$
$$\leqslant 4\frac{\beta}{\varepsilon}|F(.)|_\infty^2\int_0^{2\pi} u^2(\theta,t)\,d\theta + \beta\varepsilon\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}u(\theta,t)\right)^2 d\theta..$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2(\theta,t)d\theta \leqslant (-D_0 + \varepsilon|D'(.)|_\infty^2 + \beta\varepsilon)\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}u(\theta,t)\right)^2 d\theta$$
$$+\left(4\frac{\beta}{\varepsilon}|F(.)|_\infty^2 + \frac{1}{\varepsilon}\right)\int_0^{2\pi} u^2(\theta,t)d\theta. \tag{38}$$

By choosing $\varepsilon > 0$ small enough, we can eliminate the term of the right-hand side of (38) containing $\frac{\partial}{\partial\theta}u(\theta,t)$. We arrive at a differential inequality of the type

$$\frac{d}{dt}||u^2(.,t)||^2 \leqslant 2L_1||u^2(.,t)||^2$$

(for some constant L_1, independent on $||u(.,t)||$), which leads to

$$||u(.,t)|| \leqslant e^{L_1 t}||u_0||. \tag{39}$$

Combining (19) and (21), we have from (20)

$$||Bu(t)|| \leqslant C_1|u_0|_{\mathscr{D}(B)} + C\sqrt{t}\sup_{0\leqslant s\leqslant t}||B[D'u(s) - \beta u(s)g_F(u(s))]||, \quad \forall t\geqslant 0.$$

Then, according to part (2) of Lemma 2.1, we have

$$||Bu(t)|| \leqslant C_1|u_0|_{\mathscr{D}(B)} + C\sqrt{t}\sup_{0\leqslant s\leqslant t}[L + \beta Q||u(s)||]\sup_{0\leqslant s\leqslant t}|u(s)|_{\mathscr{D}(B)}, \quad \forall t\geqslant 0.$$

Together with (39), we arrive at

$$\|Bu(t)\| \leqslant C_1 |u_0|_{\mathscr{D}(B)} + C\sqrt{t} \sup_{0\leqslant s\leqslant t} [L + \beta Q \exp(L_1 t)\|u_0\|] \sup_{0\leqslant s\leqslant t} |u(s)|_{\mathscr{D}(B)}, \quad \forall t \geqslant 0. \tag{40}$$

Define $K(r)$, for $r \geqslant 0$, as the unique t root (positive) of the equation

$$C\sqrt{t}[L + \beta Q r \exp(L_1 t)] = \tfrac{1}{2}.$$

Using again estimates (39) and (40), we conclude that

$$|u(t)|_{\mathscr{D}(B)} \leqslant 2(C_1 + \exp(L_1 K(\|u_0\|)))|u_0|_{\mathscr{D}(B)}, \quad \text{for all } t \in [0, K(\|u_0\|)].$$

This completes the proof of the proposition, with K_1 defined as mentioned. □

2.4. Regularity

In Section 2.3, we have proved that if the initial value is in $\mathscr{D}(B)$, then the solution takes its values in $\mathscr{D}(B)$. We can achieve a higher regularity of the mild solution of (CP) if we assume more regularity for the initial value. This is done in the next theorem.

Recall that a function $h : I \to X$ is Hölder continuous with exponent $\eta \in (0, 1)$ on I, where I is an interval, if there is a constant M such that [7]

$$\|h(t) - h(s)\| \leqslant M|t - s|^\eta, \quad \text{for } s, t \in I.$$

It is locally Hölder continuous if every $t \in I$ has a neighborhood in which h is Hölder continuous.

The following result describes the regularity of a mild solution of Eq. (9).

Theorem 2.4. *For every $u_0 \in \mathscr{D}(A)$, the mild solution of Eq.* (9) *is a classical solution.*

Proof. If we show that the function

$$f(t) = B[D'u(t) - \beta u(t) g_F(u(t))]$$

is locally Hölder continuous from $(0, \infty)$ into $\mathscr{D}(B)$, then it follows from the theory of nonhomogeneous linear equations (see [7, Corollary 3.3]) that the mild solution of (9) is a classical solution. That f is locally Hölder continuous on $(0, \infty)$ will follow if we prove that the map $t \to u(t)$ is locally Hölder continuous, from $(0, \infty)$ into $\mathscr{D}(B)$.

Indeed, let $0<t<t'\leqslant t_0$, with $t_0 = K(\|u_0\|)$ as in Proposition 2.2. Then

$$\begin{aligned} f(t)-f(t') = {} & D''(u(t)-u(t')) + D'B(u(t)-u(t')) \\ & - \beta[Bu(t)-Bu(t')]g_F(u(t)) - \beta[u(t)-u(t')]Bg_F(u(t)) \\ & - \beta[g_F(u(t))-g_F(u(t'))]Bu(t') - \beta[Bg_F(u(t))-Bg_F(u(t'))]u(t'), \end{aligned}$$

which implies

$$\begin{aligned} \|f(t)-f(t')\| \leqslant {} & |u(t)-u(t')|_{\mathscr{D}(B)}[|g_F u(t)|_{\mathscr{D}(B)} + \max(|D'|_\infty, |D''|_\infty)] \\ & + |g_F u(t) - g_F u(t')|_{\mathscr{D}(B)}|u(t')|_{\mathscr{D}(B)}. \end{aligned} \tag{41}$$

In view of (14), we have

$$|g_F u(t) - g_F u(t')|_{\mathscr{D}(B)} \leqslant \delta |u(t)-u(t')|_{\mathscr{D}(B)}.$$

The fact that t, $t'\leqslant t_0$ entails

$$[|g_F u(t)|_{\mathscr{D}(B)} + \max(|D'|_\infty, |D''|_\infty)] + \delta |u(t')|_{\mathscr{D}(B)} \leqslant \xi$$

(for some constant ξ that can be chosen independent on u, t, t'). Then, inequality (41) becomes

$$\|f(t)-f(t')\| \leqslant \xi |u(t)-u(t')|_{\mathscr{D}(B)}.$$

In order to estimate $|u(t)-u(t')|_{\mathscr{D}(B)}$, we note that the quantity $u(t)-u(t')$ can be broken down into the sum of three, from which the following inequality is derived:

$$\begin{aligned} |u(t)-u(t')|_{\mathscr{D}(B)} \leqslant {} & |T(t)u_0 - T(t')u_0|_{\mathscr{D}(B)} \\ & + \left| \int_0^t (T(t-s)-T(t'-s))B[D'u(s) - \beta u(s) g_F(u(s))]ds \right|_{\mathscr{D}(B)} \\ & + \left| \int_t^{t'} T(t'-s)B[D'u(s) - \beta u(s) g_F(u(s))]ds \right|_{\mathscr{D}(B)}. \end{aligned} \tag{42}$$

The last member of the right-hand side of (42) is estimated using

$$N = \sup_{0\leqslant s\leqslant t_0} |D'u(s) - \beta u(s) g_F(u(s))|_{\mathscr{D}(B)}$$

and (21). We obtain

$$\left| \int_t^{t'} T(t'-s)B[D'u(s) - \beta u(s)g_F(u(s))]ds \right|_{\mathscr{D}(B)} \leqslant NC\sqrt{t'-t}. \tag{43}$$

In order to estimate

$$I = \left| \int_0^t (T(t-s) - T(t'-s))B[D'u(s) - \beta u(s)g_F(u(s))]ds \right|_{\mathscr{D}(B)},$$

we divide this quantity into the sum of three

$$\begin{aligned} I \leqslant & \left| \int_0^t T(s)(B[D'u(t-s) - \beta u(t-s)g_F(u(t-s))] \right. \\ & \left. - B[D'u(t'-s) - \beta u(t'-s)g_F(u(t'-s))])ds \right|_{\mathscr{D}(B)} \\ & + \left| \int_0^{t'-t} T(s)B[D'u(t'+s) - \beta u(t'-s)g_F(u(t'-s))]ds \right|_{\mathscr{D}(B)} \\ & + \left| \int_t^{t'} T(s)[D'u(t'-s) - \beta u(t'-s)g_F(u(t'-s))]ds \right|_{\mathscr{D}(B)} \\ = & I_1 + I_2 + I_3. \end{aligned}$$

In view of (21), it is clear that

$$\begin{aligned} I_1 & \leqslant C\sqrt{t'-t} \sup_{0\leqslant s\leqslant t} |u(t'-s) - u(t-s)|_{\mathscr{D}(B)} \\ & \leqslant C\sqrt{t'-t} \sup_{0\leqslant \sigma\leqslant t} |u(\sigma + t' - t) - u(\sigma)|_{\mathscr{D}(B)} \end{aligned} \tag{44}$$

$$I_2 \leqslant 2NC\sqrt{t'-t} \tag{45}$$

and

$$I_3 \leqslant 2NC\sqrt{t'-t}. \tag{46}$$

On the other hand, with $u_0 \in \mathscr{D}(A)$, we have

$$T(t')u_0 - T(t)u_0 = \int_t^{t'} T(s)Au_0\, ds,$$

$$BT(t')u_0 - BT(t)u_0 = \int_t^{t'} BT(s)Au_0\, ds.$$

Using ((17) (resp. (21)), we can see that

$$||T(t')u_0 - T(t)u_0|| \leqslant (t' - t)||Au_0||, \tag{47}$$

respectively,

$$||BT(t')u_0 - BT(t)u_0||2C(\sqrt{t'} - \sqrt{t})||Au_0|| \leqslant 2C\sqrt{t' - t}||Au_0||. \tag{48}$$

Combining (47) and (48), we obtain

$$|T(t')u_0 - T(t)u_0|_{\mathscr{D}(B)} \leqslant C_3\sqrt{t' - t}|u_0|_{\mathscr{D}(A)} \tag{49}$$

(for some positive constant C_3).

Consequently, from (43) combined with (44)–(46) and (49), we arrive at the following inequality:

$$\begin{aligned}|u(t') - u(t)|_{\mathscr{D}(B)} \leqslant{}& C_3\sqrt{t' - t}|u_0|_{\mathscr{D}(A)} + 2NC\sqrt{t' - t}\\ &+ C\sqrt{t' - t}\sup_{0 \leqslant \sigma \leqslant t} |u(\sigma + t' - t) - u(\sigma)|_{\mathscr{D}(B)}, \quad \forall 0 \leqslant t \leqslant t' \leqslant t_0.\end{aligned}$$

Then

$$\begin{aligned}\sup_{0 \leqslant \sigma \leqslant t} |u(\sigma + t' - t) - u(\sigma)|_{\mathscr{D}(B)} \leqslant{}& C_3\sqrt{t' - t}|u_0|_{\mathscr{D}(A)} + 2NC\sqrt{t' - t}\\ &+ C\sqrt{t' - t}\sup_{0 \leqslant \sigma \leqslant t} |u(\sigma + t' - t) - u(\sigma)|_{\mathscr{D}(B)},\\ &\forall 0 \leqslant t \leqslant t' \leqslant t_0.\end{aligned}$$

By choosing $h_0 > 0$ such that

$$C\sqrt{h_0} < 1,$$

we obtain

$$\sup_{0\leqslant\sigma\leqslant t} |u(\sigma+t'-t)-u(\sigma)|_{\mathscr{D}(B)}\leqslant K_2\sqrt{t'-t}, \quad \forall 0\leqslant t\leqslant t'\leqslant t_0,\ t'-t\leqslant h_0$$

(for some constant K_2 that can be expressed in the form of $K_2=K_2(|u_0|_{\mathscr{D}(A)})$). which leads to

$$|u(t)-u(t')|_{\mathscr{D}(B)}\leqslant K_2\sqrt{t'-t}, \quad \forall 0\leqslant t\leqslant t'\leqslant t_0, \text{ such that } t'-t\leqslant h_0.$$

This yields Hölder continuity as defined from $[0,t_0]$ into $\mathscr{D}(B)$, from which the preparatory remark made at the beginning of the proof leads to the conclusion that u is a classical solution. □

To conclude this section, we have proved that the Cauchy problem (9) is well posed in $\mathscr{D}(B)$. Solutions there are fixed points of strict contractions; initial values in $\mathscr{D}(A)$ yield classical solutions. This result has been obtained under a few assumptions on the operators A and B that can be satisfied by a variety of examples: in particular, there is no limitation on the dimension of the underlying physical space. The main problem we faced here was the fact that A and B do not commute. If, on the contrary, such a property is assumed (for example, by taking for B a fractional power of $(-A)$) [5] and in addition, an estimate similar t_0 (21),

$$||BT(t)u||\leqslant Ct^{-\alpha}||u||, \quad \forall t>0 \text{ for some } \alpha<1,$$

holds, then the integral equation (20) can be solved in X.

3. Stationary solutions, stability and bifurcation

In this part, we deal with steady-state proportion densities. We investigate existence and multiplicity of steady-state solutions in terms of the parameter β. We find that besides a trivial steady state which exists for all β, a branch of nontrivial ones emerges by an odd type bifurcation (see [1, Theorem 1.7]) near a value β_0. We determine the direction of the bifurcation (supercritical) and we show that the variance is going down along the branch, that is, the obtained nontrivial steady states represent the beginning of group's organization; variance of a certain arrangement is a possible measure of cost efficiency of that arrangement.

The proof requires a number of properties to be satisfied by the linearized operator. It is rather technical. In order to avoid cumbersome formulas further assumptions will be made from now on, namely: the functions D and F are supposed to be constant,

$$D(\theta)=\bar{D}, \quad F(\theta)=1.$$

3.1. Fixed point problem formulation

With the above-mentioned simplifications on F and D, we consider the problem made up of (1)–(3) and (15) and we look for equilibria, i.e., for solutions of the form $u(\theta, t) = U(\theta)$, $\forall t$. Such solutions must satisfy the system

$$\bar{D}U'(\theta) - \beta U(\theta)\left[\int_{\theta}^{\theta+\pi} U(\sigma)\,d\sigma - \int_{\theta-\pi}^{\theta} U(\sigma)\,d\sigma\right] = \gamma, \tag{50}$$

where γ is a constant to be determined, $\theta \in (0, 2\pi)$, with boundary conditions

$$U(0) = U(2\pi), \quad U'(0) = U'(2\pi) \tag{51}$$

and

$$U(\theta) \geqslant 0 \quad \text{in } (0, 2\pi) \text{ and } \int_0^{2\pi} U(\theta)\,d\theta = 1. \tag{52}$$

We introduce the following operator $\mathscr{F}$ defined by

$$\mathscr{F}(U)(\theta) = \int_0^{\theta} \frac{1}{\bar{D}}\left[\int_{\sigma}^{\sigma+\pi} U(\sigma')\,d\sigma' - \int_{\sigma-\pi}^{\sigma} U(\sigma')\,d\sigma'\right]d\sigma. \tag{53}$$

In terms of the operator g_1, defined by formula (5) for $F = 1$, $\mathscr{F}(U)$ reads

$$\mathscr{F}(U)(\theta) = \frac{1}{\bar{D}}\int_0^{\theta} g_1(U)(\sigma)\,d\sigma, \tag{54}$$

which, in particular, in view of (8), yields

$$\mathscr{F}(U)(2\pi) = 0. \tag{55}$$

In terms of $\mathscr{F}(U)$, (50) can be written as

$$\frac{d}{d\theta}U(\theta) - \beta U(\theta)\frac{d}{d\theta}\mathscr{F}(U)(\theta) = \frac{\gamma}{\bar{D}}.$$

Integrating by parts and using the boundary condition, we arrive at

$$U(\theta) = U(0)\exp\beta(\mathscr{F}(U)(\theta)) + \frac{\gamma}{\bar{D}}\int_0^{\theta}\exp\beta(\mathscr{F}(U)(\theta) - \mathscr{F}(U)(\sigma))\,d\sigma. \tag{56}$$

Using the boundary condition (51) together with (55), we obtain

$$0 = \frac{\gamma}{\bar{D}}\int_0^{2\pi}\exp\beta(\mathscr{F}(U)(2\pi) - \mathscr{F}(U)(\sigma))\,d\sigma. \tag{57}$$

from which we deduce that

$$\gamma = 0.$$

Hence, (56) reduces to

$$U(\theta) = U(0)\exp\beta(\mathscr{F}(U)(\theta))$$

which, together with the normalization condition,

$$\int_0^{2\pi} U(\theta)\, d\theta = 1,$$

leads to

$$U(\theta) = \frac{\exp\beta(\mathscr{F}(U)(\theta))}{\int_0^{2\pi}\exp\beta(\mathscr{F}(U)(s))ds} =_{\text{def}} \mathscr{H}(\beta, U)(\theta). \tag{58}$$

Then, from (58), the search for steady-state solutions is equivalent to finding fixed points of $\mathscr{H}$:

$$U = \mathscr{H}(\beta, U). \tag{59}$$

Remark 2. (i) We note that $\mathscr{H}$ is of class C^∞ in (β, U).

(ii) For any $\beta \in \mathbf{R}$, and any constant function $U = c$, we have $\mathscr{H}(\beta, U) = \frac{1}{2\pi}$. In particular, if we define $\bar{U} = \frac{1}{2\pi}$, it holds that $\mathscr{H}(\beta, \bar{U}) = \bar{U}$, for each β, that is, $U = \bar{U}$ is a trivial solution, for all β.

3.2. *Preparatory results*

We just saw that the steady-state problem comes down to a fixed point problem for a map $\mathscr{H}$ and that $\mathscr{H}$ has a branch of trivial steady states, $\bar{U} = \frac{1}{2\pi}$. We are going to determine a branch of nontrivial steady states emanating from the trivial branch at some value $\beta = \beta_0$. This requires a number of preliminary facts to be established, about the linearization of the map $\mathscr{H}(\beta, U)$ near $\bar{U}$. We start with a few notations and definitions. On occasion, we will use the following notation:

$$\mathscr{L}(\beta) = D_u\mathscr{H}(\beta, \bar{U}). \tag{60}$$

Straightforward computation based on formula (58) gives

$$[\mathscr{L}(\beta)\xi](\theta) = \frac{\beta}{2\pi}\left[\mathscr{F}(\xi)(\theta) - \frac{1}{2\pi}\int_0^{2\pi}\mathscr{F}(\xi)(s)\, ds\right]. \tag{61}$$

We denote

$$X_0 = \left\{ f \in X\colon \int_0^{2\pi} f(s)\, ds = 0 \right\}. \tag{62}$$

For any operator G with values in X, we define the mean value operator, denoted $\bar{G}$,

$$\bar{G}(x) = \frac{1}{2\pi} \int_0^{2\pi} G(x)(s)\, ds, \tag{63}$$

and we denote G_0 the projection of G onto the space X_0 parallel to the constant function space, that is to say,

$$G_0 = G - \bar{G}. \tag{64}$$

Using (61), (63) and (64), we can see that the operator $\mathscr{L}(\beta)$ reads as

$$\mathscr{L}(\beta) = \frac{\beta}{2\pi} \mathscr{F}_0. \tag{65}$$

In terms of g_1, defined by formula (5) for $F = 1$, and $\mathscr{I}$, the antiderivative operator defined by

$$(\mathscr{I} f)(\theta) = \int_0^{\theta} f(s)\, ds, \tag{66}$$

we have

$$\mathscr{F} = \frac{1}{\bar{D}} \mathscr{I} \circ g_1. \tag{67}$$

Lemma 3.1. *$\mathscr{I}$ maps X_0 into X; g_1 maps X into X_0. Moreover, it holds that*

$$\mathscr{I} \circ g_1 = g_1 \circ \mathscr{I} + \overline{(\mathscr{I} \circ g_1)}. \tag{68}$$

Proof. That $\mathscr{I}$ maps X_0 into X is a well-known fact about antiderivatives of periodic functions. It has been observed in the introduction that g_1 sends periodic functions to periodic functions and formula (8) expresses the fact that g_1 takes its values in X_0. In order to check formula (68), it is enough to compute $\frac{d}{d\theta} \circ g_1 \circ \mathscr{I}$ and notice that

$$\frac{d}{d\theta} \circ g_1 \circ \mathscr{I} = g_1.$$

Applying the operator $\mathscr{I}$ on both sides of the above identity, we conclude that, for any φ, we have

$$\mathscr{I} \circ g_1(\varphi) = g_1 \circ \mathscr{I}(\varphi) + c$$

for some constant $c = c(\varphi)$. Taking the mean value of both sides, and using the fact that g_1 maps into X_0, we arrive at the expression

$$c = \overline{(\mathscr{I}\circ g_1)}(\varphi)$$

from which the desired formula follows immediately.

Using the following obvious property, that is,

$$\overline{(\mathscr{I}\circ g_1)} = \bar{\mathscr{I}}\circ g_1 \tag{69}$$

and the fact that $g_1(1) = 0$, we get the following:

$$\mathscr{I}_0\circ g_1 = g_1\circ\mathscr{I}_0 \quad \text{on } X_0. \quad \square \tag{70}$$

Lemma 3.2. *$\mathscr{I}_0$ and g_1 map X_0 into X_0. Moreover, the following hold:*

$$(\mathscr{I}\circ g_1)_0 = \mathscr{I}_0\circ g_1, \tag{71}$$

$$(\mathscr{I}_0)^* = -\mathscr{I}_0; \quad (g_1)^* = -g_1 \text{ on } X_0. \tag{72}$$

Proof. That $\mathscr{I}_0$ takes its values in X_0 follows immediately from the defining formula (64). For g_1, see Lemma 3.1. Formula (71) can be derived from the proof of Lemma 3.1 using (69). The fact that g_1 is antisymmetric has already been mentioned in the introduction, see formula (7). Finally, the formula for $\mathscr{I}_0^*$ can be checked by a direct computation of the adjoint operator of $\mathscr{I}_0$. $\square$

We are now in position to draw the main consequences of the previous study for the operator $\mathscr{L}(\beta)$.

Proposition 3.1. *$\mathscr{L}(\beta)$ is compact and self-adjoint on X_0.*

Proof. Expression (61) shows that $\mathscr{L}(\beta)$ is the difference of a multiple of $\mathscr{F}$ and a finite rank operator. $\mathscr{F}$ takes its values in the Sobolev space $H^2(]0, 2\pi[)$, which imbeds compactly in X. Hence, the compactness of $\mathscr{L}(\beta)$. From (65), (67) and (71), we have

$$\mathscr{L}(\beta) = \frac{\beta}{2\pi\bar{D}}\mathscr{I}_0\circ g_1$$

and, in view of (70) and (72), we obtain, as a result of the direct computation of the adjoint of $\mathscr{L}(\beta)$

$$(\mathscr{L}(\beta))^* = \frac{\beta}{2\pi\bar{D}}(g_1)^*\circ(\mathscr{I}_0)^* = \frac{\beta}{2\pi\bar{D}}g_1\circ\mathscr{I}_0 = \frac{\beta}{2\pi\bar{D}}\mathscr{I}_0\circ g_1 = \mathscr{L}(\beta). \quad \square$$

We end this preparatory section with a few computational remarks regarding $\mathscr{H}$. From

$$\mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right) = \mathscr{H}\left(\beta_0, \frac{1}{2\pi} + \frac{\beta}{\beta_0} u\right),$$

we obtain

$$D^j_\beta \mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right) = D^j_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi} + \frac{\beta}{\beta_0} u\right)\left(\frac{u}{\beta_0}\right)^j,$$

$$D^j_u \mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right) = \left(\frac{\beta}{\beta_0}\right)^j D^j_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi} + \frac{\beta}{\beta_0} u\right)$$

which, in particular, leads to

$$\left(\frac{\partial}{\partial\beta}\right)^j \mathscr{H}\left(\beta, \frac{1}{2\pi}\right) = 0, \quad j \geqslant 1 \tag{73}$$

and

$$D^j_u \mathscr{H}\left(\beta, \frac{1}{2\pi}\right) = \left(\frac{\beta}{\beta_0}\right)^j D^j_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right). \tag{74}$$

We also have

$$D^2_{u\beta} \mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right) = \frac{1}{\beta_0} D_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi} + \frac{\beta}{\beta_0} u\right) + \frac{\beta}{(\beta_0)^2} D^2_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi} + \frac{\beta}{\beta_0} u\right) u,$$

which gives

$$D^2_{u\beta} \mathscr{H}\left(\beta, \frac{1}{2\pi}\right) = \frac{1}{\beta_0} D_u \mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right). \tag{75}$$

Moreover,

$$D_u \mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right)\xi = \beta \mathscr{H}\left(\beta, \frac{1}{2\pi} + u\right)\mathscr{F}(\xi) - \frac{\beta \mathscr{H}(\beta, \frac{1}{2\pi} + u) \int_0^{2\pi} \Phi(\beta \mathscr{F}(u)) \mathscr{F}(\xi)}{\tilde{\Phi}(\beta \mathscr{F}(u))} \tag{76}$$

with

$$\Phi(u) = \exp u,$$

$$\tilde{\Phi}(f) = \int_0^{2\pi} \Phi(f(x)) dx.$$

Identity (74) for $j=1$ shows that, if v denotes an eigenvector of $D_u\mathscr{H}(\beta_0,\frac{1}{2\pi})$ for the eigenvalue 1, then v is also an eigenvector of $D_u\mathscr{H}(\beta,\frac{1}{2\pi})$ for the value $\frac{\beta}{\beta_0}$. We again have

$$\begin{aligned} D_u^2\mathscr{H}\left(\beta,\frac{1}{2\pi}+u\right)\xi\zeta = {} & \beta^2\mathscr{H}\left(\beta,\frac{1}{2\pi}+u\right)\left[\mathscr{F}(\zeta)-\frac{\int_0^{2\pi}\Phi(\beta\mathscr{F}(u))\mathscr{F}(\zeta)}{\tilde{\Phi}(\beta\mathscr{F}(u))}\right] \\ & \times\left[\mathscr{F}(\xi)-\frac{\int_0^{2\pi}\Phi(\beta\mathscr{F}(u))\mathscr{F}(\xi)}{\tilde{\Phi}(\beta\mathscr{F}(u))}\right] \\ & -\beta^2\mathscr{H}\left(\beta,\frac{1}{2\pi}+u\right)\left\{\frac{\int_0^{2\pi}\Phi(\beta\mathscr{F}(u))\mathscr{F}(\zeta)\mathscr{F}(\xi)}{\tilde{\Phi}(\beta\mathscr{F}(u))}\right. \\ & \left.-\frac{(\int_0^{2\pi}\Phi(\beta\mathscr{F}(u))\mathscr{F}(\xi))(\int_0^{2\pi}\Phi(\beta\mathscr{F}(u))\mathscr{F}(\zeta))}{(\tilde{\Phi}(\beta\mathscr{F}(u)))^2}\right\}. \qquad (77) \end{aligned}$$

These formulas will be used in the sequel with appropriate values of β, ξ, ζ and u.

Furthermore, one can see that $\mathscr{H}$ is phase-invariant, that is, introducing the phase-shift ρ_φ defined on X by

$$\rho_\varphi U(.)=U(.+\varphi),$$

straightforward computation yields

$$\rho_\varphi\mathscr{H}(\beta,.)=\mathscr{H}(\beta,\rho_\varphi(.)).$$

So, if U is a fixed point of $\mathscr{H}(\beta,.)$, then $\rho_\varphi U$ is also.

Changing now U into $(\sigma U)(\theta)=U(-\theta)$, we find that

$$\sigma\mathscr{H}(\beta,U)=\mathscr{H}(\beta,\sigma U).$$

So, $\mathscr{H}(\beta,.)$ sends even functions to even functions. This property will be used in the sequel, when dealing with the bifurcation issue.

3.3. Spectrum

As a result of Proposition 3.1, we know that $D_u\mathscr{H}(\beta,\frac{1}{2\pi})$ is a compact and self-adjoint operator, so the spectrum of $\mathscr{L}(\beta)$ is reduced to a real point spectrum. Thus, the eigenfunction problem reduces to looking for $\xi\in X-\{0\}$, and $\mu\in\mathbf{R}$ such that

$$\mathscr{L}(\beta)(\xi)=\mu\xi. \qquad (78)$$

Such a function is necessarily differentiable, and taking the derivative we obtain

$$\frac{\beta}{2\pi\bar{D}}\left[\int_\theta^{\theta+\pi}\xi(\sigma)\,d\sigma-\int_{\theta-\pi}^{\theta}\xi(\sigma)\,d\sigma\right]=\mu\xi'(\theta), \qquad (79)$$

that is

$$\frac{\beta}{2\pi\bar{D}} g_1(\xi)(\theta) = \mu\xi'(\theta). \tag{80}$$

It is clear that $\mu\xi \in X_0$, which allows the possibility that $\mu = 0$ or $\int_0^{2\pi} \xi(\sigma)\, d\sigma = 0$.
Therefore,

$$\text{if } \mu \in \sigma(\mathscr{L}(\beta)),\ \mu \neq 0,\ \text{then } \frac{\beta}{2\pi\bar{D}} g_1(\xi) = \mu\xi' \text{ and } \int_0^{2\pi} \xi(\theta)\, d\theta = 0.$$

Exploiting the fact that $\xi(\theta + 2\pi) = \xi(\theta)$ and $\int_0^{2\pi} \xi(\theta) d\theta = 0$, we obtain

$$\int_\theta^{\theta+\pi} \xi(\sigma)\, d\sigma = -\int_{\theta-\pi}^{\theta} \xi(\sigma)\, d\sigma. \tag{81}$$

Consequently, Eq. (80) becomes

$$\frac{2\pi\mu\bar{D}}{\beta} \xi'(\theta) = -2\int_{\theta-\pi}^{\theta} \xi(\sigma)\, d\sigma, \tag{82}$$

which yields, using (81), that

$$\frac{2\pi\mu\bar{D}}{\beta}[\xi'(\theta) + \xi'(\theta + \pi)] = 0.$$

Then

$$\xi(\theta) + \xi(\theta + \pi) = C,$$

where C is a constant.

Since $\int_0^{2\pi} \xi(\theta) d\theta = 0$, it follows that $C = 0$, thus

$$\xi(\theta) = -\xi(\theta + \pi). \tag{83}$$

On the other hand, differentiating (82) and using (83), we deduce

$$\frac{2\pi\mu\bar{D}}{\beta} \xi''(\theta) = -4\xi(\theta). \tag{84}$$

The solutions of Eq. (84) can be written as

$$\xi(\theta) = Ae^{\omega\theta} + Be^{-\omega\theta}, \tag{85}$$

where A and B are constants and $\omega^2 = \frac{-2\beta}{\mu\pi\bar{D}}$. $\int_0^{2\pi} \xi(\theta) d\theta = 0$ yields that

$$\frac{A}{\omega}(e^{2\omega\pi} - 1) - \frac{B}{\omega}(e^{-2\omega\pi} - 1) = 0.$$

Using the fact that ξ is 2π-periodic, we get via (85)

$$A(e^{2\omega\pi}-1)e^{\omega\theta}+B(e^{-2\omega\pi}-1)e^{-\omega\theta}=0, \quad \text{for all } \theta, \text{ with } \omega\neq 0.$$

So $A(e^{2\omega\pi}-1)=0$ and $B(e^{-2\omega\pi}-1)=0$. Then

$$e^{2\omega\pi}=1.$$

Thus

$$2\omega\pi = i2k\pi, \quad k\geqslant 1,$$

then

$$\omega = ik, \quad k\geqslant 1.$$

Consequently, we arrive at

$$\mu_k=\frac{2\beta}{\bar{D}\pi k^2}.$$

Via (85), the eigenfunction corresponding to μ_k is

$$\xi_k(\theta)=Ae^{ik\theta}+Be^{-ik\theta},$$

that is

$$\xi_k(\theta)=p\cos k\theta+q\sin k\theta, \quad p,q\in\mathbf{R}.$$

Moreover,

$$\mathscr{L}(\beta)(\cos(k.))(\theta)=\frac{\beta}{\bar{D}\pi k^2}(1-(-1)^k)\cos(k\theta),$$

$$\mathscr{L}(\beta)(\sin(k.))(\theta)=\frac{\beta}{\bar{D}\pi k^2}(1-(-1)^k)\sin(k\theta),$$

which in particular leads to

$$\mathscr{L}(\beta)(\cos(2j.))=0,$$

$$\mathscr{L}(\beta)(\sin(2j.))=0.$$

Then the spectrum of $\mathscr{L}(\beta)$ is given by

$$\sigma(\mathscr{L}(\beta))=\left\{\mu_k, \mu_k=\frac{2\beta}{\bar{D}\pi(2k+1)^2}, \ k\geqslant 0\right\}\cup\{0\}, \tag{86}$$

and the eigenfunctions corresponding to μ_k are generated by $\{\cos(2k+1)\theta, \sin(2k+1)\theta\}$.

3.4. Stability and bifurcation

We will now apply the results gathered in Sections 3.2 and 3.3 to the study of stability and bifurcation of steady states of problem (59). Using an observation made at the end of Section 3.2, we are going to work in a space smaller than X_0, namely, the space, denoted X_0^{ev}, of even functions, defined as follows:

$$X_0^{\mathrm{ev}} = \{f \in X_0 \colon f \text{ is even}\}.$$

The restriction of $\mathscr{L}(\beta)$ to X_0^{ev}, denoted $\mathscr{L}_{\mathrm{ev}}(\beta)$, is still compact and self-adjoint. Then, in that case, the eigenspace for the eigenvalue μ_k is reduced to $\{\cos(2k+1)\theta\}$, and μ_k has algebraic multiplicity equal to 1 (86). We are now seeking nontrivial solutions, possibly emerging from $U = \bar{U}$ at some value of β where this solution becomes unstable. This goes through a bifurcation analysis starting from the study of the linearization of Eq. (59) near $U = \bar{U}$. Let us first recall the following definitions and a classical stability condition:

Definition 3.1 (Iooss [6]). For a linear bounded operator L, the spectral radius of L, denoted by $r(L)$, is the supremum of $\{|\lambda| \colon \lambda \in \sigma(L)\}$.

(b) The fixed point 0 of a map $K : X \to X$ is (Lyapunov) stable iff for every neighborhood $\mathscr{U}$ of 0, there exists another neighborhood $\mathscr{V} \subset \mathscr{U}$ of 0 such that $K^n(\mathscr{V}) \subset \mathscr{U}$, $\forall n \geqslant 0$.

(c) The fixed point 0 is exponentially stable iff there exists a neighborhood $\mathscr{V}$ of 0, $\gamma > 0$ and $k \in (0,1)$ such that 0 is Lyapunov stable and $\forall x \in \mathscr{V}$, $\|K^n(x)\|_X \leqslant \gamma k^n$, $n \to \infty$.

Lemma 3.3 (Iooss [6]). *Let $K : X \to X$ be differentiable at 0 and satisfy $K(0) = 0$, and let $D(K)(0) = L$ be its Frechet derivative at 0. If the spectrum of L lies in a compact subset of the open unit disc, then 0 is exponentially stable.*

From the analysis made in Section 3.3, one can see that

$$r(\mathscr{L}_{\mathrm{ev}}(\beta)) = \frac{2\beta}{\pi\bar{D}}$$

corresponding to $k = 0$. Let $\beta_0 (= \frac{\pi\bar{D}}{2})$ denote the value of the parameter for which $r(\mathscr{L}_{\mathrm{ev}}(\beta)) = 1$. Then, according to Lemma 3.3, it follows that the equilibrium $\frac{1}{2\pi}$ is exponentially stable if $\beta < \beta_0$ and unstable if $\beta > \beta_0$. We are now in a position to conclude on bifurcation.

Theorem 3.1. *$(\beta_0, \frac{1}{2\pi})$ is a bifurcation point of steady-state solutions of Eq. (59), that is to say, in each neighborhood of $(\beta_0, \frac{1}{2\pi})$ there exists a pair (β, U), $U \neq \frac{1}{2\pi}$, such that $U = \mathscr{H}(\beta, U)$.*

Proof. Differentiability of $\mathscr{H}$ together with (61) and (65) allow to write $\mathscr{H}$, near $(\beta_0, \bar{U})$ as follows:

$$\mathscr{H}(\beta, U) = \frac{1}{2\pi} + \frac{\beta}{\beta_0}\mathscr{L}_{\text{ev}}(\beta_0)(U - \bar{U}) + \mathscr{G}(\beta, U),$$

in which $\lim_{(\beta,U)\to(\beta_0,\bar{U})} \frac{\mathscr{G}(\beta,U)}{\|U-\bar{U}\|} = 0$. $\mathscr{L}_{\text{ev}}(\beta_0)$ is compact and $r(\mathscr{L}_{\text{ev}}(\beta_0)) = 1$ is an eigenvalue with odd multiplicity (equal to one). Then, the conditions of a classical bifurcation theorem [1, Theorem 1.7] are fulfilled, thus a bifurcation of nontrivial steady states takes place at $(\beta_0, \frac{1}{2\pi})$. This completes the proof of the theorem. □

3.5. *Computation of the bifurcation branch, the expectation and the variance*

We will now give more detailed information about the bifurcation branch emanating from $(\beta_0, \frac{1}{2\pi})$ and we will compute the expectation and the variance on the local branch. We will first show that one can represent the branch in terms of a smooth function $\beta = \beta(s)$. We will then compute a few derivatives of $\beta(s)$ at $s = 0$. The computation requires some care, due to the fact that $\beta(s)$ is known implicitly only and depends heavily on the preparatory results collected in Section 3.2.

3.5.1. *Computation of the bifurcation branch*

As we have already noticed in Section 3.4, it will be sufficient to restrict ourselves to X_0^{ev}. Thanks to Proposition 3.1, $\mathscr{L}_{\text{ev}}(\beta)$ is compact and self-adjoint. $N(I - \mathscr{L}_{\text{ev}}(\beta_0))$ is one dimensional generated by $v = \frac{\cos(.)}{\sqrt{\pi}}$, and so, compactness and self-adjointness imply that, if we denote $S = R(I - \mathscr{L}_{\text{ev}}(\beta_0))$, S is closed and is a supplementary subspace of $\mathbf{R}v$ in X_0^{ev}, invariant through $\mathscr{L}_{\text{ev}}(\beta_0)$, and $I - \mathscr{L}_{\text{ev}}(\beta_0)$ is an isomorphism from S onto itself. Finally, S is just characterized as the orthogonal space of $\mathbf{R}v$, $S = \{v\}^{\perp}$.

By writing $U = \frac{1}{2\pi} + u$ and $u = sv + \sigma$, with $\sigma \in S$, the equation $U = \mathscr{H}(\beta, U)$ breaks down into two equations:

$$\begin{cases} s = \langle v, \mathscr{H}(\beta, \frac{1}{2\pi} + sv + \sigma) - \frac{1}{2\pi} \rangle, \\ \sigma = \mathscr{H}(\beta, \frac{1}{2\pi} + sv + \sigma) - \frac{1}{2\pi} - \langle v, \mathscr{H}(\beta, \frac{1}{2\pi} + sv + \sigma) - \frac{1}{2\pi} \rangle v. \end{cases} \tag{87}$$

Applying the implicit function theorem to the second equation of (87), we can solve it for σ near $(\beta_0, 0)$: it yields $\sigma = \sigma(\beta, s)$, $\sigma(\beta, 0) = 0$. Differentiating the second equation of (87) with respect to s, we obtain

$$\begin{aligned} \frac{\partial \sigma}{\partial s}_{/s=0} &= D_u\mathscr{H}\left(\beta, \frac{1}{2\pi}\right)\left(v + \frac{\partial \sigma}{\partial s}_{/s=0}\right) - \left\langle v, D_u\mathscr{H}\left(\beta, \frac{1}{2\pi}\right)\left(v + \frac{\partial \sigma}{\partial s}_{/s=0}\right)\right\rangle v \\ &= D_u\mathscr{H}\left(\beta, \frac{1}{2\pi}\right)\frac{\partial \sigma}{\partial s}_{/s=0}. \end{aligned}$$

For $\frac{\beta}{\beta_0}$ close to 1, $I - D_u\mathscr{H}(\beta, \frac{1}{2\pi})$ is an isomorphism from S onto itself. Then $\frac{\partial \sigma}{\partial s}_{/s=0} = 0$. Therefore $\sigma(\beta, s) = O(s^2)$. Now, let us insert $\sigma = \sigma(\beta, s)$ in the first equation of (87), to arrive at the *bifurcation equation*

$$s = \left\langle v, \mathscr{H}\left(\beta, \frac{1}{2\pi} + sv + \sigma(\beta, s)\right) - \frac{1}{2\pi}\right\rangle.$$

We first show that one can represent the branch in terms of a function $\beta = \beta(s)$. To see this, consider the function

$$\mathscr{B}(s, \beta) = \frac{\langle v, \mathscr{H}(\beta, \frac{1}{2\pi} + sv + \sigma(\beta, s)) - \frac{1}{2\pi}\rangle}{s}, \quad s \neq 0,$$

which extends continuously to $s = 0$ as $\mathscr{B}(0, \beta) = \frac{\beta}{\beta_0}$. For $s \neq 0$, the *bifurcation equation* reads as

$$\mathscr{B}(s, \beta) = 1.$$

We have $\mathscr{B}(0, \beta_0) = 1$ and $\frac{\partial}{\partial \beta}\mathscr{B}(0, \beta_0) = \frac{1}{\beta_0} \neq 0$ and $\mathscr{B}$ is of class C^∞ (see Remark 2(i)). So, the implicit function theorem applies and insures that near $(0, \beta_0)$, the solutions (s, β) of the equation $\mathscr{B}(s, \beta) = 1$ lie in the graph of a function $\beta = \bar{\beta}(s)$ such that $\bar{\beta}(0) = \beta_0$ and $\bar{\beta}(s)$ is of class C^∞. Accordingly, we denote $\bar{\sigma}(s) = \sigma(\bar{\beta}(s), s)$. Our final goal is to compute a few derivatives of the function $\bar{\beta}(s)$. In terms of $\bar{\sigma}$ and $\bar{\beta}$, the *bifurcation equation* can be written as

$$s = \left\langle v, \mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right) - \frac{1}{2\pi}\right\rangle. \tag{88}$$

Differentiating the bifurcation equation (88) two times with respect to s, we get

$$\begin{aligned} 0 = &\left\langle v, D_u^2\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right)(v + \bar{\sigma}'(s))^2\right\rangle \\ &+ \left\langle v, D_u\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right)\bar{\sigma}''(s)\right\rangle \\ &+ \left\langle v, \frac{\partial^2}{\partial \beta^2}\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right)\left(\frac{\partial \bar{\beta}}{\partial s}\right)^2\right\rangle \\ &+ \left\langle v, \frac{\partial}{\partial \beta}\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right)\left(\frac{\partial^2 \bar{\beta}}{\partial s^2}\right)\right\rangle \\ &+ 2\left\langle v, D_{u\beta}^2\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi} + sv + \bar{\sigma}(s)\right)\left(\frac{\partial \bar{\beta}}{\partial s}\right)(v + \bar{\sigma}'(s)\right\rangle. \end{aligned} \tag{89}$$

Using (73)–(75), combined with the fact that $\bar{\sigma}(.)$ and its derivatives belong to S the supplementary subspace of $\mathbf{R}v$ in X_0^{ev}, invariant through $D_u\mathscr{H}(\beta_0, \frac{1}{2\pi})$, formula (89) comes down to

$$0 = \left\langle v, D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^2\right\rangle + \frac{2}{\beta_0}\frac{\partial\bar{\beta}}{\partial s}_{/s=0}. \tag{90}$$

On the other hand, using (77), we obtain

$$D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^2 = 2\cos^2(\theta) - 1,$$

which implies that

$$\left\langle v, D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^2\right\rangle = 0. \tag{91}$$

Together with (90), the above equality leads to $\frac{\partial\bar{\beta}(0)}{\partial s} = 0$. Then,

$$\bar{\beta}(s) = \beta_0 + O(s^2).$$

Using the Taylor expansion, up to order three, of $\mathscr{H}(\beta, \frac{1}{2\pi} + u)$ in u in the right-hand side of the *bifurcation equation*, substituting $sv + \sigma$ for u and using the fact that $\sigma(\beta, s) = O(s^2)$, then substituting $\bar{\beta}(s)$ for β, we arrive at

$$s = s\frac{\bar{\beta}(s)}{\beta_0} + \left\langle v, D_u^2\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi}\right)sv\bar{\sigma}(s)\right\rangle + \frac{s^3}{6}\left\langle v, D_u^3\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi}\right)v^3\right\rangle + o(s^3),$$

which can be rewritten as

$$s\frac{(\beta_0 - \bar{\beta}(s))}{\beta_0} = \left\langle v, D_u^2\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi}\right)sv\bar{\sigma}(s)\right\rangle + \frac{s^3}{6}\left\langle v, D_u^3\mathscr{H}\left(\bar{\beta}(s), \frac{1}{2\pi}\right)v^3\right\rangle + o(s^3). \tag{92}$$

Dividing both sides of (92) by s^3 and evaluating the limit at $s = 0$, we obtain

$$-\frac{\bar{\beta}''(0)}{2\beta_0} = \left\langle v, D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v\frac{\bar{\sigma}''(0)}{2}\right\rangle + \frac{1}{6}\left\langle v, D_u^3\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^3\right\rangle. \tag{93}$$

Let us now turn to the estimation of the first term of the right-hand side of (93). Straightforward computation gives $\bar{\sigma}''(0) = \frac{\partial^2\sigma}{\partial s^2}(\beta_0, 0)$. Differentiating the second equation of (87) with respect to s, we obtain

$$\begin{aligned}\frac{\partial^2\sigma}{\partial s^2}(\beta_0, 0) =& D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^2 + D_u\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)\frac{\partial^2\sigma}{\partial s^2}(\beta_0, 0)\\ &- \left\langle v, D_u^2\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^2\right\rangle v - \left\langle v, D_u\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)\frac{\partial^2\sigma}{\partial s^2}(\beta_0, 0)\right\rangle v.\end{aligned}$$

Via (91), combined with the fact that $\sigma(.)$ and its derivatives belong to S the supplementary subspace of $\mathbf{R}v$ in X_0^{ev}, invariant through $D_u\mathscr{H}(\beta_0, \frac{1}{2\pi})$, we arrive at

$$\frac{\partial^2\sigma}{\partial s^2}(\beta_0,0) = D_u^2\mathscr{H}\left(\beta_0,\frac{1}{2\pi}\right)v^2 + D_u\mathscr{H}\left(\beta_0,\frac{1}{2\pi}\right)\frac{\partial^2\sigma}{\partial s^2}(\beta_0,0). \tag{94}$$

Denote, for a moment, ξ the function $\frac{\partial^2\sigma}{\partial s^2}(\beta_0,0)$ and differentiate both sides of Eq. (94) with respect to θ to obtain, via (77), (61) and (54)

$$\xi'(\theta) = -2\sin(2\theta) + \tfrac{1}{4}g_1(\xi)(\theta). \tag{95}$$

Using the fact that $g_1(P) = 0$, for each polynomial or series P in $\sin(2\theta n)$ and $\cos(2n\theta)$ (see formula (5)), one can see that $\xi(\theta) = \cos(2\theta)$ is a particular solution of (95). So, the general solution of Eq. (95) is

$$\xi(\theta) = \cos(2\theta) + \xi_c(\theta), \tag{96}$$

where $\xi_c(\theta)$ denotes an arbitrary solution of the homogenous equation associated with (95). One can immediately see that this equation is analogous to the eigenfunction problem (78) for $\beta = \beta_0$. Using (78), (80) and (86), we arrive at the following expression:

$$\xi_c(\theta) = A\cos(\theta) + B\sin(\theta).$$

The additional restriction to even functions yields $B = 0$, and the requirement that ξ be in S leads to $A = 0$. So, we get

$$\frac{\partial^2\sigma}{\partial s^2}(\beta_0,0) = \cos(2\theta),$$

from which, using formula (77) with the fact that $g_1(P) = 0$ therefore $\mathscr{F}(P) = 0$, for each polynomial or series P in $\sin(2\theta n)$ and $\cos(2n\theta)$, we obtain

$$\left\langle v, D_u^2\mathscr{H}\left(\beta_0,\frac{1}{2\pi}\right)v\frac{1}{2}\frac{\partial^2\sigma}{\partial s^2}(\beta_0,0)\right\rangle = 0.$$

Therefore

$$-\frac{\bar{\beta}''(0)}{2\beta_0} = \frac{1}{6}\left\langle v, D_u^3\mathscr{H}\left(\beta_0,\frac{1}{2\pi}\right)v^3\right\rangle.$$

Using (77) and (76) again, we get

$$D_u^3\mathscr{H}\left(\beta_0,\frac{1}{2\pi}\right)v^3 = 4\pi^2v^3 - 6\pi v,$$

which yields that

$$\left\langle v, D_u^3\mathscr{H}\left(\beta_0, \frac{1}{2\pi}\right)v^3\right\rangle = -3\pi.$$

Then,

$$\bar{\beta}(s) = \beta_0 + \frac{\pi}{2}\beta_0 s^2 + o(s^2). \tag{97}$$

The result reads, in terms of $\lambda = \bar{\lambda}(s) = \frac{\bar{\beta}(s)}{D}$, as

$$\bar{\lambda}(s) = \frac{\pi}{2} + \frac{\pi^2}{4}s^2 + o(s^2). \tag{98}$$

Consequently the branch of nontrivial solutions emanating from $(\frac{\pi}{2}, \frac{1}{2\pi})$ is supercritical (i.e., takes place for $\lambda > \frac{\pi}{2}$).

3.5.2. Expectation and variance

In this part, we will compute the expectation and the variance on the local branch. Let $U_s(.)$ be the bifurcated solution of (59) for $s > 0$ small. The expected value of U_s, E_s is by definition

$$\begin{aligned} E_s &= \int_{-\pi}^{+\pi} \theta U_s(\theta)\, d\theta \\ &= \int_{-\pi}^{+\pi} \theta\left(\frac{1}{2\pi} + s\cos\theta + o(s)\right) d\theta. \end{aligned}$$

Then

$$E_s = o(s).$$

The variance V_s is the integral

$$\begin{aligned} V_s &= \int_{-\pi}^{+\pi} (\theta - E_s)^2 U_s(\theta)\, d\theta \\ &= \int_{-\pi}^{+\pi} (\theta - o(s))^2\left(\frac{1}{2\pi} + s\cos\theta + o(s)\right) d\theta \\ &= \frac{\pi^2}{3} - 4\pi s + o(s). \end{aligned}$$

The expectation and the variance can be computed in terms of λ. In view of (98), one then obtains the following approximate expressions:

$$E(\lambda) = o(\sqrt{\lambda - \lambda_0}); \quad V(\lambda) = \frac{\pi^2}{3} - 4\pi\sqrt{\frac{(\lambda - \lambda_0)}{\lambda_0^2}}. \tag{99}$$

On the other hand, the computation of the same quantities on the trivial branch leads, respectively, to

$$E = 0 \quad \text{and} \quad V = \frac{\pi^2}{3},$$

that is a slight deviation from uniform distribution can be seen in the angular mean value becoming possibly nonzero, while the angular dispersion is going below the value $\frac{\pi}{\sqrt{3}}$, giving a rough estimate of the gain incurred by the fish when playing the gregarious strategy.

So far we have only considered local results, i.e., existence of solutions in a small neighborhood of a bifurcation point. However, it is possible to show existence of an extended branch of nontrivial solutions, as a result of a global bifurcation theorem (see [8, Theorem 1.6]). In fact, in view of $U \geqslant 0$ and $\int_0^{2\pi} U(\theta)\, d\theta \doteq 1$, it may be easily seen that $\mathscr{H}(\beta, U)$ is uniformly bounded on bounded β intervals; on the other hand, $\mathscr{H}(0, U) = \frac{1}{2\pi}$ (immediate from (58)), that is, the only feasible fixed point of $\mathscr{H}(0, U)$ is $\bar{U}$. So, Theorem 1.6 in [8] allows us to conclude that, if $\mathscr{C}$ denotes the connected component of the set of nontrivial fixed point having $(\beta_0, \bar{U})$ in its closure, then $\mathscr{C}$ is unbounded in β: for each $\beta > \beta_0$, one can show that $\mathscr{H}(\beta, U)$ has a fixed point $U \neq \bar{U}$.

4. Concluding remarks

Let us first consider the results of the above two sections together. In Section 2, we proved that the Cauchy problem associated with the system of equations (1)–(3) is well posed in a suitable function space. Indeed, we have discussed existence, uniqueness and positivity of solutions. We have also seen that the value of the integral of the solution is a constant, so that the solution remains a proportion density for all positive time. Section 3 focused on the study of the stability of the trivial solution as a function of the parameter β. Assuming that $D(\theta) = \bar{D}$ and $F(\theta) = 1$, the trivial steady state is the constant function $\bar{U} = \frac{1}{2\pi}$, that is, an individual's orientation is uniformly distributed in all directions. It was shown that stability is lost as the parameter β crosses a certain threshold β_0 and a set of nontrivial steady states emerges near this value. In fact, below that value β_0, the dispersion dominates and the population organizes itself asymptotically as if there were no gregarism. We have shown that the branch of nontrivial steady-state solutions emerging from that trivial solution is supercritical. Close to $\beta = \beta_0$, the bifurcated solutions read as: $U(\theta) = \frac{1}{2\pi} + C(\beta)\cos\theta + o(C(\beta))$ with a constant $C(\beta) > 0$ for $\beta > \beta_0$.

As a final remark, we comment on the interpretation to be given to the solutions in the context of the physical environment of the fish. The angle θ characterizes the angular orientation of an individual fish body, lying supposedly in a horizontal plane, with respect to a fixed direction, arbitrarily chosen. This has nothing to do with the actual location of the fish in the water volume. In what concerns the steady-state solutions, a privileged direction has emerged from the preparatory study, namely a symmetry axis has been found which has been used as the origin for the orientation of angles. But, this axis is by no means connected to a specific horizontal direction in the sea. Not only could this symmetry axis be supported by any one of the horizontal directions but there is also no relationship between a given orientation and the actual location of fish having this orientation. Introducing some rules linking spatial locations and tail-to-head orientations of fish will be the subject of future research.

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