MATH 334, MIDTERM EXAM II, FALL 2020

Name	BYU ID

- This is a closed-book exam. Calculators are not allowed. You have 4 hours to work on the exam.
- For Problems 15-18, you must show valid arguments with all necessary steps. Mysterious answers will receive little or no credit.

Problem	Possible points	Earned points
1 - 13	26	
14	5	
15	13	
16	13	
17	13	
18	13	
Total	83	

$f(t) = \mathcal{L}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n e^{at}$, n a positive integer	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t) = \begin{cases} 0 & \text{if} \quad t < c, \\ 1 & \text{if} \quad t \geq c \end{cases}$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
\ln(1+x) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \\
\sin x &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} \\
\cos x &= \sum_{k=0}^{\infty} (-1)^{k} \frac{x^{2k}}{(2k)!} \\
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k
\end{aligned}
$$

Problem 1. (2 points) I acknowledge that I will not use notes, books, calculators, internet sources, or any assistance from other individuals as I complete this exam. I will complete the exam in one sitting. I will not discuss the exam with any other class members until after the exam period is completed.

a. True
b. False

Problem 2. (2 points) The general solution to the differential equation $y'' - 5y' + 4y = e^t$ is of the form

a. $c_1e^{-t} + c_2e^{4t} + Ae^t$

b. $c_1e^t + c_2e^{4t} + Ae^t$

c. $c_1e^t + c_2e^{4t} + Ate^t$

d. $c_1e^{-t} + c_2e^{4t} + Ate^t$

Problem 3. (2 points) Suppose that a linear, homogeneous ODE with constant coefficients has the characteristic polynomial $(r^2 + 1)(r + 2)^2$. What is the order of the ODE?

a. 2
b. 3
c. 4
d. 5

Problem 4. (2 points) What is the general solution to the ODE in the previous problem?

a. $c_1e^{-2t} + c_2 \cos t + c_3 \sin t$

b. $c_1e^{-2t} + c_2te^{-2t} + c_3 \cos t + c_4 \sin t$

c. $c_1e^{-2t} + c_2e^t \cos t + c_3e^t \sin t$

d. $c_1e^{-2t} + c_2te^{-2t} + c_3e^t \cos t + c_4e^t \sin t$

Problem 5. (2 points) The Laplace transform of the function $5e^{-2t} - 3\sin(4t)$ is equal to

a. $\frac{5}{s+2} - \frac{12}{s^2+16}$

b. $\frac{5}{s+2} - \frac{3}{s^2+16}$

c. $\frac{5}{s-2} - \frac{3}{s^2+16}$

d. $\frac{5}{s-2} - \frac{12}{s^2+16}$

Problem 6. (2 points) The inverse Laplace transform of the function $\frac{3}{s^2+4}$ is equal to

a. $\sin(2t)$

b. $\sin\left(\frac{3t}{2}\right)$

c. $\frac{2}{3}\sin(2t)$

d. $\frac{3}{2}\sin(2t)$

Problem 7. (2 points) The ODE

$$(x^2 - 1)y''' + (\ln x)y'' + e^{-x}y = 1$$

is sure to have a solution on the interval

a. $(-1, 1)$

b. $(-1, 0)$

c. $(0, 1)$

d. $[0, 1]$

Problem 8. (2 points) The Wronskian of the functions e^t, 1, $\cos t$ (in that order) is equal to

a. 0

b. $e^t(\sin t + \cos t)$

c. $e^t(\sin t - \cos t)$

d. $e^t(\cos t - \sin t)$

Problem 9. (2 points) The initial value problem

$$\begin{cases} (1-x)\,y'' + y & = & x \\ y(0) & = & 1 \\ y'(0) & = & 1 \end{cases}$$

has a power series solution $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ The first four coefficients a_0, a_1, a_2, a_3 are

a. 1, 1, $-1/2$, $-1/6$

b. 1, -1, -1, 1

c. 1, 1, $-1/2$, $1/3$

d. 1, 1, $1/2$, $1/3$

Problem 10. (2 points) The radius of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2+1}$$

is equal to

a. 0
b. 1
c. 2
d. 3

Problem 11. (2 points) The function $\frac{5}{2-3x^2}$ can be written as a power series centered at 0. Which of the following is the correct power series?

a. $\frac{5}{2} \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k x^{2k}$

b. $\frac{5}{2} \sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^k x^{2k}$

c. $\frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{5}{2}\right)^k x^{2k}$

d. $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k x^{2k}$

Problem 12. (2 points) A mass m is hanged on a spring with spring coefficient $k > 0$. Suppose the damping coefficient is $\gamma > 0$. Denote by $y(t)$ the displacement of the mass from its equilibrium position, measured positive in the downward direction, at time t. Then y satisfies the equation

a. $my'' - \gamma y' + ky = mg$

b. $my'' + \gamma y' + ky = 0$

c. $my'' + \gamma y' - ky = mg$

d. $my'' + \gamma y' - ky = 0$

Problem 13. (2 points) In case $m = 1$, $\gamma = 3$, $k = 2$, the motion described in the previous problem is

a. Overdamped
b. Underdamped
c. Critically damped
d. Undamped

Problem 14. (5 points) Choose ALL the improper integrals that converge.

a. $\int_1^\infty \frac{t+1}{t^2+1} dt$

b. $\int_1^\infty e^{-t^2} dt$

c. $\int_1^\infty \frac{\ln t}{t} dt$

d. $\int_0^1 \frac{\sin t}{\sqrt{t}} dt$

e. $\int_0^7 \frac{1}{\sqrt{7-t}} dt$

Problem 15. (13 points) Solve the initial value problem

$$y'' + 2y' = 3 + 4\sin(2t),\ y(0) = 1,\ y'(0) = 1$$

$$y = y_c + y_p$$

Solve the homogeneous ODE for y_c: the characteristic eq. is

$$r^2 + 2r = 0$$

which gives $r = 0$ and $r = -2$. Thus,

$$y_c = c_1 e^{0t} + c_2 e^{-2t} = c_1 + c_2 e^{-2t}.$$

To find a particular solution, we split the equation into two to find a particular solution of each:

$$\left.\begin{array}{ll} y'' + 2y' = 3 & (1) \longrightarrow y_{p1} \\ y'' + 2y' = 4\sin 2t & (2) \longrightarrow y_{p2} \end{array}\right\} y_p = y_{p1} + y_{p2}$$

We look for a particular sol. to (1) of the form

$$y = At$$

Substitute this function into (1): $\quad 2A = 3 \longrightarrow A = \frac{3}{2}.$

Thus, $\quad y_{p1} = \frac{3}{2}t$

We look for a particular sol. to (2) of the form

$$y = A\cos 2t + B\sin 2t$$

$$y' = -2A\sin 2t + 2B\cos 2t$$

$$y'' = -4A\cos 2t - 4B\sin 2t$$

$$y'' + 2y' = (-4A + 4B)\cos 2t + (-4B - 4A)\sin 2t$$

We want

$$\begin{cases} -4A + 4B = 0 \\ -4B - 4A = 4 \end{cases} \rightsquigarrow \text{get } A = B = -\frac{1}{2}$$

Thus, $y_{p2} = -\frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$

We get $y_p = y_{p1} + y_{p2} = \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$

The general sol. to the ODE is

$$y = y_c + y_p = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

$$y' = -2c_2 e^{-2t} + \frac{3}{2} + \sin 2t - \cos 2t$$

$$y(0) = 1 \rightsquigarrow c_1 + c_2 - \frac{1}{2} = 1 \rightarrow c_1 + c_2 = \frac{3}{2}$$

$$y'(0) = 1 \rightsquigarrow -2c_2 + \frac{3}{2} - 1 = 1 \rightsquigarrow c_2 = -\frac{1}{4}$$

$$\rightsquigarrow c_1 = \frac{3}{2} - c_2 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

Therefore, $y = \frac{7}{4} - \frac{1}{4}e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$

Problem 16. (13 points) Find the general solution to the ODE
$$y'' + 4y' + 4y = t^{-2}e^{-2t}.$$

Characteristic eq: $r^2 + 4r + 4 = 0$

Double root $r = -2$

The complementary is $y_c = c_1 \underbrace{e^{-2t}}_{y_1} + c_2 \underbrace{te^{-2t}}_{y_2}$

To find a particular solution, we use the method of variation of parameters:

$$y_p = c_1(t)\, y_1 + c_2(t)\, y_2$$

The system to solve for c_1 and c_2 is

$$\begin{cases} c_1' y_1 + c_2' y_2 = 0 \\ c_1' y_1' + c_2' y_2' = t^{-2}e^{-2t} \end{cases}$$

$$c_1' = \frac{\begin{vmatrix} 0 & y_2 \\ t^{-2}e^{-2t} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-t^{-2}e^{-2t}y_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$c_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & t^{-2}e^{-2t} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 t^{-2}e^{-2t}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-2t} e^{-2t} \begin{vmatrix} 1 & t \\ -2 & 1-2t \end{vmatrix}$$

$$= e^{-4t}(1-2t+2t) = e^{-4t}$$

Thus, $$c_1' = \frac{-t^{-2} e^{-2t} y_2}{e^{-4t}} = \frac{-t^{-2} e^{-2t} te^{-2t}}{e^{-4t}} = -\frac{1}{t}$$

$$c_2' = \frac{t^{-2} e^{-2t} y_1}{e^{-4t}} = \frac{t^{-2} e^{-2t} e^{-2t}}{e^{-4t}} = \frac{1}{t^2}$$

We get $$c_1 = -\ln t, \quad c_2 = -\frac{1}{t}$$

Then $$y_p = c_1 y_1 + c_2 y_2 = -(\ln t) e^{-2t} - e^{-2t}$$

Therefore, the general sol. to the ODE is

$$y = y_c + y_p = c_1 e^{-2t} + c_2 te^{-2t} - (\ln t) e^{-2t} - e^{-2t}$$

Problem 17. (13 points) Solve the initial value problem

$$y''' + y' = 1, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 2.$$

Characteristic eq. $\underbrace{r^3 + r}_{r(r^2+1)} = 0$

It has three roots $r = 0$, $r = \pm i$.

The complementary sol. is $y_c = c_1 e^{0t} + c_2 \cos t + c_3 \sin t$

$= c_1 + c_2 \cos t + c_3 \sin t$

We guess a particular sol. of the form

$y_p = At$

Plug into the eq: $A = 1$

Therefore, the ODE has a general sol

$y = c_1 + c_2 \cos t + c_3 \sin t + t$

Apply initial conditions to find c_1, c_2, c_3.

$y(0) = 0 \rightsquigarrow c_1 + c_2 = 0$

$y' = -c_2 \sin t + c_3 \cos t + 1$

$y'(0) = 1 \rightsquigarrow c_3 + 1 = 1 \rightsquigarrow c_3 = 0$

$y'' = -c_2 \cos t - c_3 \sin t$

$y''(0) = 2 \rightsquigarrow -c_2 = 2 \rightsquigarrow c_2 = -2$

Then $c_1 = -c_2 = 2$.

Therefore, $y = 2 - 2\cos t + t$

Problem 18. (13 points) Solve the initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = y'(0) = 0,$$

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 < t \le 1, \\ t^2-1 & \text{if} \quad t > 1. \end{cases}$$

Use Laplace transform:

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = sY - y(0) = sY$$

$$\mathcal{L}\{y''\} = s^2 Y - sy(0) - y'(0) = s^2 Y$$

$$f(t) = u_1(t)(t^2-1)$$

$$\mathcal{L}\{f\} = \mathcal{L}\{u_1(t)\underbrace{(t^2-1)}_{g(t-1)}\} = e^{-s}\mathcal{L}\{g\}(s)$$

Put $x = t-1$. Then $g(x) = t^2 - 1 = (x+1)^2 - 1 = x^2 + 2x$

Then $\mathcal{L}\{g\}(s) = \mathcal{L}\{x^2\} + 2\mathcal{L}\{x\} = \frac{2}{s^3} + \frac{2}{s^2} = \frac{2+2s}{s^3}$

Thus,

$$(s^2+3s+2)Y = e^{-s}\frac{2+2s}{s^3}$$

$$\longrightarrow \quad Y = \frac{e^{-s}(2+2s)}{s^3(s^2+3s+2)} = \frac{e^{-s}2(\cancel{1+s})}{s^3(\cancel{s+1})(s+2)} = e^{-s}\frac{2}{s^3(s+2)}$$

Partial decomposition:

$$\frac{2}{s^3(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} \qquad (*)$$

Multiply by $s+2$:

$$\frac{2}{s^3} = \left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3}\right)(s+2) + D$$

Plug $s=-2$: $\quad \frac{2}{(-2)^3} = D \rightsquigarrow D = -\frac{1}{4}.$

Multiply $(*)$ by s^3:

$$\frac{2}{s+2} = As^2 + Bs + C + \frac{Ds^3}{s+2} \qquad (**)$$

Plug $s=0$: $\quad C=1$

Plug $s=2$: $\quad \underbrace{\frac{2}{4}}_{1/2} = 4A + 2B + \underbrace{C}_{1} + \underbrace{\frac{D(8)}{4}}_{-1/2}$

Then $4A+2B=0 \rightsquigarrow 2A+B=0.$

Plug $s=1$ in $(**)$:

$$\frac{2}{3} = A+B+\underbrace{C}_{1}+\underbrace{\frac{D}{3}}_{-1/12} \rightsquigarrow A+B = -\frac{1}{4}$$

$A = \frac{1}{4}, \quad B = -\frac{1}{2}$

Thus, $\displaystyle \frac{2}{s^3(s+2)} = \frac{1/4}{s} + \frac{-1/2}{s^2} + \frac{1}{s^3} + \frac{-1/4}{s+2}$

Then

$$Y = e^{-s}\frac{2}{s^3(s+2)} = \frac{1}{4}\,\frac{e^{-s}}{s} - \frac{1}{2}\,\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s^3} - \frac{1}{4}\,\frac{e^{-s}}{s+2}$$

We get

$$y = \mathcal{L}^{-1}\{Y\} = \frac{1}{4}\mathcal{L}\left\{\frac{e^{-s}}{s}\right\} - \frac{1}{2}\mathcal{L}\left\{\frac{e^{-s}}{s^2}\right\} + \mathcal{L}\left\{\frac{e^{-s}}{s^3}\right\} - \frac{1}{4}\mathcal{L}\left\{\frac{e^{-s}}{s+2}\right\}$$

$$= \frac{1}{4}u_1(t) - \frac{1}{2}u_1(t)(t-1) + u_1(t)\frac{(t-1)^2}{2} - \frac{1}{4}u_1(t)e^{-2(t-1)}$$