MATH 334, MIDTERM EXAM I, FALL 2021

INSTRUCTOR: TUAN PHAM

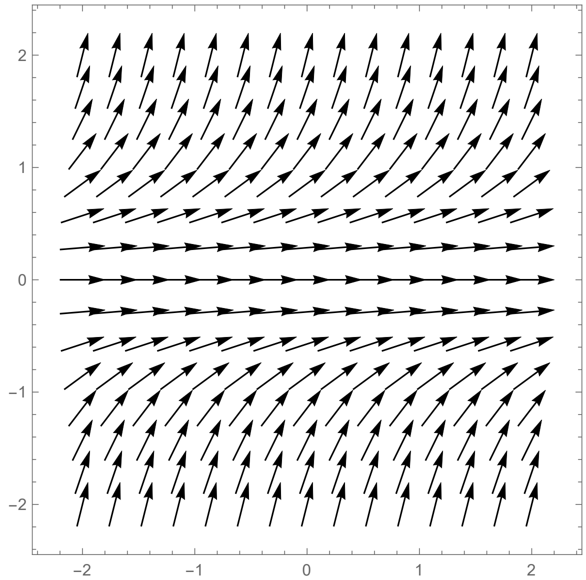
Name	Section # (Sec. 2: 11-12PM, Sec. 3: 12 - 1PM)

Instructions:

- This is a closed-book exam, 2 hours long. Non-graphing calculators are allowed.
- For Problems 13, 14, 15, you must show valid arguments with all necessary steps. Mysterious answers will receive little or no credit.
- Do not discuss the exam with anyone before the exam window (Oct 6-8) is closed.

Problem	Possible points	Earned points
1-12	24	
13	10	
14	10	
15	10	
Total	54	

Problem 1. (2 points) Which of the following ODE corresponds to the given direction field?



A. $y' = y^2$
B. $y' = ty^2$
C. $y' = t$
D. $y' = y/2$

Problem 2. (2 points) Choose the correct type for the ODE $y'y'' = y^3$.

A. Linear autonomous of second order
B. Nonlinear autonomous of third order
C. Nonlinear autonomous of second order
D. Nonlinear non-autonomous of second order

Problem 3. (2 points) Every solution to the ODE $y' + y = t$ approaches to which of the following curves as $t \to \infty$?

A. $y = t$
B. $y = t - 1$
C. $y = 0$
D. $y = e^{-t}$

Problem 4. (2 points) The initial value problem $ty'' + 3y' + 2y = e^{-t}$, $y(1) = 1$, $y'(1) = 2$ is guaranteed to have a solution on which interval? (Choose the largest possible.)

A. $(0, \infty)$
B. $(1, \infty)$
C. $(-\infty, 1)$
D. $(-\infty, \infty)$

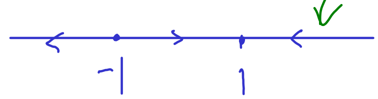
Problem 5. (2 points) The ODE $xy' + e^x y = 2$ is an exact differential equation.

A. True
B. False

$N = x \quad M = e^x y \qquad N_x = 1 \neq M_y = e^x$

Problem 6. (2 points) Let y be a solution to the initial value problem $y' = e^{-y}(1-y^2)$, $y(-2) = 2$. What is the limit of $y(t)$ as $t \to \infty$?

A. $-\infty$
B. ∞
C. -1
D. 1



initial condition falls on this region

Problem 7. (2 points) If an autonomous ODE $y' = f(y)$ has exactly two equilibrium states, then they cannot be both stable.

A. True
B. False

Problem 8. (2 points) For two functions u and v, the Wronskian determinant $W[u, v]$ is either zero for every t or nonzero for every t.

A. True
B. False

Problem 9. (2 points) Consider the ODE $y'' + y' - 2y = 0$. Which of the following statements is false?

A. The sum of two solutions is another solution.
B. The difference of two solutions is another solution.
C. The product of two solutions is another solution.
D. The derivative of a solution is another solution.

Problem 10. (2 points) According to Abel's theorem, the Wronskian determinant of two solutions of the ODE $ty'' + (t-1)y' + y = 0$ is equal to

A. Ce^{t-1}
B. $C(e^{-t} + t)$
C. Cte^{-t}
D. $Ce^{\frac{t^2}{2}-t}$

standard form : $y'' + \underbrace{\frac{t-1}{t}}_{p(t)} y' + \frac{1}{t} y = 0$

$W = C\exp\left(-\int p(t)dt\right) = C\exp\left(-\int (1-\frac{1}{t})dt\right) = C\exp(-t+\ln t)$
$= Cte^{-t}$.

Problem 11. (2 points) The function $y = te^{2t}$ solves which of the following ODE?

A. $y'' + 4y' + 4y = 0$
B. $y'' - 4y' + 4y = 0$
C. $y'' + 2y' = 0$
D. None of the above

2 is a double root of the char. eq.

Problem 12. (2 points) Let y be solution to the ODE $y'' + 2y' + 2y = 0$. What is the limit of $y(t)$ as $t \to \infty$?

A. Does not exist
B. Exists and depends on the initial conditions
C. ∞
D. 0

$y = e^{-t}(c_1 \cos t + c_2 \sin t) \longrightarrow 0$ as $t \to \infty$

Problem 13. (10 points) Solve the following initial value problem *using the exact differential method.*

$$y' = \frac{2x - y}{2y + x}, \quad y(1) = 1$$

$$(2y + x)y' = 2x - y$$

$$\rightsquigarrow \underbrace{(2y + x)}_{N}y' + \underbrace{y - 2x}_{M} = 0 \qquad (*)$$

$M_y = 1 = N_x \rightsquigarrow$ the eq $(*)$ is in exact diff. form

Find ϕ: $\phi_x = M = y - 2x \rightsquigarrow \phi = xy - x^2 + C(y)$

$$\phi_y = x + C'(y)$$
$$\|$$
$$N = x + 2y$$

$\rightsquigarrow C'(y) = 2y \rightsquigarrow C(y) = y^2 \rightsquigarrow \phi = xy - x^2 + y^2$

Thus, $(*)$ implies $xy - x^2 + y^2 = C$.

Initial condition: $x = y = 1 \rightsquigarrow C = 1$. Thus,

$$xy - x^2 + y^2 = 1$$

Optional: $y^2 + xy - x^2 - 1 = 0 \rightsquigarrow \Delta = x^2 - 4(-x^2 - 1) = 5x^2 + 4$

$$y = \frac{x \pm \sqrt{5x^2 + 4}}{2}$$

Choose the plus sign due to initial condition: $y = \dfrac{x + \sqrt{5x^2 + 4}}{2}$.

Problem 14. (10 points) Solve the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = y'(0) = 1.$$

Char. eq. $r^2+2r+1=0 \rightsquigarrow r=-1$ is a double root

$$y = e^{-t}(at+b)$$

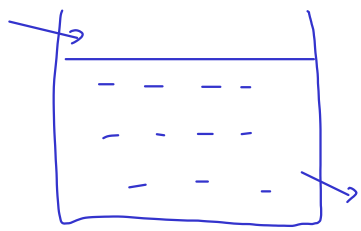
$$y(0) = e^{0}(a0+b) = b \rightsquigarrow b=1$$

$$y' = e^{-t}a - e^{-t}(at+b) = e^{-t}(a-b-at)$$

$$y'(0) = e^{0}(a-b-a0) = a-b = a-1 \rightsquigarrow a=2$$

Conclusion: $\boxed{y = e^{-t}(2t+1)}$

Problem 15. (10 points) A container initially contains 100 gallons of seawater with salt concentration 0.25 lbs/gal. Fresh water is pumped into the container at the rate of 2 gal/min. At the same time, the well-mixed liquid is pumped out from the container at the same rate. How long does it take for the salt concentration in the container to reach 0.1 lbs/gal? Write your answer in term of minutes and round it up to 2 decimal points.



Volume = 100 gallons (doesn't change in time)

$y = y(t)$: the amount of salt at time t.

$$y' = 2 \times 0 - 2 \times \frac{y}{100} = \frac{-y}{50}$$

$$\rightsquigarrow \quad y = C e^{-t/50}.$$

$\frac{y(0)}{100}$ = initial concentration = 0.25 $\rightsquigarrow y(0) = 25 \rightsquigarrow C = 25$.

For the concentration to be 0.1 lbs/gal, we need $\frac{y(t)}{100} = 0.1$

$$\rightsquigarrow \quad y(t) = 10.$$

Solve for t from $25 e^{-t/50} = 10$

$$\rightsquigarrow \quad e^{-t/50} = \frac{10}{25} = \frac{2}{5} \rightsquigarrow -\frac{t}{50} = \ln\left(\frac{2}{5}\right)$$

$$\rightsquigarrow \quad t = -50 \ln\left(\frac{2}{5}\right) \approx 45.81 \text{ (minutes)}$$