Midterm I: Some problems for review

The exam is two and a half hours long and taken at the Testing Center between May 19 and May 21. It is a closed book exam, covering Sections 12.1 - 14.6. No calculators or notes are allowed. You will be provided the following formula on the exam:

$$\kappa = \frac{|r' \times r''|}{|r'|^3}, \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}, \quad N = \frac{T'}{|T'|}, \quad a_T = \frac{r' \cdot r''}{|r'|}, \quad a_N = \frac{|r' \times r''|}{|r'|}.$$

You should review the homework problems, especially the odd problems (solution in the back of the textbook), and the examples given in class. Also, take a look at the practice exam posted on Learning Suite. It is always a good idea to study with someone. You should know how to solve following problems:

- Do basic operations on vectors: addition, scaling, dot product, cross product.
- Find equations of lines and planes.
- Parametrize a curve. Find length, curvature, torsion.
- Find velocity, speed, acceleration, tangential and normal components of acceleration.
- Find limit of a function.
- Find partial derivatives of a function.
- Write equation of a tangent plane to a surface at a given point.

In Problems 1-10, u, v, w are vectors in 3D. Determine whether the statement is true or false. Give reason for your answers.

1) $|u + v| = |u| + |v|$

2) $|-2u| = 2|u|$

3) $|u \times v| \leq |u||v|$

4) $|u \cdot v| \leq |u||v|$

5) $|u \times v| \leq |u \cdot v|$

6) $u \cdot v = v \cdot u$

7) $u \times v = v \times u$

8) $(u \times v) \times w = u \times (v \times w)$

9) $(u \times v) \cdot u = 0$

10) The vector $(3, -1, 2)$ is parallel to the plane $6x - 2y + 4z = 1$.

In Problems 11-15, $r(t)$ is a vector function of single variable. Determine whether the statement is true or false. Give reason for your answers.

11) The curve $r(t) = (0, t^2, 4t)$ is a parabola.

12) The curve $r(t) = (2t, 3 - t, 0)$ is a curve passing through the origin.

13) $\frac{d}{dt}|r(t)| = |r'(t)|$

14) The projection of the curve $r(t) = (\cos 2t, t, \sin 2t)$ onto the xz-plane is a circle.

15) If the curvature is equal to 0 everywhere on the curve then the curve must be a straight line.

In Problems 16-20, classify the given surfaces (cylinder/ ellipsoid/ elliptic paraboloid/ hyperbolic paraboloid/ etc).

16) In $\mathbb{R}^3$, the graph of $y = x^2$ is a/an ______________.

17) The set of points $\{(x, y, z) | x^2 + y^2 = 1\}$ is a/an ______________.

18) In $\mathbb{R}^3$, $x^2 + 4y^2 + z^2 = 1$ is the equation of a/an ______________.

19) The set of points $\{(x, y, z) | x^2 + 4y^2 - z = 0\}$ is a/an ______________.

20) The set of points $\{(x, y, z) | x^2 - 4y^2 - z = 0\}$ is a/an ______________.

Write solutions to the following problems.

21) Write the equation of the plane passing through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$.

22) Write the equation of the plane passing through $(3, -1, 1)$, $(4, 0, 2)$, $(6, 3, 1)$.

23) Find the area of the triangle with vertices at $(3, -1, 1)$, $(4, 0, 2)$, $(6, 3, 1)$.

24) Write the equation of the plane passing through $(1, 2, -2)$ and containing the line $x = 2t, y = 3 - t, z = 1 + 3t$.

25) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

26) Find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.

27) Write the equation of the tangent plane to the surface $z = 3x^2 - y^2 + 2x$ at point $(1, -2, 1)$.

28) The rate of change of function $f(x, y) = xy + y^2$ in the direction of vector $\langle 0, 1\rangle$ at point $(2, 1)$ is ______. At this point, the function increases the fastest in the direction of (unit) vector ______.

29) A function $f(x, y)$ satisfying $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$ is said to be ____________ at (x_0, y_0).

30) Where is the function $f(x, y) = \frac{e^x+e^y}{e^{xy}-1}$ continuous?

31) Along a level set of a function, the rate of change of the function is _________.

32) The graph of $f(x, y)$ is __________________ of $g(x, y, z) = z - f(x, y)$.

33) Let $u = \ln(1 + se^t)$. Express the total differential du in terms of ds and dt.

34) By Clairaut's Theorem, a smooth function $f(x, y)$ has at most _________ different partial derivatives of third order.

35) Let $f(x, y) = ax(1 + y) + by$. If $\nabla f(1, 1) = \langle 2, 1\rangle$ then $a =$______ and $b =$______.

36) Find linear approximation of $f(x, y) = x^3 - 2xy^2$ around $(1, 1)$.

37) If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y) = L$. True or false?

38) $\lim_{(x,y)\to(1,1)} \frac{2xy^2}{x^2+y^2} =$ ______ (or write DNE if the limit doesn't exist.)

39) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} =$ ______

40) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} =$ ______

Solution keys:

1) False
2) True
3) True
4) True
5) False
6) True
7) False
8) False
9) True
10) False
11) True
12) False
13) False
14) True
15) True
16) parabolic cylinder
17) circular cylinder
18) ellipsoid
19) elliptic paraboloid
20) hyperbolic paraboloid
21) $x + 4y - 3z = 6$
22) $-4x + 3y + z + 14 = 0$
23) $\frac{\sqrt{26}}{2}$
24) $6x + 9y - z = 26$
25) $r(t) = \langle 4\cos t, 4\sin t, 5 - 4\cos t\rangle$
26) $\frac{2}{5^{3/2}}$
27) $z = 8x + 4y + 1$
28) 4 and $\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$
29) continuous
30) Everywhere in $\mathbb{R}^2$ except for the x-axis and the y-axis
31) zero
32) the 0-level set
33) $du = \frac{e^t}{1+se^t} ds + \frac{se^t}{1+se^t} dt$
34) 4
35) $a = 1$ and $b = 0$
36) $f(x, y) \approx -1 + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = x - 4y + 2$
37) False. Can you give an example?
38) 1
39) 0
40) DNE